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MATHEMATICS

# Finite Difference Scheme for Barotropic Gas Equations

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**Abstract**—An implicit finite difference scheme approximating the equations of barotropic gas flow is proposed. This scheme ensures the positivity of density and the validity of an energy inequality and the mass conservation law. The continuity equation is approximated implicitly. It is proved that the resulting system of nonlinear equations has a solution for any time and space stepsizes. An iterative method for solving the system of nonlinear equations at each time step is proposed.

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Constructing finite difference schemes for gas dynamics equations and using these schemes for numerical modelling of compressible ideal (viscous) gas flows is the subject of a huge number of publications (see, for example, [1, 2] and references therein). However, theoretical studies are almost absent. For example, the natural question as to whether a finite difference scheme ensures the positivity of density (which follows from physics) remains open. An exception is the monograph [3], where the positivity of density is proved for the equations of viscous compressible gas in the one-dimensional case, and Zlotnik's papers (see, for example, [4]), where the underlying system of equations is transformed assuming that the density is positive and a finite difference scheme is constructed for this system, yielding the positivity of density.

In the present paper, we propose a finite difference scheme approximating the gas dynamics system of equations that ensures the positivity of density and the conservation of mass balance. The continuity equation is approximated implicitly with respect to the density. Additionally, we prove the existence of a solution to the finite difference problem for any relations between the time and space stepsizes, and we also prove an energy inequality. An iterative process is proposed for solving the system of nonlinear equations arising at every time step.

## 1. STATEMENT OF THE PROBLEM AND A PRIORI ESTIMATES

The system of equations describing flows of ideal barotropic compressible gas has the form (see, for example, [5])

$$\begin{gathered}\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0,\\ \frac{\partial(\rho u)}{\partial t} + \frac{(\rho u^2)}{\partial x} + \frac{\partial p}{\partial x} = 0,\\ p = a\rho^{\gamma}, \quad \gamma = \text{const} > 1.\end{gathered} \tag{1}$$

For simplicity, below we set $a = 1$. Equations (1) are considered in the cylinder $Q_T = [0,1]\times[0,T]$. They are supplemented by the boundary and initial conditions

$$\begin{gathered}u(0,t) = u(1,t) = 0, \quad u(x,0) = u_0(x),\\ \rho(x,0) = \rho_0(x) > 0.\end{gathered} \tag{2}$$

Taking the scalar product of the first equation in (1) (the continuity equation) with $-\frac{1}{2}u^2$ and the second equation in (1) with $u$ and summing up the results, we obtain

$$-\frac{1}{2}(\rho_t, u^2) - \frac{1}{2}((\rho u)_x, u^2) + ((\rho u)_t, u) + ((\rho u^2)_x, u) + (p_x, u) = 0.$$

After simple transformations and integration with respect to time, we have the "energy identity"

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$$\frac{1}{2}\int_0^1 \rho(t)u^2(x,t)dx + \frac{1}{\gamma-1}\int_0^1 p(x,t))dx = \frac{1}{2}\int_0^1 \rho_0 u_0^2 dx + \frac{1}{\gamma-1}\int_0^1 p_0(x)dx \quad (3)$$

(see [6]). Note that quotation marks were used, because the positivity of density has not been proved.

**Remark**. If the equation of motion (the second equation in (1)) contains viscous terms, Eq. (3) becomes an inequality.

Our purpose is to construct a finite difference scheme approximating problem (1), (2) and satisfying the following conditions.

1. If the density $\rho$ is positive at all grid nodes at the initial moment, then this property will also hold at all the subsequent moments.

2. Assuming that the density is positive, relation (3) means that the norms $\|\rho(t)\|_{L_1[0,1]}, \|\rho^{1/2}(t)u(t)\|_{L_2[0,1]}$ are uniformly bounded in time. Hence, we require the solution of the grid problem to be uniformly bounded in time as well.

