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MATHEMATICS

Fixed-Point and Coincidence Theorems in Ordered Sets

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Abstract—The paper is devoted to the problem of the existence of common fixed points and coincidence points of a family of set-valued maps of ordered sets. Fixed-point and coincidence theorems for families of set-values maps are presented, which generalize some of the known results. The presented theorems, unlike previous ones, do not assume the maps to be isotone or coverable. They require only the existence of special chains having lower bounds with certain properties in the ordered set.

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Classical results of the fixed-point theory of maps of ordered sets are the Knaster–Tarski, Smithson, and Zermelo theorems, which have extensive applications. This paper considers the problems of the existence of common fixed points and coincidence points for set-values maps of ordered sets. First, we recall the classical results. We state them in terms of the dual order (with respect to that in the statements given in [1]).

Theorem 1 (Knaster–Tarski theorem [1, Chapter 18, Theorem 2.1]). *Let $(X, \preceq)$ be a partially ordered set in which each chain has an infimum. Suppose that $f : X \to X$ is an isotone map and there exists an element $x_0 \in X$ for which $f(x_0) \preceq x_0$. Then f has a fixed point.*

Theorem 2 (Smithson's theorem [2; 1, Chapter 18, Theorem 2.21]). *Let $(X, \preceq)$ be a partially ordered set in which each chain has an infimum. Let $F : X \rightrightarrows X$ be an isotone set-valued map such that, for each chain $C \subseteq X$, the map F has an isotone selection $g : C \to X$, i.e., there exists a $\xi \in F(\inf C)$ such that $\xi \preceq g(x)$ for each $x \in C$. Suppose that, for some element $x_0 \in X$, there exists an $x' \in F(x_0)$, $x' \preceq x_0$, then the fixed point set* $\mathrm{Fix}(F)$ *of the map F is nonempty and contains a minimal element.*

Theorem 3 (Zermelo's theorem, [1, Chapter 18, Theorem 3.1]). *Let $(X, \preceq)$ be a partially ordered set in which each chain has an infimum. Suppose that $f : X \to X$ is a map satisfying the condition $f(a) \preceq a$, for any $a \in X$. Then f has a fixed point.*

The following generalization of Theorem 3 is due to Yachymski; we state it again in terms of the dual order.

Theorem 4 [1, Chapter 18, Theorem 3.13]. *Let $(X, \preceq)$ be a partially ordered set, and let $f : X \to X$ be a regressive map*, i.e., *$f(x) \preceq x$, for any $\forall x \in X$. Suppose that each chain in X has a lower bound. Then f has a fixed point.*

Results on the existence of points of coincidence of two maps of ordered sets were obtained (possibly for the first time) in [3–6] and then developed and generalized in [7–9].

In this paper, we present fixed-point and coincidence theorems for families of set-valued maps (Theorems 5–7 below), which generalize the results cited above, and their corollaries. Unlike those in papers [3–9], the theorems presented here do not require the given maps to be isotone or covering. They require only the existence of special chains having lower bounds with certain properties in the ordered set.

We use the following notation and terminology (see [3–9]). By $\rightrightarrows$ we denote a set-valued map. Let $(X, \preceq)$ be an ordered set, and let $\mathcal{F} = \{F_\alpha\}_{\alpha \in A}$ be a family of set-valued maps $F_\alpha : X \rightrightarrows X, \alpha \in A \neq \varnothing$. For each element $x \in X$, we set $\mathbb{O}_X(x) = \{x' \in X \mid x' \preceq x\}$. By a set of $\mathcal{F}$-values at a point $x \in X$ we mean any set $\{y_\alpha\}_{\alpha \in A} \subseteq X$, where $y_\alpha \in F_\alpha(x)$ and $\alpha \in A$. The set of common fixed points of a family $\mathcal{F}$ is denoted by

$$\mathrm{Comfix}(F) := \left\{ x \in X \mid x \in \bigcap_{\alpha \in A} F_\alpha(x) \right\}.$$

