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MATHEMATICS

Fischer Decomposition of the Space of Entire Functions for the Convolution Operator

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Abstract—It is known that any function in a Hilbert Bargmann–Fock space can be represented as the sum of a solution of a given homogeneous differential equation with constant coefficients and a function being a multiple of the characteristic function of this equation with conjugate coefficients. In the paper, a decomposition of the space of entire functions of one complex variable with the topology of uniform convergence on compact sets for the convolution operator is presented. As a corollary, a solution of the de la Vallée Poussin interpolation problem for the convolution operator with interpolation points at the zeros of the characteristic function with conjugate coefficient is obtained.

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1. PROBLEM STATEMENT

Let $H(\mathbb{C}^n)$ be the space of entire functions with the topology of uniform convergence on compact sets. By M we denote the set of all homogeneous polynomials in variables $z_1, z_2, \dots, z_n$. In 1917, Fischer obtained the following result [1].

Theorem 1. *Let $P(z) \in M$, and let $P^*(z)$ denote the polynomial conjugate to $P(z)$, i.e., $P^*(z) = \overline{P(\bar{z})}$. Then the representation*

$$M = M_1 \oplus M_2$$

holds, where M_1 is the kernel of the operator $P^*\left(\frac{\partial}{\partial z}\right)$ and the polynomials in M_2 are divisible by $P(z)$.

The polynomials $P(z)$ and $P^*(z)$ are said to form a Fischer pair. In [2], it was shown that, given any polynomial $P(z)$, $(P(z), P^*(z))$ is a Fischer pair in the Bargmann–Fock space. In 1989, Shapiro [3] proved that, given any $P(z) \in M$, the polynomials $P(z)$ and $P^*(z)$ form a Fischer pair in the space $H(\mathbb{C}^n)$.

In what follows, we study the space $H(\mathbb{C})$. Let us introduce the space $H^*(\mathbb{C})$ dual to $H(\mathbb{C})$ and the space $P_{\mathbb{C}}$ of entire functions with exponential growth. The Laplace transform

$$\hat{F}(\lambda) = (F, e^{\lambda z}) = \varphi(\lambda), \quad F \in H^*(\mathbb{C}),$$

establishes a topological isomorphism between the spaces $H^*(\mathbb{C})$ and $P_{\mathbb{C}}$. According to [4], a sequence $\{q_n(z)\}_{n=1}^{\infty}$, $q_n(z) \in P_{\mathbb{C}}$ converges in the topology of $P_{\mathbb{C}}$ if and only if the sequence of $q_n(z)$ converges uniformly on each compact set in the plane $\mathbb{C}$ and there exist constants $c, \sigma > 0$ such that

$$|q_n(z)| \le c e^{\sigma |z|}, \quad z \in \mathbb{C}, \tag{1}$$

uniformly in n.

Consider the convolution operator with a characteristic function $\varphi(z)$ on $H(\mathbb{C})$:

$$M_\varphi[f(z)] = (F_t, f(z+t)),$$
$$f(z) \in H(\mathbb{C}), \quad t \in \mathbb{C}.$$

This paper is devoted to the Fischer decomposition of the space $H(\mathbb{C})$ in the case of the convolution operator

$$f(z) = w(z) + \varphi^*(z) \cdot l(z),$$
$$w(z) \in \ker M_\varphi, \quad l(z) \in H(\mathbb{C}), \quad \varphi^*(z) = \overline{\varphi(\bar{z})}. \tag{2}$$

The Fischer decomposition is applied to solve Goursat and Dirichlet problems, the Cauchy problem and the de la Vallée Poussin interpolation pro-blem.

2. MAIN RESULT

Together with the operator M_φ, consider the continuous linear operator

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$$M_\varphi[\varphi^*\cdot]: H(\mathbb{C}) \to H(\mathbb{C}).$$

Lemma 1 [6]. *Relation* (2) *is equivalent to the surjectivity of the operator* $M_\varphi[\varphi^*\cdot]$.

Thus, the Fischer decomposition reduces to the surjectivity of the operator $M_\varphi[\varphi^*\cdot]$ According to the Dieudonné–Schwartz theorem, the operator $M_\varphi[\varphi^*\cdot]$ is surjective if and only if the adjoint operator $K_{\varphi^*}[\varphi\cdot]$. is injective and has closed image.

Consider the convolution operator K_{φ^*} in the space $P_\mathbb{C}$:

$$K_{\varphi^*}[\psi(\lambda)] = \frac{1}{2\pi i}\int_C e^{\lambda z}\varphi^*(z)g_\psi(z)dz, \quad \psi(\lambda) \in P_\mathbb{C},$$

where $\varphi^*(z)$ is the characteristic function of the operator, $g_\psi(z)$ is the Borel transform of $\psi(z)$, and the contour C encloses all singularities of $g_\psi(z)$.

The kernel of K_{φ^*} consists of functions of the form (see [7])

$$r(z) = \sum_{i=1}^{Q} c_i e^{\bar{\lambda}_i z},$$

where $\bar{\lambda}_1, \cdots, \bar{\lambda}_Q$ are the zeros of $\varphi^*(z)$ enclosed by the integration contour C.