3. A grid analogue of the mass conservation law must hold.

4. The solution of the finite difference scheme must exist.

## 2. APPROXIMATION OF THE CONTINUITY EQUATION

On the real line, let us introduce a uniform grid with the stepsize $h = \frac{1}{N}, x_i = ih$, and let $\tau$ be the time step. Define the grid functions $\rho_i^n, 0 \le i \le N-1$; $u_i^n, 0 \le i \le N$, $u_0^n = u_N^n = 0$. Let us assume that, beyond the boundaries of the space indices, the grid functions are defined by zero. As usual, we use the notations

$$v_x = \frac{v_{i+1} - v_i}{h}, \quad v_{\bar{x}} = \frac{v_i - v_{i-1}}{h}, \quad v_t = \frac{v^{n+1} - v^n}{\tau}.$$

The nonlinear term in the continuity equation (the first equation in (1)) at the $i$th node is approximated as follows:

$$(\rho u)_x \sim A(u)\rho$$

$$\equiv \begin{cases} \dfrac{\rho_i u_{i+1} - \rho_{i-1}u_i}{h} & \text{if} \quad u_i,\, u_{i+1} > 0, \\ \dfrac{\rho_{i+1}u_{i+1} - \rho_i u_i}{h} & \text{if} \quad u_i,\, u_{i+1} < 0, \\ \dfrac{\rho_i(u_{i+1} - u_i)}{h} & \text{if} \quad u_i < 0,\, u_{i+1} > 0, \\ \dfrac{\rho_{i+1}u_{i+1} - \rho_{i-1}u_i}{h} & \text{if} \quad u_i > 0,\, u_{i+1} < 0. \end{cases} \quad (4)$$

One can check directly that, for smooth functions, this approximation has the order $O(h)$.

The continuity equation is approximated by a two-level implicit difference scheme in time. We use the standard notation $\rho^n = \rho, \rho^{n+1} = \hat{\rho}$, and $\rho_t = \frac{\hat{\rho} - \rho}{\tau}$, where $\tau$ is the time step. The first equation in (1) is approximated as follows:

$$\rho_t + A(u)\hat{\rho} = 0. \quad (5)$$

Let us write (5) in a more convenient form:

$$\frac{\rho_i^{n+1} - \rho_i^n}{\tau} + \frac{-\rho_{i+1}^{n+1}(-u_{i+1} + |u_{i+1}|) + \rho_i^{n+1}(u_{i+1} + |u_{i+1}| - u_i + |u_i|) - \rho_{i-1}^{n+1}(u_i + |u_i|)}{2h} = 0, \quad (6)$$
$$0 \le i \le N-1, \quad n \ge 0.$$

Recall that the grid function $\rho_i^{n+1}$ is defined to be zero for $i = -1, i = N$.

Equation (6) is implicit in $\hat{\rho}$. The matrix of the operator $A$ (up to the factor $\frac{1}{2h}$) is given by

$$A\begin{pmatrix} u_1 + |u_1| & u_1 - |u_1| & 0 & \dots & 0 \\ -u_1 - |u_1| & u_2 + |u_2| - u_1 + |u_1| & u_2 - |u_2| & 0 & \dots \\ 0 & -u_2 - |u_2| & u_3 + |u_3| - u_2 + |u_2| & u_3 - |u_3| & 0 \\ 0 & \dots & -u_3 - |u_3| & \ddots & 0 \\ \dots & \dots & \dots & \ddots & A_{N-2,N-1} \\ 0 & \dots & 0 & A_{N-1,N-2} & A_{N-1,N-1} \end{pmatrix}.$$

The matrix $A$ possesses the following properties:

—its diagonal elements are nonnegative;

–the off-diagonal elements are nonpositive;

–the sum of the elements in a column equals zero.

It follows from (5) that $(I + \tau A(u))\hat{\rho} = \tau\rho$. Then the matrix $I + \tau A(u)$ is an $M$-matrix for any grid function $u$; hence, the inverse matrix to it has all nonnegative entries, and its diagonal elements are strictly positive. This implies the following result.

**Theorem 1**. *Let* $\rho_j^0 > 0, j = 0,1,\ldots,N-1$. *Then, for any* $n > 0$, *one has the inequality* $\rho_j^n > 0$, *i.e., the approximation* (5) *of the continuity equation ensures that the density is positive*.

The form of the matrix $A$ implies the following assertion.

**Theorem 2**. *The approximation* (5) *of the continuity equation ensures the validity of the mass conservation law in the grid case. Namely*,

$$\sum_{j=0}^{N-1} h\rho_j^n = \sum_{j=0}^{N-1} h\rho_j^0. \tag{7}$$

**Proof**. Taking the scalar product of (8) (see the next section) with the unit grid function, using the summation-by-parts formula and the impermeability conditions at the boundary $u_0^n = u_N^n = 0$, we obtain the mass conservation condition at each step, which implies the statement of the theorem.

