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MATHEMATICS

General Theorem on a Finite Support of Mixed Strategy in the Theory of Zero-Sum Games

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Abstract—A theorem related to the theory of zero-sum games is proved. Rather general assumptions on the payoff function are found that are sufficient for an optimal strategy of one of the players to be chosen in the class of mixed strategies concentrated in at most $m + 1$ points if the opponent chooses a pure strategy in a finite-dimensional convex compact set and m is its dimension. This theorem generalizes results of several authors, starting from Bohnenblust, Karlin, and Shapley (1950).

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This paper generalizes the result obtained by Bohnenblust, Karlin, and Shapley in [1] (see also [2, Chapter 2, Subsections 3.5, 3.6; 3, Chapter 20, p. 736]). The redundant assumption that the function f is continuous is replaced with semicontinuity, so that our assumptions are similar to those made in Kneser's [4] and Fan's [5] theorems. Moreover, we use quasiconvexity in the spirit of Sion's work [6]. In fact, the present result is an improvement of these theorems for the case when one of the arguments of the payoff function (involved in the theorems in [4–6]) takes values in a finite-dimensional convex compact set. In style, our proof is close to Davydov's [7], which, in turn, relies on Shnirel'man's ideas [8]. However, the function f in [7, 8] is assumed to be jointly continuous with respect to its arguments. Interestingly, our result was obtained in studying a financial problem, namely, the problem of pricing and hedging of a contingent claim on option, formalized with the help of guaranteed estimation method [9], i.e., the solution of the corresponding Bellman–Isaacs equation relies on the proved theorem. The proof of the theorem is of interest. Moreover, the following result is of interest in itself.

Lemma. *Let*

(i) *X be a nonempty compact convex subset of* $\mathbb{R}^m$, $m \geq 1$, *Y be an arbitrary nonempty set*;

(ii) $f(x, y)$ *be a scalar function of arguments* $x \in X$ *and* $y \in Y$ *such that, for any* $y \in Y$, *the functions* $x \mapsto f(x, y)$ *are*

(a) *lower semicontinuous and*

(b) *quasiconvex.*

Define[1]

$$W = \min_{x \in X} \sup_{y \in Y} f(x, y), \tag{1}$$

$$W_m = \sup_{y_1 \in Y, \ldots, y_{m+1} \in Y} \min_{x \in X} \bigvee_{i=1}^{m+1} f(x, y_i);$$

then $W = W_m$.

Proof. It is easy to check that $W_m \leq W$. Indeed, for all $x \in X$ and $y_i \in Y$, $i = 1, 2, \ldots, m + 1$,

$$\bigvee_{i=1}^{m+1} f(x, y_i) \leq \sup_{y \in Y} f(x, y).$$

Therefore, for all $y_i \in Y$, $i = 1, 2, \ldots, m + 1$,

$$\min_{x \in X} \bigvee_{i=1}^{m+1} f(x, y_i) \leq \min_{x \in X} \sup_{y \in Y} f(x, y)$$

1 Here and below, the symbol $\vee$ is used to denote the binary associative (and commutative) operation $\vee$ of taking the maximum, which justifies notation of the form $\bigvee_{i=1} a_i$. The use of the minimum in (1) is correct, since the function $x \mapsto \sup_{y \in Y} f(x, y)$ is lower semicontinuous; moreover, this function can take the value of $+\infty$ (here, by neighborhoods of $+\infty$, we mean intervals of the form (a, $+\infty$], $a \in \mathbb{R}$), so that the minimum is reached at some point of the compact set X.

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and, hence,

$$\sup_{y_1\in Y,\dots,y_{m+1}\in Y}\ \min_{x\in X}\bigvee_{i=1}^{m+1} f(x,y_i) \le \min_{x\in X}\sup_{y\in Y} f(x,y).$$

Let us prove the reverse inequality $W \le W_m$. It is nontrivial only if Y contains more than $m+1$ elements. Indeed, since the operation $\vee$ of taking the maximum is jointly continuous with respect to its arguments, the function $x \mapsto \bigvee_{i=1}^{m+1} f(x,y_i)$ is lower semicontinuous, so that its minimum is reached at some point of X (depending on $y_1, \dots, y_{m+1}$). Therefore, if we define

$$C_y = \{x\colon f(x,y) \le W_m\},$$

then

$$\bigcap_{i=1}^{m+1} C_{y_i} \ne \phi$$

for any $y_1, \dots, y_{m+1}$ from Y.

