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MATHEMATICS

On Isomorphism of Reproducing Kernel Hilbert Spaces

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Abstract—Reproducing kernel Hilbert spaces having orthogonally similar decomposition systems are considered. Conditions under which such spaces coincide or are isomorphic are found.

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1. INTRODUCTION

In this paper, we present statements concerning reproducing kernel Hilbert spaces. These statements are useful for solving problems of describing the space dual to the Hilbert space of analytic functions, for constructing Riesz bases, and in other problems of complex analysis.

Let H_1 be a reproducing kernel Hilbert space consisting of functions defined on a set M of points. Such a space is characterized by the condition that, for any $t_0 \in M$, the functional $f \to f(t_0)$ is a continuous linear functional on H_1 (see [1]). The elements of the space H_1 are functions of a variable $t \in M$. Let $\Omega \subset \mathbb{C}$ be a space with measure μ. As Ω we can take, e.g., a domain in the complex plane $\mathbb{C}$. The role of Ω can also be played by a countable set of points; then μ, can be, e.g., the counting measure.

Let $\{e_1(\cdot,\xi)\}_{\xi\in\Omega}$ be an orthosimilar decomposition system with measure μ in the space H (see [2]), i.e., $\{e_1(\cdot,\xi)\}_{\xi\in\Omega}$ is contained in H and any function $f \in H_1$ can be written in the form

$$f(t) = \int_{\mathbb{C}} (f(\cdot), e_1(\cdot,\eta))_{H_1} e_1(t,\eta)\,d\mu(\eta), \qquad t \in M. \tag{1}$$

For the definition of an orthosimilar decomposition system in a reproducing kernel Hilbert space, see also [3].

Obviously, the system of functions $\{e_1(\cdot,\xi)\}_{\xi\in\Omega}$, is complete in the space H_1. To each continuous linear functional on H_1 generated by a function $f \in H_1$ we assign the function

$$\tilde{f}(\xi) \overset{\text{def}}{=} (e_1(\cdot,\xi), f)_{H_1}, \qquad \xi \in \Omega.$$

The functions $\tilde{f}, f \in H_1$ form the Hilbert space

$$\tilde{H}_1 = \{\tilde{f}, f \in H_1\}$$

with inner product

$$(\tilde{f},\tilde{g})_{\tilde{H}_1} \overset{\text{def}}{=} (g,f)_{H_1}, f, g \in H_1$$

and norm

$$\|\tilde{f}\|_{\tilde{H}_1} \overset{\text{def}}{=} \|f\|_{H_1}.$$

Note that the function $e_1(\tau,\eta), \eta \in \Omega$ has the property

$$\int_{\Omega} |e_1(\tau,\eta)|^2 d\mu(\eta) < \infty \qquad \forall \tau \in M. \tag{2}$$

Indeed, since $\{e_1(\cdot,\xi)\}_{\xi\in\Omega}$ is an orthosimilar system decomposition with measure μ in the space H_1, it follows by virtue of the analogue of Parseval's identity for orthosimilar decomposition systems (see [2]) that

$$\begin{aligned} \infty > \|K_{H_1}(\cdot,\tau)\|_{H_1}^2 &= \int_{\Omega} |(K_{H_1}(\cdot,\tau), e_1(\cdot,\eta))_{H_1}|^2 d\mu(\eta) \\ &= \int_{\Omega} |e_1(\tau,\eta)|^2 d\mu(\eta), \qquad \forall \tau \in M. \end{aligned} \tag{3}$$

Let $R_1(\Omega,\mu)$ denote the closure of the linear span of the system of functions $\{e_1(\tau,\cdot)\}_{\tau\in M}$ with respect to the norm

$$\|\cdot\|_I \overset{\text{def}}{=} \sqrt{\int_{\Omega} |\cdot|^2 d\mu}.$$

Obviously, the system $\{e_1(\tau,\cdot)\}_{\tau\in M}$ is contained in $\tilde{H}_1$. In [3], the following assertion was proved (see Lemma 1 in [3]).

