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MATHEMATICS

On the Regularity of a Boundary Point for the $p(x)$-Laplacian

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Abstract—The Dirichlet problem for the $p(x)$-Laplacian with a continuous boundary function is considered, and a sufficient condition is found for the continuity of its solution at a boundary point, assuming that the exponent $p(x)$ satisfies the log-Hölder continuity condition at this point.

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In a bounded domain $D \subset \mathbb{R}^n$, $n \geq 2$, we consider the equation

$$Lu = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = 0, \tag{1}$$

where $p(x)$ is a measurable function in D that satisfies the condition

$$1 < \alpha \leq p(x) \leq \beta \quad \text{for almost all} \quad x \in D. \tag{2}$$

In the case $p = \text{const}$, this equation has been studied in detail. Specifically, Ladyzhenskaya and Uraltseva established the Hölder continuity of its solutions [1], and Serrin proved the Harnack inequality for nonnegative solutions [2].

For the first time, equations of form (1) were considered by Zhikov [3, 4] in the context of homogenizing integrands of the form $|\nabla u|^{p(x)}$ as applied to elasticity problems. Such equations also arise in the mathematical simulation of fluids with properties varying under the influence of an electromagnetic field or temperature [5, 6].

To define the solution of Eq. (1), we introduce the class of functions

$$W(D) = \{u \in W^{1,1}(D) \colon |\nabla u|^{p(x)} \in L^1(D)\},$$

where $W^{1,1}(D)$ is the Sobolev space of functions that are summable in D together with their generalized first derivatives. A sequence $u_j \in W(D)$ is said to converge in $W(D)$ to a function $u \in W(D)$ if $u_j \to u$ in $L^1(D)$ and

$$\lim_{j\to\infty}\int_D |\nabla u - \nabla u_j|^{p(x)}dx = 0. \tag{3}$$

We say that $u \in W(D)$ belongs to the class $W_0(D)$ if there exists a sequence of functions $u_j \in W(D)$ with a compact support in D such that (3) holds. A sequence $u_j \in W_0(D)$ is said to converge in $W_0(D)$ to a function $u \in W_0(D)$ if (3) holds.

We are interested in the classes of functions $H(D)$ and $H_0(D)$ that are the completions, in $W(D)$ and $W_0(D)$, of smooth functions in D with respect to the introduced convergence properties. Namely, let

$$H(D) = \{u \in W(D)\colon$$
$$\exists u_j \in C^\infty(D) \cap W(D), u_j \to u \quad \text{in} \quad W(D)\},$$
$$H_0(D) = \{u \in W(D)\colon$$
$$\exists u_j \in C_0^\infty(D), u_j \to u \quad \text{in} \quad W_0(D)\}.$$

Zhikov's results [4] imply that assumption (2) alone is insufficient for smooth functions to be dense in the classes $W(D)$ and $W_0(D)$. The density of smooth functions in these classes is ensured by the well-known logarithmic condition

$$|p(x) - p(y)| \leq k_0\left(\ln\frac{1}{|x-y|}\right)^{-1} \quad \text{for} \quad x, y \in D, \ |x-y| < \frac{1}{2}, \tag{4}$$

which was found by Zhikov [7].

Let us define H-solutions and W-solutions of Eq. (1). A function $u \in H(D)$ (respectively, $u \in W(D)$) is

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called an H-solution (W-solution) of Eq. (1) if it satisfies the integral identity

$$\int_D |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\psi dx = 0 \tag{5}$$

for test functions $\psi \in H_0(D)$ ($\psi \in W_0(D)$, respectively).

In this paper, we consider only H-solutions of Eq. (1), which are hereafter referred to as solutions.

Consider the Dirichlet problem

$$\begin{gathered} Lu = 0 \quad \text{in} \quad D, \quad u \in H(D), \\ f \in H(D), \quad (u - f) \in H_0(D). \end{gathered} \tag{6}$$

Its solution is related to the minimizer w of the variational problem

$$\inf_{w \in C_0^\infty(D)} F(w + f) = \min_{w \in H_0(D)} F(w + f),$$

$$\text{where} \quad F(v) = \int_D \frac{|\nabla v|^{p(x)}}{p(x)} dx,$$

by the formula $u = w + f$.

The unique solvability of problem (6) in a Lipschitz domain D follows from Theorem 5.2 in [7]. For an arbitrary bounded domain D, the required result is contained in Theorem 3.1 in [8].