Note that, by assumption (a) in (ii), C_y are closed, hence compact, while, by assumption (b) in (ii), they are convex; thus, the Helly theorem [10] is applicable:

$$\bigcap_{y\in Y} C_y \ne \phi.$$

Therefore, there is a point $x^*\in X$ such that $f(x^*,y) \le W_m$ for all $y \in Y$. Consequently,

$$W = \min_{x\in X}\sup_{y\in Y} f(x,y) \le \sup_{y\in Y} f(x^*,y) \le W_m.$$

Now we formulate and prove the theorem. Define

$$S_m = \left\{ q = (q_1,\dots,q_{m+1}) \in \mathbb{R}^{m+1}\colon q_i \ge 0,\ i = 1,2,\dots,m+1;\ \sum_{i=1}^{m+1} q_i = 1 \right\}, \quad m \ge 0.$$

Theorem. *Consider a zero-sum game satisfying the following assumptions*:

(I) *The first player chooses pure strategies in a space X that is a compact convex subset of $\mathbb{R}^m$, $m \ge 1$, while the second player chooses a mixed strategy, namely, a distribution*[2] *on a nonempty set Y.*

(II) *The payoff function $f(x,y)$, $x \in X$, $y \in Y$ is such that, for any $y \in Y$, the functions $x \mapsto f(x,y)$*

(a) *are lower semicontinuous*,

(b) *convex combinations of functions from the family $x \mapsto f(x,y)$, $y \in Y$, are quasiconvex.*[3]

Then the following equality is valid:

$$\min_{x\in X}\sup_{y\in Y} f(x,y) = \sup_{Q\in\mathcal{P}_Y^m}\min_{x\in X}\int f(x,y)Q(dy), \quad (*)$$

where $\mathcal{P}_Y^m$ is the class of all probability measures on Y concentrated in at most $m+1$ points, i.e.,

$$\mathcal{P}_Y^m = \left\{ Q = \sum_{i=1}^{m+1} q_i\delta_{y_i}\colon q \in S_m,\ y_i \in Y,\ i, i+1,\dots,m+1 \right\},$$

and δ_y denotes a probability measure concentrated at the point $y \in Y$.

(III) *Under the additional assumption that Y is a compact Hausdorff topological space and the functions $y \mapsto f(x,y)$ are upper semicontinuous for all x, the supremum on both sides of* (*) *can be replaced by the maximum.*

Proof. Note that, for any $x \in X$, $u \ge 0$, and $y_i \in Y$, $i = 1, 2, \dots, n+1$, the function

$$q = (q_1,\dots,q_{n+1}) \mapsto \sum_{i=1}^{n+1} q_i f(x,y_i), \quad q \in S_n,$$

is affine and continuous and, by assumption (IIb), the convex combination

$$x \mapsto \sum_{i=1}^{n+1} q_i f(x,y_i), \quad x \in X,$$

is a quasiconvex function and, as a consequence of assumption (IIa), is lower semicontinuous for any $q \in S_n$. Applying Sion's theorem (see [6, Corollary 3.3]), we obtain

$$\min_{x\in X}\max_{q\in S_n}\sum_{i=1}^{n+1} q_i f(x,y_i) = \max_{q\in S_n}\min_{x\in X}\sum_{i=1}^{n+1} q_i f(x,y_i). \quad (2)$$

Moreover, it is obvious that

$$\bigvee_{i=1}^{m+1} f(x,y_i) = \max_{q\in S_n}\sum_{i=1}^{n+1} q_i f(x,y_i). \quad (3)$$

If $Y = \{y_1, \dots, y_{n+1}\}$ for $n \ge 0$ and $n \le m$, then it is obvious that $\mathcal{P}_Y^n = \mathcal{P}_Y^m$; in view of (2) and (3), we obtain (*).

If Y contains more than $m+1$ elements, we resort to the lemma proved above. Fix an arbitrary $\varepsilon > 0$ and choose $y_1^\varepsilon, \dots, y_{m+1}^\varepsilon$ from Y such that

[2] That is, the probability measure on $(Y, \mathcal{A})$, where the σ-algebra $\mathcal{A}$ contains all singletons Y and the functions $y \mapsto f(x,y)$ must be measurable with respect to $\mathcal{A}$. In fact, a particular choice of $\mathcal{A}$ is not used in the proof of the first part (i) of the theorem, which does not rely on assumption (III), so, in this part, we can assume, for example, it $\mathcal{A}$ is the σ-algebra of all at most countable subsets of Y. Then the measurability condition is satisfied for any function, while we discuss only discrete distributions.

[3] That is, functions of the form $x \mapsto \sum_{i=1}^{n} p_i f(x,y_i)$ for any $p = (p_1,\dots,p_n) \in S_{n-1}$, $n \ge 1$, and arbitrary $y_i \in Y$, $i = 1, 2, \dots, n$, are quasiconvex. Specifically, the functions $x \mapsto f(x,y)$, $y \in Y$ are quasiconvex.

$$\min_{x\in X}\bigvee_{i=1}^{m+1} f(x,y_i^\varepsilon) \geq \left[\sup_{y_1\in Y,\ldots,y_{m+1}\in Y}\min_{x\in X}\bigvee_{i=1}^{m+1} f(x,y_i)\right]-\varepsilon = W-\varepsilon, \tag{4}$$

where W is given by formula (1). Applying (2) with $n = m$ and $y_i = y_i^\varepsilon$, $i = 1, 2, \ldots, m + 1$, we note that the maximum over q on the right-hand side of (2) is reached at some $q^\varepsilon \in S_m$. Defining $Q^\varepsilon = \sum_{i=1}^{m+1} q_i^\varepsilon \delta_{y_i^\varepsilon} \in \mathcal{P}_Y^m$ and taking into account (2)–(4) yields

$$\min_{x\in X}\int f(x,y)Q^\varepsilon(dy) \geq W-\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\sup_{Q\in\mathcal{P}_Y^m}\min_{x\in X}\int f(x,y)Q(dy) \geq W.$$

The reverse inequality follows from an inequality valid for any distribution $Q \in \mathcal{P}_Y^m$, namely,

$$\min_{x\in X}\int f(x,y)Q(dy) \leq \min_{x\in X}\sup_{y\in Y} f(x,y) = W,$$

since the mean of the function does not exceed its supremum.