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Let $x_0 \in X$. By $\mathscr{C}_1(x_0;\mathscr{F})$ we denote the set of pairs of the form (S, f), where $S \subseteq \mathbb{O}_X(x_0)$ is a chain in X and f is a special $\mathscr{F}$-selector on S, i.e., $f = \{f_\alpha\}_{\alpha\in A}, f_\alpha : S \to X$, $f_\alpha(x) \in F_\alpha(x)$, $x \succeq f_\alpha(x)$ for any $x \in S$ and $\alpha \in A$, and, for any $\alpha \in A$ and $u, v \in S$, $v \prec u \Rightarrow v \preceq f_\alpha(u)$. The subset $\tilde{\mathscr{C}}_1(x_0;\mathscr{F}) \subseteq \mathscr{C}_1(x_0;\mathscr{F})$ consists of pairs (S, f) such that, for any $x \in S$, there exists an element $x' \in \mathbb{O}_X(x_0)$ and a set $\{y_\alpha\}_{\alpha\in A}$ of $\mathscr{F}$-values at x' such that $x \preceq y_\alpha \preceq x'$ for any $\alpha \in A$.

Theorem 5. *Suppose that, for an ordered set $(X, \preceq)$, a family $\mathscr{F} = \{F_\alpha\}_{\alpha\in A}$, $F_\alpha : X \to X$, $\alpha \in A$, and a point $x_0 \in X$, the set $\tilde{\mathscr{C}}_1(x_0;\mathscr{F})$ is nonempty and, for any $(S, f) \in \tilde{\mathscr{C}}_1(x_0;\mathscr{F})$, there exist a common lower boundary $w \in X$ of the chains $f_\alpha(S)$, $\forall\alpha \in A$, and a set $\{w_\alpha\}_{\alpha\in A}$ of $\mathscr{F}$-values at w such that $w \succeq w_\alpha$, $\forall\alpha \in A$. Suppose also that the existence of a $\tilde{\beta} \in A$ for which $w_{\tilde{\beta}} \prec w$ implies the existence of an element $\hat{w} \in X, \hat{w} \preceq w_\alpha, \forall\alpha \in A$, and there exists a set $\{z_\alpha\}_{\alpha\in A}$ of $\mathscr{F}$-values at $\hat{w}$ such that $z_\alpha \preceq \hat{w}, \forall\alpha \in A$. Then the set* Comfix$(\mathscr{F})$ *is nonempty, and it has a minimal element.*

Now, suppose that the set A is endowed with a total order $\preceq_1$. A set $\{y_\alpha\}_{\alpha\in A}$ of $\mathscr{F}$-values at a point $x \in X$ is called a nonincreasing chain of $\mathscr{F}$-values (with respect to the order $\preceq_1$) at x if $\alpha \preceq_1 \beta \Rightarrow y_\beta \preceq y_\alpha$ for any $\alpha, \beta \in A$.

We say that a special $\mathscr{F}$-selector $f = \{f_\alpha\}_{\alpha\in A}, f_\alpha : S \to X$, $\alpha \in A$, on a chain $S \subseteq X$ is a special chain $\mathscr{F}$-selector if, for any $x \in S, \{f_\alpha(x)\}_{\alpha\in A}$ is a nonincreasing chain of $\mathscr{F}$-values at the point x.

Let $x_0 \in X$. By $\mathscr{C}_2(x_0;\mathscr{F})$ we denote the set of pairs (S, f), where $S \subseteq \mathbb{O}_X(x_0)$ is a chain, $f = \{f_\alpha\}_{\alpha\in A}$ is a special chain $\mathscr{F}$-selector on S, and, for any $x \in S$, there exists an $x' = x'(x) \in \mathbb{O}_X(x_0)$ and a nonincreasing chain of $\mathscr{F}$-values at $\{y_\alpha\}_{\alpha\in A}$ in x' such that $x \preceq y_\alpha \preceq x'$, $\alpha \in A$,.