## 3. APPROXIMATION OF THE EQUATION OF MOTION

Let us write the approximation of the continuity equation in another form. Namely, set

$$\{\rho\} = [\rho_i] - \frac{h}{2}\rho_{\bar{x}}\operatorname{sgn}(u), \quad \text{where} \quad [\rho_i] = \frac{\rho_i + \rho_{i-1}}{2}.$$

Note that the operator $\{\cdot\}$ depends on the grid function $u$ and, unless otherwise specified, this operator is determined by the function $\hat{u}$. By analogy with (5), the fully implicit finite difference scheme for the continuity equation takes the form

$$\rho_t + (\{\hat{\rho}\}\hat{u})_x = 0, \quad x = ih, \quad i = 0,1,\ldots,N-1. \tag{8}$$

Since

$$-\frac{1}{2}\left(\frac{\partial(\rho u)}{\partial x}, u^2\right) + \left(\frac{\partial(\rho u^2)}{\partial x}, u\right) = 0, \tag{9}$$

we require that the approximation of the term $\frac{\partial(\rho u^2)}{\partial x}$ in the equation of motion preserve the grid analogue of (9), assuming that the approximation of $\frac{\partial(\rho u)}{\partial x}$ corresponds to (8). Introducing the notation $u_i^+ = u_{i+1}$, we approximate the expression $I \equiv \partial_x(\rho u^2) = \partial_x(\rho u)u + \rho u \partial_x u$ in the following way [7]:

$$I \sim (\{\rho\}u)_x u + \frac{1}{2}[\{\rho\}u \cdot u_{\bar{x}} + \{\rho\}^+ u^+ \cdot u_x]. \tag{10}$$

One can check directly that this approximation satisfies the grid analogue of (9).

Then the fully implicit finite difference approximation of the equation of motion becomes

$$\begin{gathered}(\rho u)_t + (\{\hat{\rho}\}\hat{u})_x\hat{u} + \frac{1}{2}[\{\hat{\rho}\}\hat{u} \cdot \hat{u}_{\bar{x}} + \{\hat{\rho}\}^+\hat{u}^+ \cdot \hat{u}_x] \\ + \frac{\gamma}{\gamma - 1}\{\hat{\rho}\}(\hat{\rho}^{\gamma-1})_{\bar{x}} = 0, \\ x = ih, \quad i = 1,2,\ldots,N-1.\end{gathered} \tag{11}$$

The unknowns in Eqs. (8), (11) are the functions $\hat{\rho}$ and $\hat{u}$, i.e., this system of equations contains $2N - 1$ equations and the same number of unknowns. Let us prove the uniform boundedness of the norms $\|(\rho^n)^{1/2}u^n\|$. Let us introduce the scalar product (recall that the functions $\rho$ and $u$ are extended by zero to the whole grid domain) by the formula $(u, v) = \sum_{i=0}^{N} hu_i v_i$.

Taking the scalar product of (8) with $\frac{\tau}{2}\hat{u}^2$ and of (11) with $\tau\hat{u}$ and subtracting the first relation from the second one, after transformations, we obtain

$$\begin{gathered}\frac{1}{2}[(\hat{\rho}, \hat{u}^2) - (\rho, u^2)] + \frac{\tau^2}{2}(\rho, u_t^2) \\ + \frac{\tau\gamma}{\gamma - 1}(\{\hat{\rho}\}(\hat{\rho}^{\gamma-1})_{\bar{x}}, \hat{u}) = 0.\end{gathered} \tag{12}$$

Since

$$\frac{\tau\gamma}{\gamma - 1}(\{\hat{\rho}\}(\hat{\rho}^{\gamma-1})_{\bar{x}}, \hat{u}) \geq \frac{1}{\gamma - 1}[(\hat{p}, 1) - (p, 1)],$$

Eq. (12) implies the following result.

**Theorem 3**. *The solution of the finite difference scheme* (8), (11) *satisfies the estimate*

$$\begin{gathered}\frac{1}{2}(\rho^n, (u^n)^2) + \frac{1}{\gamma - 1}(p^n, 1) \\ \leq \frac{1}{2}(\rho^0, (u^0)^2) + \frac{1}{\gamma - 1}(p^0, 1).\end{gathered} \tag{13}$$

