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MATHEMATICS

On Eigenfunctions of a Convolution Operator on a Finite Interval for Which the Fourier Image of the Kernel Is the Characteristic Function

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Abstract—An asymptotic representation of the eigenfunctions of a convolution-type completely continuous operator for which the image of the kernel is the characteristic function of the interval is constructed.

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Papers [1, 2] studied an asymptotics of the spectrum of the convolution integral operator on a finite interval with kernel

$$K(x) = |x|^{-\alpha}, \quad 0 < \alpha < 1.$$

In [3], the asymptotic behavior of the spectra of weakly polar integral operators was considered. In [4], the author found an asymptotics of the spectrum of a convolution operator such that the Fourier image of its kernel is the characteristic function of the interval, i.e., has two finite points of discontinuity. In this paper, we write out asymptotic representations of the eigenfunctions of this operator.

Consider the equation

$$\varphi(x) = \lambda \int_{-1}^{1} K(x-t)\varphi(t)\,dt, \quad x \in [-1, 1], \tag{1}$$

where $K(x) = (\pi x)^{-1}\sin(lx)$, $l > 0$, and λ is a spectral parameter.

Lemma 1. *The integral operator K determined by the right-hand side of Eq.* (1) *is nonnegative on the space* $L_2(-1, 1)$.

In what follows, we assume that λ is positive and sufficiently large. Note that, by Mercer's theorem, the eigenvalues satisfy the relation

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = 2K(0) = \frac{2l}{\pi}.$$

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Lemma 2. *The eigenfunctions of the problem are either even or odd.*

First, consider the case of an odd eigenfunction. Extending (1) to the entire real line and setting

$$\varphi(x) = 0, \quad |x| > 1;$$

$$\psi(x-1) = -\lambda \int_{-1}^{1} K(x-t)\varphi(t)\,dt, \ x \geq 1;$$

$$\psi(x) = 0, \quad x < 0,$$

we obtain

$$\varphi(x) = \lambda \int_R K(x-t)\varphi(t)dt$$
$$+\psi(x-1) - \psi(-1-x), \quad x \in \mathrm{R}.$$

Passing to the Fourier images

$$\hat{\varphi}(p) = \int_R e^{ipx}\varphi(x)\,dx = \int_{-1}^{1} e^{ipx}\varphi(x)\,dx,$$

$$\hat{\psi}(p) = \int_0^{+\infty} e^{ipx}\psi(x)\,dx, \quad \hat{K}(p) = \begin{cases} 1, & -l < p < l \\ 0, & |p| > l, \end{cases}$$

we arrive at the equation

$$(1 - \lambda \hat{K}(p))\hat{\varphi}(p) = e^{ip}\hat{\psi}(p) - e^{-ip}\hat{\psi}(-p), \quad p \in \mathrm{R}.$$

Note that the function $\hat{\psi}(z)$ is analytic in the upper half-plane and vanishes at infinity.

We set $T = \{x \in \mathrm{R} \mid |x| > l\}$ and

$$\Phi(z) = \begin{cases} e^{iz}(1-\lambda)\hat{\varphi}(z) - e^{2iz}\hat{\psi}(z), & \operatorname{Im} z > 0 \\ -\hat{\psi}(-z), & \operatorname{Im} z < 0. \end{cases}$$

The function $\Phi(z)$ is analytic in the entire complex plane cut along T; it may have singularities of at most logarithmic order at the points $\pm l$ and vanishes at infinity. In a standard way [5, p. 200], we obtain the following conjugation condition with respect to the functions $\Phi^{\pm}$:

$$\Phi^+(x) = (1-\lambda)\Phi^-(x) + \lambda e^{2ix}\Phi^-(-x), \quad x \in T. \quad (2)$$

In what follows, we assume that $x \in T$, unless otherwise specified. We set

$$\mu = \frac{1}{2\pi}\ln(\lambda - 1), \quad \nu = \frac{\lambda}{1-\lambda}.$$

We seek solutions of the conjugation problem (2) in the form

$$\Phi(z) = \sqrt{l^2 - z^2}\left(\frac{l-z}{l+z}\right)^{i\mu} Y(z).$$

Setting

$$\Theta(x) = e^{2ix}\left(\frac{x+l}{x-l}\right)^{2i\mu},$$

we obtain the main equation for the function $Y(z)$:

$$Y^-(x) = \frac{\nu}{2}\Theta(x)Y^-(-x) - \frac{\nu}{2\pi i}\int_T \frac{\Theta(t)Y^-(-t)\,dt}{t-x}. \quad (3)$$

In the case of an even eigenfunction, the same argument yields the equation

$$Y^-(x) = -\frac{\nu}{2}\Theta(x)Y^-(-x) + \frac{\nu}{2\pi i}\int_T \frac{\Theta(t)Y^-(-t)\,dt}{t-x}, \quad (3')$$

which differs from Eq. (3) in the sign of ν. In what follows, we write $Y^-(x)$ instead of $Y(x)$.