Theorem 6. *Suppose given an ordered set $(X, \preceq)$, a totally ordered set $(A, \preceq_1)$, and a family $\mathscr{F} = \{F_\alpha\}_{\alpha\in A}$, $F_\alpha : X \to X \preceq X, \alpha \in A$. Suppose that, for each $x \in X$, any nonincreasing chain of F-values at x has an infimum. Suppose also that the set $\mathscr{C}_2(x_0;\mathscr{F})$ is nonempty and, for any pair $(S, f) \in \mathscr{C}_2(x_0;\mathscr{F})$, where $f = \{f_\alpha\}_{\alpha\in A}$, , there exists a common lower bound $w \in X$ for all chains $f_\alpha(S)$, $\alpha \in A$, and a nonincreasing chain $\{w_\alpha\}_{\alpha\in A}$ of $\mathscr{F}$-values at w such that $w \succeq w_\alpha, \alpha \in A$. Finally, suppose that if $\hat{w} = \inf_{\alpha\in A}\{w_\alpha\} \prec w$, then, in turn, at the point $\hat{w}$ there exists a nonincreasing chain $\{z_\alpha\}_{\alpha\in A}$ of $\mathscr{F}$-values such that $z_\alpha \preceq \hat{w}, \alpha \in A$. Then the set* Comfix$(\mathscr{F})$ *is nonempty and contains a minimal element.*

Now, suppose that $(X, \preceq), (Y, \preceq)$ are ordered sets, , $x_0 \in X$, $\mathscr{F} = \{F_1, \ldots, F_n\}$, and $F_1, \ldots, F_n : X \rightrightarrows Y$, $n \geq 2$.

Let $\mathscr{C}_3(x_0;\mathscr{F})$ denote the set of pairs of the form (S, f), where $S \subseteq \mathbb{O}_X(x_0)$ is a chain and $f = \{f_i\}_{1\leq i\leq n}$, $f_i : S \to Y$ is a chain $\mathscr{F}$-selector on S such that, for any $x \in S$, we have

$$f_i(x) \in \left(\bigcap_{j=i+1}^{n} F_j(\mathbb{O}_X(x_0))\right) \bigcap F_i(x),$$
$$i = 1, 2, \ldots, n-1, \quad f_n(x) \in F_n(x),$$

and there exists an $x' = x'(x) \in \mathbb{O}_X(x_0)$ and a nonincreasing chain $\{y_j\}_{1\leq j\leq n}$ of $\mathscr{F}$-values at x' such that $f_1(x) = y_n$. Suppose also that, for any $x, z \in S$, $x \prec z \Rightarrow f_1(x) \preceq f_n(z)$.

Theorem 7. *Suppose that $(X, \preceq), (Y, \preceq)$ are ordered sets, $x_0 \in X$, $\mathscr{F} = \{F_1, \ldots, F_n\}$, $F_1, \ldots, F_n : X \rightrightarrows Y$, and $n \geq 2$. Suppose also that $\mathscr{C}_3(x_0;\mathscr{F}) \neq \phi$ and, for any pair (S, f) in $\mathscr{C}_3(x_0;\mathscr{F})$, where $f = \{f_j\}_{1\leq j\leq n}$, the chain S has a lower bound $w \in X$, and at the point w there exists a nonincreasing chain $z = \{z_j\}_{1\leq j\leq n}$ of $\mathscr{F}$-values such that each z_j is a lower bound for the set $\{f_j(x) \mid x \in S\}, j = 1$, 2, ..., n. Finally, suppose that if $z_n \prec z_1$, then $\mathbb{O}_X(x_0)$ contains a chain $\{w_k\}_{1\leq k\leq n-1}, w_{n-1} \preceq \cdots \preceq w_1 \preceq w$ such that $z_n \in \bigcap_{k=1}^{n-1} F_{n-k}(w_k) \cap F_n(w)$ and there exists a nonincreasing chain $v_{n-1} \preceq \cdots \preceq v_1 \preceq z_n$ of $\mathscr{F}$-values at w_{n-1}, i.e., $z_n \in F_1(w_{n-1}), v_k \in F_{k+1}(w_{n-1}), k = 1, 2, \ldots, n-1$. Then the set* $\mathrm{Coin}(F_1, \ldots, F_n) = \left\{x \in X \mid \bigcap_{i=1}^{n} F_i(x) \neq \phi\right\}$ *of the coincidence points of the family $\mathscr{F}$ is nonempty.*