## 4. EXISTENCE OF A SOLUTION OF THE FINITE DIFFERENCE SCHEME

Let us prove that the solution of (8), (11) always exists for any grid steps $\tau$ and $h$. Introduce the new variable $v = \rho u$. Then Eqs. (8), (11) take the form

$$\hat{\rho} = \rho - \tau\left(\frac{\{\hat{\rho}\}\hat{v}}{\hat{\rho}}\right)_x \equiv F_1(\hat{\rho}, \hat{v}),$$

$$\hat{v} = v - \tau\left(\frac{\{\hat{\rho}\}\hat{v}}{\hat{\rho}}\right)_x \frac{\hat{v}}{\hat{\rho}} - \frac{\tau}{2}\left[\frac{\{\hat{\rho}\}\hat{v}}{\hat{\rho}} \cdot \left(\frac{\hat{v}}{\hat{\rho}}\right)_{\bar{x}} + \frac{\{\hat{\rho}\}^+\hat{v}^+}{\hat{\rho}^+} \cdot \left(\frac{\hat{v}}{\hat{\rho}}\right)_x\right] \quad (14)$$

$$-\tau\frac{\gamma}{\gamma - 1}\{\hat{\rho}\}(\hat{\rho}^{\gamma-1})_{\bar{x}} \equiv F_2(\hat{\rho}, \hat{v}).$$

The existence of a solution of (14) follows from Theorem 3 and the Leray–Schauder theorem (see, for example, [7]). Thus, we have proved the following result.

**Theorem 4**. *A solution of problem* (8), (11) *exists for any* $\tau$ *and* $h$.

Note that, by what was proved above, $\hat{\rho} > 0$, hence, $\hat{u} = \hat{v}/\hat{\rho}$ also exists.

To find the solution $\rho^{n+1}, u^{n+1}$ of problem (8), (11), one can use the following linear iteration process, in which the bracket operator $\{\cdot\}$ is determined by the function $u^k$,

$$\frac{\rho^{k+1} - \rho^n}{\tau} + (\{\rho^{k+1}\}u^k)_x = 0,$$

$$\frac{\rho^{k+1}u^{k+1} - \rho^n u^n}{\tau} + (\{\rho^{k+1}\}u^k)_x u^{k+1}$$

$$+\frac{1}{2}[\{\rho^{k+1}\}u^k \cdot u^{k+1}_{\bar{x}} + \{\rho^{k+1}\}^+(u^k)^+ \cdot u^{k+1}_x] \quad (15)$$

$$+\frac{\gamma}{\gamma - 1}\{\rho^{k+1}\}((\rho^{k+1})^{\gamma-1})_{\bar{x}} = 0, \quad u^k|_{k=0} = u^n;$$

here, $k = 0, 1, \ldots$ is the iteration number.

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## REFERENCES

1. O. M. Belotserkovskii, M. O. Vasil'ev, A. B. Vedernikov, V. P. Dymnikov, B. V. Zamyshlyaev, B. Yu. Krysanov, N. V. Kovshov, V. E. Kunitsyn, E. A. Molokov, A. Yu. Repin, N. A. Sidorenkova, E. L. Stupitskii, Ya. A. Kholodov, and A. S. Kholodov, "On the numerical simulation of certain problems concerning the interaction of the lithosphere, hydrosphere, and atmosphere of the Earth," in *Fragments of the History and Achievements of the Institute for Analytical Instrumentation of the Russian Academy of Sciences 1986–2011* (Inst. Analit. Pribor. Ross. Akad. Nauk, Moscow, 2011), pp. 14–71 [in Russian].
2. I. V. Popov and I. V. Fryazinov, *Method of Adaptive Numerical Viscosity for the Numerical Solution of Gas Dynamics Equations* (URSS, Moscow, 2015) [in Russian].
3. S. N. Antontsev, A. V. Kazhikhov, and V. N. Monakhov, *Boundary Value Problems in Mechanics of Nonhomogeneous Fluids* (Nauka, Novosibirsk, 1983; North Holland, Amsterdam, 1990).
4. A. A. Amosov and A. A. Zlotnik, USSR Comput. Math. Math. Phys. **27** (4), 46–57 (1987).
5. A. G. Kulikovskii, N. V. Pogorelov, and A. Yu. Semenov, *Mathematical Aspects of Numerical Solution of Hyperbolic Systems* (Fizmatlit, Moscow, 2001; Chapman and Hall/CRC, London, 2001).
6. P.-L. Lions, *Mathematical Topics in Fluid Mechanics*, Vol. 2: *Compressible Models* (Clarendon, Oxford, 1998).
7. O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow* (Gordon and Breach, New York, 1969; Nauka, Moscow, 1970).

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