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Lemma 1. *The space $\tilde{H}_1$ coincides with $R_1(\Omega,\mu)$, i.e., the spaces $\tilde{H}_1$ and $R_1(\Omega,\mu)$ consist of the same functions and the inner products in $\tilde{H}_1$ and $R_1(\Omega,\mu)$ coincide, i.e.,*

$$(h,q)_{\tilde{H}_1} = (h,q)_{R_1(\Omega,\mu)}, \quad h,q \in \tilde{H}_1 = R_1(\Omega,\mu).$$

Next, let H_2 be a reproducing kernel Hilbert space consisting of functions defined on a set M of points, and let $\{e_2(\cdot,\xi)\}_{\xi\in\Omega}$ be an orthosimilar decomposition system with measure μ in H_2, , i.e., $\{e_2(\cdot,\xi)\}_{\xi\in\Omega}$ is contained in H_2 and any function $f \in H_2$ can be represented in the form

$$f(t) = \int\limits_{\mathbb{C}} (f(\cdot), e_2(\cdot,\eta))_{H_2} e_2(\cdot,\eta) d\mu(\eta), \quad t \in M. \tag{4}$$

The system of functions $\{e_2(\cdot,\xi)\}_{\xi\in\Omega}$ is complete in the space H_2. To each continuous linear functional on H_2 generated by a function $f \in H_2$ we assign the function

$$\hat{f}(\xi) \overset{\text{def}}{=} (e_2(\cdot,\xi), f)_{H_2}, \quad \xi \in \Omega.$$

The functions $\hat{f}, f \in H_2$ form the Hilbert space

$$\hat{H}_2 = \{\hat{f}, f \in H_2\}$$

with inner product

$$(\hat{f},\hat{g})_{\widehat{H}_2} \overset{\text{def}}{=} (g,f)_{H_2}, \quad f,g \in H_2$$

and norm

$$\|\hat{f}\|_{\widehat{H}_2} \overset{\text{def}}{=} \|f\|_{H_2}, \quad f \in H_2.$$

The system $\{e_2(\tau,\cdot)\}_{\tau\in M}$ is contained in $\hat{H}_2$, and the function $e_2(\tau,\eta), \eta \in \Omega$ has the property

$$\int\limits_{\Omega} |e_2(\tau,\eta)|^2 d\mu(\eta) < \infty, \quad \forall \tau \in M. \tag{5}$$

Let $R_2(\Omega,\mu)$ denote the closure of the linear span of the system $\{e_2(\tau,\cdot)\}_{\tau\in M}$ with respect to the norm

$$\|\cdot\|_I = \sqrt{\int\limits_{\Omega} |\cdot|^2 d\mu}.$$

The following lemma is valid.

Lemma 2. *The space $\hat{H}_2$ coincides with the space $R_2(\Omega,\mu)$, i.e., the spaces $\hat{H}_2$ and $R_2(\Omega,\mu)$ consist of the same functions, and the inner products in $\hat{H}_2$ and $R_2(\Omega,\mu)$ coincide.*

In what follows, we state assertions concerning the spaces H_1 and H_2introduced above.

2. MAIN RESULTS

Theorem 1. *The spaces H_1 and H_2 coincide, i.e., these spaces consist of the same functions and*

$$(f,g)_{H_1} = (f,g)_{H_2}, \quad f,g \in H_1 = H_2,$$

if and only if there exists a unitary one-to-one continuous linear operator B from $\tilde{H}_1$ onto $\hat{H}_2$ such that

$$B\colon e_1(\tau,\cdot) \to e_2(\tau,\cdot), \quad \tau \in M. \tag{6}$$

Scheme of the proof. The system of functions $\{e_1(\cdot,\eta)\}_{\eta\in\Omega}$ is an orthosimilar decomposition system in the space H_1. This means that any function f in H_1 can be represented in the form

$$f(t) = \int\limits_{\mathbb{C}} (f(\cdot), e_1(\cdot,\eta))_{H_1} e_1(t,\eta) d\mu(\eta), \quad t \in M.$$