GENERALIZED DIRICHLET PROBLEM

Below, we consider the generalized Dirichlet problem

$$Lu_f = 0 \quad \text{in} \quad D, \quad u_f|_{\partial D} = f \tag{7}$$

with a continuous function f on ∂D and examine the behavior of its solution at a fixed boundary point $x_0 \in \partial D$, assuming that the exponent p satisfies condition (2) and has a logarithmic modulus of continuity at x_0, i.e.,

$$\begin{gathered} |p(x) - p(x_0)| \le k_0 \left(\ln \frac{1}{|x - x_0|} \right)^{-1} \\ \text{for} \quad x \in D, \quad |x - x_0| < \frac{1}{2}. \end{gathered} \tag{8}$$

The solution of problem (7) is defined as follows. The boundary function $f \in C(\partial D)$ is extended by continuity to $\overline{D}$ with the same notation retained for the extension. Consider a sequence of infinitely differentiable functions f_k in $\mathbb{R}^n$ that converge uniformly on $\overline{D}$ to f. Solve the Dirichlet problems

$$\begin{gathered} Lu_k = 0 \quad \text{in} \quad D, \\ u_k \in H(D), \quad (u_k - f_k) \in H_0(D). \end{gathered}$$

By the maximum principle, the sequence $\{u_k\}$ converges uniformly in D to a function u belonging to $H(D')$ for all $D' \Subset D$. Since the sequence ∇u_k is uniformly bounded in $L^{p(x)}_{\text{loc}}(D)$, we can assume that ∇u_k converges weakly to ∇u in $L^{p(x)}_{\text{loc}}(D)$ and $|\nabla u_k|^{p(x)-2}\nabla u_k$ converges weakly in $L^{p'(x)}_{\text{loc}}(D)$ to an element $\xi \in L^{p'(x)}_{\text{loc}}(D)$. Let us show that $\xi \equiv |\nabla u|^{p(x)-2}\nabla u$ by applying the methods of [9]. Let $0 \le \eta \in C_0^\infty(D)$. By virtue of integral identity (5) for u_k, we have

$$\int_D (|\nabla u_k|^{p(x)-2}\nabla u_k - |\nabla u|^{p(x)-2}\nabla u) \cdot (\nabla u_k - \nabla u)\eta dx$$

$$= -\int_D (u_k - u)|\nabla u_k|^{p(x)-2}\nabla u_k \cdot \nabla\eta dx$$

$$- \int_D \eta|\nabla u|^{p(x)-2}\nabla u \cdot (\nabla u_k - \nabla u) dx \to 0 \quad \text{as} \quad k \to \infty.$$

With the help of the monotonicity

$$\begin{gathered} (|\xi|^{p(x)-2}\xi - |\eta|^{p(x)-2}\eta) \cdot (\xi - \eta) > 0 \\ \text{for all} \quad \xi, \eta \in \mathbb{R}^n, \quad \xi \ne \eta, \end{gathered}$$

we find that $\nabla u_k(x)$ converges to ∇u almost everywhere in D. Therefore, $|\nabla u_k|^{p(x)-2}\nabla u_k$ converges to $|\nabla u|^{p(x)-2}\nabla u$ weakly in $L^{p'(x)}(D')$ for any $D' \Subset D$. Passing to the limit for u_k in integral identity (5), we see that (5) holds for the limit function u on test functions $\psi \in H_0(D)$. The limit function, which is bounded in D and independent of the extension method or the approximation of the boundary function f, is called a weak solution of Dirichlet problem (7). In this construction, the exponent p is only required to have property (2).

REGULARITY OF A BOUNDARY POINT

Definition. A boundary point $x_0 \in \partial D$ is called *regular* if

$$\operatorname*{ess\,lim}_{D \ni x \to x_0} u_f(x) = f(x_0)$$

for any function f continuous on ∂D.

For the Laplace equation, a regularity criterion for a boundary point was obtained by Wiener in [10]. For linear divergence uniformly elliptic equations of the second order, a similar result was obtained in [11]. A sufficient condition for the regularity of a boundary point of the Wiener type and an estimate for the modulus of continuity of the solution near the boundary for the p-Laplacian were found by Maz'ya [12]. Kilpeläinen and Malý [13] proved that this sufficient condition is also necessary. A criterion for the regularity of a boundary point for Eq. (1) under condition (4) was obtained in [8].

To formulate the result, we introduce the concept of capacity. Preliminarily, we extend $p(x)$ to the entire

$\mathbb{R}^n$ with the preservation of properties (2), (8) and denote by B_R an open ball of radius R. The capacity of a compact set $K \subset B_R$ with respect to the ball B_R is the number

$$C_p(K, B_R) = \inf \int_{B_R} \frac{|\nabla\varphi|^{p(x)}}{p(x)} dx,$$

where the infimum is taken over the set of functions $\varphi \in C_0^\infty(B_R)$ equal to unity in a neighborhood of K.

For $x_0 \in \partial D$, let

$$p_0 = p(x_0),$$
$$\gamma(t) = (C_p(\overline{B}_t^{x_0} \backslash D, B_{2t}^{x_0})t^{p_0 - n})^{1/(p_0 - 1)},$$

where $B_t^{x_0}$ is the ball of radius t centered at x_0. The main result of this work is the following theorem.