Let $N_\varphi = \{\lambda_k\}_{k=1}^{\infty}$ denote the set of zeros of the function $\varphi(z)$.

Definition 1. We say that $N_\varphi \subset \mathbb{C}$ is a sequentially sufficient set in the space $\ker K_{\varphi^*}$ if the convergence to zero of any sequence of functions in $\ker K_{\varphi^*}$ on compact subsets of N_φ implies the convergence of this sequence to zero at all points z belonging to any compact set $K \subset C$ in the space $\ker K_{\varphi^*}$.

Theorem 2. *The set* N_φ *is sequentially sufficient in* $\ker K_{\varphi^*}$.

Proof. Estimate (1) implies $|\bar{\lambda}_i| \le \sigma$. We assume that the closed disk of radius σ centered at the origin contains finitely many different points $\bar{\lambda}_i, i = 1, 2, \ldots, p$ (some of the points may fall on the boundary). These points determine the kernel of the operator K_{φ^*}. Consider the sequence

$$r_n(z) = \sum_{i=1}^{p_n} c_i^n e^{\bar{\lambda}_i z}. \qquad (3)$$

Let us show that if the sequence of $r_n(z) \in \ker K_{\varphi^*}$ tends to zero on any compact set in N_φ, then this sequence tends to zero on any compact set $K \subset \mathbb{C}$ in the space $\ker K_{\varphi^*}$. Consider the system

$$\sum_{i=1}^{p_n} c_i^n e^{\bar{\lambda}_i \lambda_j} = r_n(\lambda_j), \quad j = 1, 2, \ldots, p, \quad \forall n. \qquad (4)$$

Let us show that the uniform convergence

$$r_m(\lambda_k) \rightrightarrows 0 \quad \text{as} \quad m \to \infty \qquad (5)$$

on any compact set in N_φ implies that the coefficients c_i^n tend to zero.

The matrix A has the form

$$A = \begin{pmatrix} e^{\bar{\lambda}_1\lambda_1} & \ldots & e^{\bar{\lambda}_p\lambda_1} \\ \ldots & \ldots & \ldots \\ e^{\bar{\lambda}_1\lambda_p} & \ldots & e^{\bar{\lambda}_p\lambda_p} \end{pmatrix}.$$

According to a theorem in [8, p. 226 of the Russian original], the determinant Δ of this matrix is nonzero and, therefore, the coefficients c_i^n are solutions of the system of equations (4). Cramer's rule yields the coefficients

$$c_i^n = \frac{\Delta_i}{\Delta},$$

where Δ_i is the determinant of the matrix obtained from Δ by replacing the ith column by the column of constant terms. According to condition (5), as $n \to \infty$, the column of constant terms tends to zero; therefore, for any i, $\Delta_i \to 0$ as $n \to \infty$, and, thereby, $c_i^n \to 0$ as $n \to \infty$.

Given any compact set $K \subset \mathbb{C}$, the functions $e^{\bar{\lambda}_i z}$ are bounded on this set, the coefficients c_i^n tend to zero, and, hence, the whole linear combination (3) tends to zero as $n \to \infty$. Therefore, $r_n(z) \to 0$ as $n \to \infty$ on any compact set $K \subset \mathbb{C}$.

Let us show that, for any sequence of functions $r_n(z) \in \ker K_{\varphi^*}$, there exist numbers $c, \sigma > 0$ for which estimate (1) holds. Since all $\bar{\lambda}_i, i = 1, 2, \ldots, p$, lie in the disk of radius σ, it follows that $|\bar{\lambda}_i| \le \sigma$, which implies

$$|r_n(z)| \le \sum_{i=1}^{p_n} |c_i^n||e^{\bar{\lambda}_i z}| \le p_n \max_i \{c_i^n\} e^{|\bar{\lambda}_i||z|} \le ce^{\sigma|z|}, \quad \forall z \in K \subset \mathbb{C}. \qquad (6)$$

Taking into account the uniform convergence of $r_n(z)$ on compact sets K and estimate (6), we see that $r_n(z) \to 0$ on any compact set $K \subset \mathbb{C}$ as $n \to \infty$. Therefore, the set N_φ is sequentially sufficient in the space $\ker K_{\varphi^*}$. This completes the proof of the theorem.

Theorem 3. *If* N_φ *is a sequentially sufficient set in the kernel of the operator* K_{φ^*}, *then the operator* $M_\varphi[\varphi^*\cdot]$ *is surjective in the space* $H(\mathbb{C})$.

The proof of this theorem is similar to that of Theorem 5 in [9].

Finally, we state the main result of this paper.

Theorem 4. *If the set of zeros of the characteristic function* $\varphi(z)$ *of an operator* M_φ *is sequentially sufficient in* $\ker K_{\varphi^*}$, *then there exists a Fischer decomposition of the space* $H(\mathbb{C})$ *for the operator* M_φ.

The de la Vallée Poussin problem consists in finding a function in the kernel of the convolution operator which takes preset values at points from a given sequence.

Corollary 1. *The results obtained above make it possible to solve the de la Vallée Poussin problem with interpolation points at* $\overline{\lambda}_k$ *in the kernel of the operator* M_φ.

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