Substituting the reproducing kernel for f into this relation, we obtain

$$K_{H_1}(t,\xi) = \int\limits_{\mathbb{C}} (K_{H_1}(\cdot,\xi), e_1(\cdot,\eta))_{H_1} e_1(t,\eta) d\mu(\eta)$$

$$= \int\limits_{\mathbb{C}} \overline{e_1(\xi,\eta)} e_1(t,\eta) d\mu(\eta) = (e_1(t,\eta), e_1(\xi,\eta))_{R_1}, \tag{7}$$

$$t,\xi \in M$$

(see the definition of the space $R_1(\Omega,\mu)$). In a similar way, we calculate the reproducing kernel of the space H_2:

$$\begin{aligned} K_{H_2}(t,\xi) &= \int\limits_{\mathbb{C}} \overline{e_2(\xi,\eta)} e_2(t,\eta) d\mu(\eta) \\ &= (e_2(t,\eta), e_2(\xi,\eta))_{R_2}, \quad t,\xi \in M. \end{aligned} \tag{8}$$

Suppose that there exists a unitary one-to-one continuous linear operator B from $\tilde{H}_1$ onto $\hat{H}_2$ for which relation (6) holds. Note that $\tilde{H}_1 = R_1$ and $\hat{H}_2 = R_2$ (see Lemmas 1 and 2). Since the operator B is unitary and (6) holds, using (7) and (8), we obtain

$$\begin{aligned} K_{H_1}(t,\xi) &= (e_1(t,\eta), e_1(\xi,\eta))_{R_1} \\ &= (Be_1(t,\eta), Be_1(\xi,\eta))_{R_2} \\ &= (e_2(t,\eta), e_2(\xi,\eta))_{R_2} = K_{H_2}(t,\xi), \quad t,\xi \in M. \end{aligned} \tag{9}$$

According to the Moore–Aronszajn theorem, the spaces H_1 and H_2 coincide.

Let us prove the converse. Suppose that the spaces H_1 and H_2 coincide. Then the reproducing kernels of these spaces coincide as well. Relations (7) and (8) imply

$$\begin{aligned} (e_1(t,\cdot), e_1(\xi,\cdot))_{R_1} &= K_{H_1}(t,\xi) \\ = K_{H_2}(t,\xi) &= (e_2(t,\cdot), e_2(\xi,\cdot))_{R_2}, \quad \xi,t \in M. \end{aligned} \tag{10}$$

Let

$$p(w) = \sum_{k=1}^{m} c_k e_1(\xi_k, w), w \in \Omega, \tag{11}$$

be any finite linear combination of elements of the system of functions $\{e_1(\xi,\cdot)\}_{\xi\in M}$. We set

$$Bp(w) \stackrel{\text{def}}{=} \sum_{k=1}^{m} c_k e_2(\xi_k, w), \quad w \in \Omega.$$

It is easy to derive from (10) that

$$\|p\|^2_{R_1} = (p, p)_{R_1} = (Bp, Bp)_{R_2} = \|Bp\|^2_{R_2}. \quad (12)$$

The system $\{e_1(\xi, \cdot)\}_{\xi \in M}$ is complete in R_1, and the system $\{e_2(\xi, \cdot)\}_{\xi \in M}$ is complete in R_2. Relation (12) holds for any function of the form (11). Therefore, the operator B can be extended to a unitary continuous linear operator from R_1 onto R_2, where $R_1 = \tilde{H}_1$ and $R_2 = \hat{H}_2$.

A similar argument proves the following theorem.

Theorem 2. *The spaces H_1 and H_2 are isomorphic, i.e., these spaces consist of the same functions and*

$$C_1\|f\|_{H_1} \le \|f\|_{H_2} \le C_2\|f\|_{H_1}, \quad f \in H_1 = H_2,$$

where $C_1, C_2 > 0$ are some constants, if and only if there exists a one-to-one continuous linear operator B from $\tilde{H}_1$ onto $\hat{H}_2$ such that

$$B: e_1(\tau, \cdot) \to e_2(\tau, \cdot), \quad \tau \in M. \quad (13)$$

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