Theorem 1. *If conditions* (2) *and* (8) *hold and*

$$\int_0 \gamma(t)t^{-1}dt = \infty, \tag{9}$$

then the boundary point $x_0 \in \partial D$ *is regular.*

If p satisfies logarithmic condition (4), then (9) is also a necessary condition for the regularity of the point x_0 [8].

In the case $p_0 > n$, condition (9) always holds, which follows from the capacity estimate $C_p(\{x_0\}, B_t^{x_0}) \geq C(n, p)t^{n - p_0}$. Any point $x_0 \in \partial D$ at which $p(x_0) > n$ and (8) holds is regular. With the help of the Sobolev embedding theorem, for sufficiently small ρ and $r \leq \frac{\rho}{4}$, we can prove the estimate

$$\operatorname*{ess\,sup}_{D \cap B_r^{x_0}} |u_f(x) - f(x_0)| \leq \operatorname*{osc}_{\partial D \cap B_\rho^{x_0}} f + C(n, p) \operatorname*{osc}_{\partial D} f \left(\frac{r}{\rho}\right)^{1 - n/p_0},$$

which implies that u_f is continuous at x_0.

In the case $p_0 \leq n$, a key role in the proof of Theorem 1 is played by the following estimate for the oscillation of the solution u to problem (6) with a smooth (in $\overline{D}$) boundary function f in balls of sufficiently small radius $r < \frac{1}{16}$:

$$\operatorname*{ess\,osc}_{D \cap B_r^{x_0}} u \leq (1 - \delta\gamma(r)) \operatorname*{osc}_{D \cap B_{4r}^{x_0}} u + \delta\gamma(r) \operatorname*{osc}_{\partial D \cap B_{4r}^{x_0}} f + r, \tag{10}$$

where $\delta = \delta(n, p, M) > 0, M = \max_{\partial D} |f|$.

Here, the dependence on p is determined by the constants from conditions (2) and (8). From this, we obtain the following result.

Theorem 2. *Let* $p_0 \leq n$. *There exist positive constants* θ *and* C *depending only on* n, p, *and* M *such that, for* $\rho \leq \rho_0(n, p)$ *and* $r \leq \frac{\rho}{4}$, *we have the inequality*

$$\operatorname*{ess\,sup}_{D \cap B_r^{x_0}} |u_f(x) - f(x_0)| \leq C\left(\operatorname*{osc}_{\partial D \cap B_\rho^{x_0}} f + \rho + \exp\left(-\theta\int_r^\rho \gamma(t)t^{-1}dt\right)\right).$$

The proof of estimate (10) is based on a weak Harnack inequality of special form for the solution u of problem (6) with a smooth boundary function f in $\overline{D}$ such that $0 \leq f \leq M$ on ∂D. Define

$$m = \inf_{\partial D \cap B_{4R}^{x_0}} f, \quad \text{where} \quad R < \frac{1}{16},$$

$$u_m(x) = \begin{cases} \min(u(x), m) & \text{if} \quad x \in D \cap B_{4R}^{x_0}, \\ m & \text{if} \quad x \in B_{4R}^{x_0} \backslash D, \end{cases}$$

$$v_m(x) = u_m(x) + R.$$

Theorem 3. *For any* $0 < q < \frac{n(p_0 - 1)}{n - 1}$,

$$\left(R^{-n} \int_{B_{3R}^{x_0}} v_m^q dx\right)^{1/q} \leq C(n, p, q, M) \operatorname*{ess\,inf}_{B_R^{x_0}} v_m. \tag{11}$$

Estimate (11) is given in Theorem 6.1 in [8], where the proof relies heavily on logarithmic condition (4) holding in a neighborhood of the point x_0. We propose a fundamentally different proof technique whereby Theorem 3 can be proved under the weaker condition (8). The proof is based on a modification of the technique from [14], in which a novel method was used to prove a weak Harnack inequality for nonnegative solutions of nonuniformly degenerate linear elliptic equations of divergence type.

Let us present a geometric condition for the regularity of a boundary point x_0, which is assumed to coincide with the origin O. Suppose that $\mathbb{R}^n \backslash \overline{D}$ in a neighborhood of O has the form

$$\left\{x\colon 0 < x_n < a, \sum_{i=1}^{n-1} x_i^2 < g^2(x_n)\right\},$$

where $g(t)$ is a continuous nondecreasing function such that $t^b \leq g(t) \leq t$ for $t \in [0, a]$, where $b > 1$ is a constant.

Theorem 4. *If* $n - 1 \leq p_0 \leq n$, *then the boundary point* O *is regular, while, for* $p_0 < n - 1$, *a sufficient con-*

dition for its regularity is the divergence of the following integral at zero:

$$\int_0 \left(\frac{g(t)}{t}\right)^{\frac{n-1-p_0}{p_0-1}} \frac{dt}{t} = \infty.$$

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