Let $\chi(z) = z + \mu\ln\left(\frac{z+l}{z-l}\right)$; then $\Theta(x) = \exp(2i\chi(x))$. We denote the positive stationary point of the exponent by

$$p_0 = \sqrt{2\mu l + l^2} : \chi'(\pm p_0) = 0, \quad \chi_0 = \chi(p_0) \sim 2p_0.$$

We set $E = R \cap \{l < |x| < p_0 - \sqrt{p_0}\}$, $S = R \cap \{|x| > p_0 + \sqrt{p_0}\}$, and $P = T \setminus (E \cup S)$.

We refer to bounded infinitely differentiable nonoscillating functions which can be represented in the form

$$f_\pm\left(\frac{x \pm p_0}{\sqrt{p_0}}\right), \quad |x \pm p_0| \le \sqrt{p_0},$$

where $f_\pm \in C^\infty[-1, 1]$ do not depend on λ, as regular functions.

Lemma 3. *The representation*

$$\frac{1}{\pi i}\int_T \frac{\Theta(t)\,dt}{t-x} = \Lambda(x)\Theta(x) + \Omega(x)$$

holds, where $\Lambda(x)$ *is a regular function,* $\Lambda(-x) = \overline{\Lambda(x)}$, $\Lambda(x) = -\operatorname{cotanh}(2\pi\mu)$ *for* $x \in E$, $\Lambda(x) = 1$ *for* $x \in S$, $\Omega(x) = O\left(\frac{1}{\sqrt{p_0}}\right)$, *and* $\Omega(\infty) = 0$.

Again applying the operator from Eq. (3) and using Lemma 3, we obtain the equation

$$Y(x) = \frac{\nu^2}{4\pi i}\int_T \left((1-\Lambda(x))\frac{\Theta(x)}{\Theta(t)} - (1-\Lambda(t))\right)\frac{Y(t)\,dt}{t-x} + \frac{\nu^2}{4\pi i}\int_T \frac{\Omega(t)-\Omega(x)}{t-x}\frac{Y(t)\,dt}{\Theta(t)}. \quad (4)$$

Equation (3') leads to the same equation (4).

To determine the oscillating terms in the solution, we factorize the singular part of Eq. (4), i.e., represent it as the superposition of integral operators whose inverses can be written explicitly.

Consider the two singular operators

$$L_-u(x) \equiv u(x) - \frac{1}{\pi i}\int_T \frac{\Theta(x)}{\Theta(t)}\frac{B(x)u(t)\,dt}{t-x},$$

$$L_+u(x) \equiv u(x) + \frac{1}{\pi i}\int_T \frac{B(t)u(t)\,dt}{t-x},$$

where $B(x)$ is a function defined later on.

By virtue of Lemma 3, the superposition of these operators equals

$$L_-L_+u(x) = (1-B^2(x))u(x) - \frac{1}{\pi i}\int_T \left((1+B(t)\Lambda(-t))B(x)\frac{\Theta(x)}{\Theta(t)} - (1+B(x)\Lambda(-x))B(t)\right) + \frac{u(t)\,dt}{t-x}$$
$$+ \frac{1}{\pi i}\int_T \Theta(x)B(x)B(t)\frac{\Omega(-x)-\Omega(-t)}{t-x}u(t)\,dt.$$

Now, we choose $B(x)$ so that

$$\frac{B(x)(1+B(x)\Lambda(-x))}{1-B^2(x)} = \frac{\nu^2}{4}(1-\Lambda(x)).$$

For an appropriate solution of this quadratic equation, we set

$$B(x)=\frac{\frac{\nu^2(1-\Lambda(x))}{2}}{1+\sqrt{1+\nu^2(1-\Lambda(x))\left(\Lambda(-x)+\frac{\nu^2(1-\Lambda(x))}{4}\right)}}.$$

Lemma 4. $B(x)\neq\pm1$ for $x\in T$.