It can be shown that Theorems 5–7 imply, respectively, Theorems 1–3 in [8, 9]. Note also that, for n = 2, Theorem 3 in [8, 9] implies Theorem 1 in [5, 6] (see a pertinent remark in [9]). For $n = 2$, Theorem 7 is stated as follows.

Theorem 8. *Suppose that $(X, \preceq), (Y, \preceq)$ are partially ordered sets, $x_0 \in X$, and $\mathscr{F} = \{F_1, F_2\}$, $F_1, F_2 : X \to Y \preceq Y$ are set-valued maps. Suppose also that the set $\mathscr{C}_3(x_0;\mathscr{F})$ is nonempty and, for any pair $(S, f) \in \mathscr{C}_3(x_0;\mathscr{F})$, where $f = \{f_1, f_2\}$, the chain S has a lower bound $w \in X$ and there exist $\mathscr{F}$-values $\{z_1, z_2\}$ at w such that $z_j \in F_j(w)$, $j = 1, 2$, and $z_1 \succeq z_2$, where z_j is a lower bound for the set $\{f_j(x) \mid x \in S\}$, $j = 1, 2$. Finally, suppose that if $z_2 \prec z_1$, then there exists a $w_1 \in X, w_1 \preceq w$ such that $z_2 \in F_1(w_1) \cap F_2(w)$ and*

$\exists v \in F_2(w_1)$, $v \preceq z_2$. *Then* $\mathrm{Coin}(F_1, F_2) = \{x \in X \mid F_1(x) \cap F_2(x) \neq \phi\} \neq \phi$.

Setting $X = Y$, $F_1 = \mathrm{Id}_X, F_2 = F : X \rightrightarrows X$, and hence $\mathcal{C}_3(x_0; \mathrm{Id}_X, F) = \mathcal{C}_2(x_0; \{F\})$ in the statement of Theorem 8, we obtain the following result.

Theorem 9. *Suppose that* $(X, \preceq)$ *is a partially ordered set*, $x_0 \in X$, $F : X \rightrightarrows X$, *and* $\mathcal{C}_2(x_0; \{F\}) \neq \phi$. *Suppose also that, for each pair* $(S, f) \in \mathcal{C}_2(x_0; \{F\})$, *there exists a lower bound* w *for the chain* $f(S)$ *and a* $z \in F(w)$, $z \preceq w$. *Finally, suppose that the relation* $z \prec w$ *implies the existence of a* $z' \in F(z), z' \preceq z$. *Then* $\mathrm{Fix}(F) \neq \phi$.

Theorem 9 follows also from Theorems 5 and 6 with $n = 1$. Moreover, Theorems 5 (with $n = 1$), and 9 imply Theorem 5 in [7].

Let us compare Theorem 9 with Theorems 4 and 3.

Proposition 1. *Theorem* 4 *follows from Theorem* 9.

The idea of the proof is that, under the assumptions of Theorem 4, any iteration sequence of the form $C = \{f(x_0), f^2(x_0), \ldots\}$ for any $x_0 \in X$ is a chain, it has a lower bound, and $(C, f) \in \mathcal{C}_2(x_0; \{F\})$.

Thus, Theorems 7 and 8 can be regarded as generalizations of Theorem 4 (and, hence, of (Zermelo's) Theorem 3, which follows from Theorem 4) to the case of coincidences. We also mention that Theorems 3 and 4 do not follow from Theorems 1–3 of [8, 9] and from the corresponding assertions in [3–6].

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