If (III) is assumed, the maximum on the left-hand side of (∗) is attained by the compactness of Y and the upper semicontinuity of the functions $y \mapsto f(x,y)$.

By A.D. Alexandroff's theorem[4], for every $x \in X$, the function $Q \mapsto \int f(x,y)Q(dy)$ is upper semicontinuous in the weak topology on the space $\mathcal{P}_Y$ of all Borel probability measures[5] on Y (i.e., a minimum topology such that the functions $Q \mapsto \int g(y)Q(dy)$ are continuous for any continuous scalar function g on Y).

The set $\mathcal{P}_Y^m \subseteq \mathcal{P}_Y$ is closed. Indeed, if the net Q^α, $\alpha \in I$, of measures from $\mathcal{P}_Y^m$ converges to a measure $Q \in \mathcal{P}_Y$ in the weak topology, then Q^α can be represented as $Q^\alpha = \sum_{i=1}^{m+1} q_i^\alpha \delta_{y_i^\alpha}$ and, since Y and the simplex S_m are compact, there exist subnets $q^\beta = (q_1^\beta,\ldots,q_{m+1}^\beta) \in S_m$ and $y_1^\beta \in Y,\ldots,y_{m+1}^\beta \in Y$ converging to the points $q^* = (q_1^*,\ldots,q_{m+1}^*) \in S_m$ and $y_1^* \in Y,\ldots,y_{m+1}^* \in Y$, respectively; moreover, it is obvious that the subnet Q^β converges weakly to the measure $Q^* = \sum_{i=1}^{m+1} q_i^* \delta_{y_i^*}$, so that $Q = Q^* \in \mathcal{P}_Y^m$. Since $\mathcal{P}_Y$ is compact in the weak topology, so is $\mathcal{P}_Y^m$.

Thus, the function $Q \mapsto \int f(x,y)Q(dy)$ reaches a maximum on $\mathcal{P}_Y^m$.

Remark 1. Assumption (IIb) in the theorem is satisfied, for example, in the one-dimensional case (when $m = 1$) for $X = [a,b] \subseteq \mathbb{R}$ and a family of unimodal functions $x \mapsto f(x,y)$, $y \in Y$, with a common minimizer (i.e., when there exists an argument $x^* \in [a,b]$ such that the functions $x \mapsto f(x,y)$, $y \in Y$, are nonincreasing for $x \leq x^*$ and nondecreasing for $x \geq x^*$).

Remark 2. The convexity of the functions $x \mapsto f(x,y)$, $y \in Y$, is sufficient for the validity of assumption (IIb) of the theorem.

Remark 3. Assumption (IIb) can also be replaced by following one: there is a strictly monotone and continuous scalar function φ such that the composition $\varphi \circ f$ satisfies the conditions in Remark 2, i.e., the function $x \mapsto \varphi(f(x,y))$ is convex for all $y \in Y$ (specifically, the functions $x \mapsto f(x,y)$ are quasiconvex[6] for all $y \in Y$).

Remark 4. The class $\mathcal{P}_Y^m$ is not convex if Y contains more than $m + 1$ points.

Remark 5. A simple example shows that only the quasi-convexity of the functions $x \mapsto f(x,y)$, $y \in Y$ (rather than assumption (IIb) of the theorem) is not sufficient for the validity of (∗) and that the condition in Remark 1 on a common minimizer for the family of unimodal functions $x \mapsto f(x,y)$, $y \in Y$ is essential. Let

$$f(x,y) = 4-(x-y)^2,$$
$$x \in [-1,1] = X, \quad y \in \{-1,1\} = Y.$$

Here, the functions $x \mapsto f(x,y)$, $y \in Y$, are quasiconvex and unimodal (with unique, but different minimizers). It is easy to see that

$$\min_{x\in X}\max_{y\in Y} f(x,y) = 3,$$

but

$$\max_{Q\in\mathcal{P}_Y}\min_{x\in X}\int f(x,y)Q(dy) = 2.$$

[4] See Theorem 2 in [11, Section 16] (more precisely, a lower (upper) semicontinuous function whose argument runs over a set of probability measures and whose value is the measure of an open (closed) set is considered in [11]. However, this case is easy to extend to an integral of a semicontinuous function (see, e.g., [12, Theorem 9.1.5]).

[5] Here, it is natural to define the domain $\mathcal{A}$ of the measures Q as a Borel σ-algebra. The functions $y \mapsto f(x,y)$ are measurable due to their upper semicontinuity.

[6] Note that a quasiconvex function is not necessarily representable as a composition of a strictly monotone scalar function and a convex function. For the first time, this was noted by Fenchel [13].

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