Corollary. *The Cauchy index of the expression* $\frac{1+B(x)}{1-B(x)}$ *is defined.*

Note that

$$B(x)=\frac{\lambda}{\lambda-2},\quad x\in E;\quad B(x)=0,\quad x\in S.$$

In this case, the superposition of operators can be rewritten in the form

$$L_-L_+Y(x)=W(x),$$

$$W(x)=\frac{1+B(x)\Lambda(-x)}{\pi i}$$
$$\times\int_T\left(\frac{B(t)}{1-\Lambda(t)}-\frac{B(x)}{1-\Lambda(x)}\right)\frac{(1-\Lambda(t))Y(t)\,dt}{t-x}$$
$$+\frac{\Theta(x)B(x)}{\pi i}\int_T\frac{B(x)\Lambda(-x)-B(t)\Lambda(-t)}{t-x}\frac{Y(t)\,dt}{\Theta(t)}$$
$$+(1-B^2(x))\frac{\nu^2}{4\pi i}\int_T\frac{\Omega(t)-\Omega(x)}{t-x}\frac{Y(t)\,dt}{\Theta(t)}$$
$$-\frac{\Theta(x)B(x)}{\pi i}\int_T\frac{\Omega(-t)-\Omega(-x)}{t-x}B(t)Y(t)\,dt.$$

Remark. In the class of functions under consideration, we have $W(x)=O^*\left(\frac{1}{x}\right)$ as $x\to\infty$.

Now, let us invert the singular operators by the standard method of reduction to Riemann's conjugation problem [6, p. 176].

Lemma 5. *A general solution of the equation*

$$L_-u(x)=v(x)$$

in the class of functions allowed to have singularities of integrable order at $\pm l$ *has the form*

$$u(x)=C_-\frac{B(x)\Theta(x)X_-^+(x)}{1+B(x)}+\frac{v(x)}{1-B^2(x)}$$
$$+\frac{B(x)}{1+B(x)}\frac{1}{\pi i}\int_T\frac{\Theta(x)X_-^+(x)}{\Theta(t)X_-^+(t)}\frac{v(t)}{1-B(t)}\frac{dt}{t-x},$$

where

$$X_-(z)=\frac{1}{l+z}\exp\left(\frac{1}{2\pi i}\int_T\ln G(t)\frac{dt}{t-z}\right)$$
$$=\frac{1}{\sqrt{l^2-z^2}}\exp\left(\frac{1}{2\pi i}\int_{E\cup P}\ln(-G(t))\frac{dt}{t-z}\right).$$

Lemma 6. *A general solution of the equation*

$$L_+u(x)=v(x)$$

in the class of functions allowed to have singularities of integrable order at $\pm l$ *has the form*

$$u(x)=C_+\frac{X_+^+(x)}{1-B(x)}+\frac{v(x)}{1-B^2(x)}$$
$$-\frac{1}{1-B(x)}\frac{1}{\pi i}\int_T\frac{X_+^+(x)}{X_+^+(t)}\frac{B(t)v(t)}{1+B(t)}\frac{dt}{t-x},$$

where

$$X_+(z)=\frac{1}{l-z}\exp\left(-\frac{1}{2\pi i}\int_T\ln G(t)\frac{dt}{t-z}\right)$$
$$=\frac{1}{\sqrt{l^2-z^2}}\exp\left(-\frac{1}{2\pi i}\int_{E\cup P}\ln(-G(t))\frac{dt}{t-z}\right).$$

Lemma 7. *A solution of Eq.* (4) *can be represented in the form*

$$Y(x)=\frac{B(x)\Theta(x)X_-^+(x)}{1+B(x)}R_1(x)$$
$$+\frac{X_+^+(x)}{1-B(x)}R_2(x)+\frac{X_+^+(x)}{1-B(x)}\Xi(x)R_3(x),$$

where the $R_j(x)$, $j=1,2,3$, *are regular functions and*

$$\Xi(x)=\frac{1}{\pi i}\int_T\frac{\Theta(t)X_-^+(t)}{X_+^+(t)}\left(\frac{B(t)}{1+B(t)}\right)^2\frac{ds}{s-x}.$$

Theorem 1. *A solution of Eq.* (3) *can be represented in the form*

$$Y(x)=\frac{2}{\nu}\frac{B(x)}{1+B(x)}\frac{\Theta(x)X_+^+(-x)R(-x)}{1-B(-x)}$$
$$+\frac{X_+^+(x)R(x)}{1-B(x)},$$

where $R(x)$ *is a regular function.*

Lemma 8. *A condition determining the series of eigenvalues for odd eigenfunctions has the form*

$$\mathrm{Re}\left(\frac{2}{\nu}C_1+\Theta(p_0)C_2\right)=0,$$

where C_1 *and* C_2 *do not depend on* λ.

Considering the case of an even eigenfunction in a similar way, we obtain the second series of eigenvalues.

Theorem 2. *The spectrum of problem* (1) *consists of two alternating sequences of the form*

$$\lambda_n^+ = \exp\left(\pi l^{-1}\left(\frac{\pi n}{4} + \vartheta_+ + \ldots\right)^2\right), \quad n = 1, 3, \ldots;$$

$$\lambda_n^- = \exp\left(\pi l^{-1}\left(\frac{\pi n}{4} + \vartheta_- + \ldots\right)^2\right), \quad n = 2, 4, \ldots,$$

where $\vartheta_\pm$ *are constants.*

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