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MATHEMATICS

On the Stability of a Periodic Hamiltonian System with One Degree of Freedom in a Transcendental Case

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Abstract—The stability of an equilibrium of a nonautonomous Hamiltonian system with one degree of freedom whose Hamiltonian function depends 2π-periodically on time and is analytic near the equilibrium is considered. The multipliers of the system linearized around the equilibrium are assumed to be multiple and equal to 1 or –1. Sufficient conditions are found under which a transcendental case occurs, i.e., stability cannot be determined by analyzing the finite-power terms in the series expansion of the Hamiltonian about the equilibrium. The equilibrium is proved to be unstable in the transcendental case.

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Consider a mechanical system whose motion is described by the canonical equations

$$\frac{dq}{dt} = \frac{dH}{dp}, \quad \frac{dp}{dt} = -\frac{dH}{dq}. \tag{1}$$

Assume that the origin $q = p = 0$ is an equilibrium of this system and that the Hamiltonian function H in a sufficiently small neighborhood of the origin can be represented in the form of a convergent series

$$H = \sum_{k=2}^{\infty} H_k(q, p, t), \tag{2}$$

where H_k is a homogeneous form of degree k in q and p with continuous coefficients being 2π-periodic functions of t.

If the multipliers of the linear system with Hamiltonian H_2 are not equal to unity in absolute value, then the equilibrium of system (1) is Lyapunov unstable. However, if they are, we have the critical case, when stability cannot be determined by analyzing the linear system [1].

According to the general technique for solving the stability problem in the critical case, system (1) has to be normalized, i.e., we need to construct a canonical change of variables reducing the Hamiltonian function (2) to its normal form. The Hamiltonian has to be normalized up to terms with powers of q and p such that the stability problem for the truncated system with the Hamiltonian being the normalized part of the Hamiltonian function is equivalent to the stability problem for the original system. Stability can be concluded on the basis of sufficient conditions written as inequalities for the coefficients of the normal form [2].

In the general case, it suffices to normalize the Hamiltonian function up to terms of the fourth power inclusive. However, there are degenerate cases in which normalization has to be performed up to terms of higher powers. In such degenerate cases, the stability problem was studied in detail in recent works [3–5].

In this paper, we consider a special degenerate case in which no strict stability conclusions can be drawn by normalizing the Hamiltonian of the problem up to terms of arbitrarily high (but finite) power, i.e., when the above-mentioned general approach to stability analysis cannot be applied.

1. FORMULATION OF THE PROBLEM

Let the multipliers of the linear system with Hamiltonian H_2 be multiple and equal to 1 or –1. In this case, the characteristic exponents $\pm i\lambda$ are such that either $\lambda = N$, where N is an integer (there is a first-order resonance) or $2\lambda = 2N + 1$ (there is a second-order resonance). The stability analysis of an equilibrium under first- and second-order resonances and a qualitative analysis of the motion in its neighborhood have been addressed in numerous works (see, e.g., [3, 6, 7] and references therein).

Assume that the elementary divisors of the characteristic matrix of the linear system are not simple.

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Then, by using a linear real change of variables $q, p \to x, y$ that is T-periodic in t (where $T = 2\pi$ in the case of first-order resonance and $T = 4\pi$ in the case of second-order resonance), Hamiltonian (2) can be reduced to the form (see [8, 9])

$$H = \frac{1}{2}\delta y^2 + H_3 + \ldots + H_k + \ldots, \qquad (3)$$

where $H_k = \sum_{\nu+\beta=k} h_{\nu\beta} x^\nu y^\beta$ and the coefficients $h_{\nu\beta}(t)$ are continuous T-periodic functions of t. The value of δ is equal to 1 or -1.

With the help of a canonical close-to-identity analytical change of variables $x, y \to \xi, \eta$ (constructed, for example, with the help of the Deprit–Hori method [2, 10]) Hamiltonian (3) can be brought to the form (see [8, 11])

$$\Gamma = \frac{1}{2}\delta\eta^2 + a_M\xi^M + \cdots (M \geq 3), \qquad (4)$$

where M is an integer and the coefficient a_M is a constant. If $a_M \neq 0$, then the stability of the equilibrium of the system with Hamiltonian (4) (and, hence, of the original system (1)) can be determined using the following criterion [8].

Criterion. *If M is an even number and* $\delta a_M > 0$, *then the equilibrium is Lyapunov stable. If M is an odd number or M is an even number, but* $\delta a_M < 0$, *then the equilibrium is Lyapunov unstable.*

In the study of the dynamic stability of particular mechanical systems, one can encounter a special case where $a_M = 0$ for any M. Following the terminology proposed by Lyapunov [12], this case in the stability problem will be called transcendental. Obviously, in the transcendental case, the above criterion is not applicable. Transcendental cases are of interest both theoretically and in applications. Specifically, they occur in classical and celestial mechanics. It was established in [13] that the transcendental case takes place in the problem of determining the orbital stability of periodic pendulum motions of a heavy rigid body in the Goryachev–Chaplygin case. Due to the presence of a first integral, it was shown that the periodic pendulum motions are orbitally unstable in the indicated case. Note also that a transcendental situation also arises in the Lagrange integrable case [14].

The stability of an equilibrium of an autonomous Hamiltonian system with two degrees of freedom in the transcendental case when the characteristic equation of the linearized system has a multiple zero at the origin was studied in [15].

The goal of this work is to obtain sufficient conditions for the existence of a transcendental situation and to study the stability of the equilibrium of system (1) in the transcendental case.

2. CONDITIONS FOR THE EXISTENCE OF A TRANSCENDENTAL CASE

In [11] a transcendental case was considered in connection with the existence of a family of periodic solutions for system (1). Specifically, in the transcendental case, the system was shown to have a one-parameter (analytical with respect to the parameter) family of T-periodic solutions emanating from the equilibrium. In other words, the existence of this family of periodic solutions is a necessary condition for the transcendental situation occurring in system (1). Let us show that this condition is also sufficient. Indeed, suppose that system (1) admits a family of periodic solutions that, in the variables x, y, have the form

$$x = g(t, \alpha), \quad y = f(t, \alpha), \qquad (5)$$

where $g(t, \alpha)$ and $f(t, \alpha)$ are T-periodic functions of t that are analytic with respect to the parameter α and vanish identically at $\alpha = 0$.

Let us construct a canonical T-periodic (in t) change of variables $x, y \to \alpha, p_\alpha$ such that solution (5) in the new variables α, p_α becomes

$$\alpha = \text{const}, \quad p_\alpha = 0, \qquad (6)$$

i.e., in the variables α, p_α, the one-parameter family of T-periodic solutions (5) is associated with a family of equilibrium positions. The generating function $S(\alpha, y)$ of the above-mentioned canonical transformation is determined by the conditions

$$x = -\frac{\partial S}{\partial y}, \quad p_\alpha = -\frac{\partial S}{\partial \alpha}. \qquad (7)$$

Simple computations show that S can be defined as

$$S = -yg(\alpha, t) + \int_0^\alpha \frac{\partial g}{\partial u}(t, u) f(t, u) du. \qquad (8)$$

The Hamiltonian function written in the variables α, p_α becomes

$$\Gamma = \frac{1}{2} p_\alpha^2 + p_\alpha^2 F(\alpha, p_\alpha, t), \qquad (9)$$

where F is a T-periodic function of t that is analytic near $\alpha = p_\alpha = 0$; moreover, $F(0, 0, t) \equiv 0$.

The structure of Hamiltonian function (9) guarantees that, after normalization, the coefficients a_M of normal form (4) vanish for any M. Indeed, let the series expansion of F in powers of α and p_α begins with terms of some power k $(k \geq 1)$. We perform a canonical change of variables $\alpha, p_\alpha \to \alpha_*, p_{\alpha*}$ given by

$$\alpha_* = \frac{\partial W}{\partial p_{\alpha*}}, \quad p_\alpha = \frac{\partial W}{\partial \alpha} \qquad (10)$$

with generating function

$$W = \alpha p_{\alpha *} + p_{\alpha *}^2 W_k(\alpha, p_{\alpha *}, t), \qquad (11)$$

where $W_k(\alpha, p_{\alpha *}, t)$ is a form of degree k in α and $p_{\alpha *}$ with coefficients being T-periodic functions of t. It is easy to show that these coefficients can be chosen so that the Hamiltonian function in the new variables $\alpha_*, p_{\alpha *}$ becomes

$$\Gamma_* = \frac{1}{2} p_{\alpha *}^2 + p_{\alpha *}^2 F_*(\alpha_*, p_{\alpha *}, t), \qquad (12)$$

where F_* is a T-periodic function of t and its series expansion about $\alpha_* = p_{\alpha *} = 0$ begins with terms whose powers are at least $k+1$. This means that terms with any finite power M $(M \geq 3)$ can be eliminated from the Hamiltonian function by applying a sequences of canonical changes of form (10), i.e., the transcendental case occurs in the system with Hamiltonian (12).

3. STABILITY ANALYSIS OF THE EQUILIBRIUM IN THE TRANSCENDENTAL CASE

Let us show that, in the transcendental case, the equilibrium $q = p = 0$ of system (1) is unstable. Since the stability problem in the original variables q, p is equivalent to one in the variables α, p_α, it is sufficient to show the instability of the system with Hamiltonian (9). This can be done by applying Chetaev's theorem.

Consider the Chetaev function $V = \alpha p_\alpha$. It is positive in the domain $\alpha > 0, p_\alpha > 0$ and vanishes on its boundary. The derivative of V can be calculated using the canonical equations with Hamiltonian (9):

$$\frac{dV}{dt} = \frac{d\alpha}{dt} p_\alpha + \alpha \frac{dp_\alpha}{dt}$$
$$= p_\alpha^2 \left(1 + 2F(\alpha, p_\alpha, t) + p_\alpha \frac{\partial F}{\partial p_\alpha} - \alpha \frac{\partial F}{\partial \alpha}\right).$$

For sufficiently small α and p_α, the derivative $\frac{dV}{dt}$ is positive in the domain $V > 0$. Thus, by Chetaev's theorem, the equilibrium $\alpha = p_\alpha = 0$ is unstable, which obviously implies that the equilibrium of the original system is unstable as well. Combining the results of this and the preceding sections, we obtain the following theorem.

Theorem. *If system* (1) *admits a one-parameter family of T-periodic solutions* ($T = 2\pi$ *or* $T = 4\pi$) *emanating from the equilibrium* $q = p = 0$, *then a transcendental case occurs and the equilibrium is unstable.*

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