ISSN 1064-5624, Doklady Mathematics, 2017, Vol. 95, No. 2, pp. 176–177. © Pleiades Publishing, Ltd., 2017.
Published in Russian in Doklady Akademii Nauk, 2017, Vol. 95, No. 6, pp. 640–642.

Left-Invariant Riemannian Problems on the Groups of Proper Motions of Hyperbolic Plane and Sphere[1]

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Presented by Academician of the RAS R.V. Gamkrelidze October 13, 2016

Received November 28, 2016

Abstract– On the Lie groups $PSL_2(\mathbb{R})$ and SO_3 we consider left-invariant Riemannian metrics with two equal eigenvalues. The global optimality of geodesics is investigated. We find the parametrization of geodesics, the cut locus and the equations for the cut time. When the third eigenvalue of a metric tends to the infinity the cut locus and the cut time converge to the cut locus and the cut time of the sub-Riemannian problem.

DOI: 10.1134/S1064562417020223

Let X be a two dimensional manifold of a constant curvature $\kappa = \pm 1$, i.e., a hyperbolic plane or a sphere. Let G be the group of proper motions (preserving the orientation isometries) of X (i.e., G is isomorphic to $PSL_2(\mathbb{R})$ or SO_3 when κ equals -1 or 1, respectively).

Identify the group G (as a manifold) with the bundle of unit tangent vectors on X. This bundle is a weak symmetric homogeneous space

$$(G \times SO_2)/SO_2,$$

where the first multiplier naturally acts on X, and the second one (SO_2) acts by rotations of all tangent vectors by the same angle. The stabilizer SO_2 is injected to the product in the anti-diagonal way. Note that the homogeneous space $(PSL_2(\mathbb{R}) \times SO_2)/SO_2$ is the first original example of A. Selberg who introduced the weak symmetric spaces [3].

We consider a $(G \times SO_2)$-invariant Riemannian metric on such homogeneous spaces. Other words, a left-invariant Riemannian metric on G with eigenvalues $I_1 = I_2$, $I_3 > 0$ of the restriction of the metric to the Lie algebra $\mathfrak{g}$ of G.

To describe shortest arcs of a Riemannian metric we need to find a parametrization of geodesics, the cut times of geodesics (the value of the parameter after which the geodesic is not optimal), the cut points of geodesics. Due to the left-invariance of the metric, it is enough to consider only geodesics starting at the identity point of G. The set of the cut points of such geodesics is called the cut locus.

Introduce the following notation. Let $\eta = \kappa \frac{I_1}{I_3} - 1$ be a parameter which measures the prolateness of small spheres of the metric. Note that $\eta \in (-\infty, -1)$ for $G = PSL_2(\mathbb{R})$ and $\eta \in (-1, +\infty)$ for $G = SO_3$. Let $e_1, e_2, e_3 \in \mathfrak{g}$ be the eigenvectors of the metric with the eigenvalues $I_1 = I_2, I_3$. We will identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form Kil. Let $p = p_1e_1 + p_2e_2 + p_3e_3 \in \mathfrak{g}^*$ be an initial momentum of a geodesic.

Proposition 1. *A geodesic starting from the identity point of G with an initial momentum p is the product of the two one-parameter subgroups*:

$$q(t) = \exp\left(\frac{tp}{I_1}\right)\exp\left(\frac{t\eta p_3 e_3}{I_1}\right).$$

The cut locus has the following geometric description.

Theorem 1. *The cut locus of the left-invariant Riemannian metric on G with the eigenvalues* $I_1 = I_2$, $I_3 > 0$ *is*:

(1) *the set P of central symmetries of X, when* $2I_1 \geq I_3$;

(2) *the union of P and the interval* J_η *of rotations of X around a fixed point* $x_0 \in X$ *by angles* $\pm\varphi$, *where* $\varphi \in [2\kappa\pi(1+\eta), \pi]$, *when* $2I_1 < I_3$.

Note that the set of central symmetries of a hyperbolic plane is homeomorphic to a plane, and the set of central symmetries of a sphere is homeomorphic to a

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projective plane. So, $P \simeq \mathbb{R}^2$ for $G = PSL_2(\mathbb{R})$ and $P \simeq \mathbb{R}P^2$ for $G = SO_3$.

In the case of $G = SO_3$ Riemannian geodesics correspond to trajectories of a rigid body with a fixed point and the inertia momenta I_1, I_2, I_3. There is such a mechanical interpretation only when there is a triangle with sides of lengths I_1, I_2, I_3. The additional component J_η of the cut locus appears when there is no such a rigid body.

To describe the cut time as a function of an initial momentum p of ageodesic that starts from the identity, denote $|p| = \sqrt{|\mathrm{Kil}(p)|}$. Let $t_0(p)$ be the first positive root of the equation (depending on the parameters η and p):

$$C(t,p)\cos\left(\frac{t\eta p_3}{2I_1}\right) - S(t,p)\sin\left(\frac{t\eta p_3}{2I_1}\right) = 0,$$

where

$$C(t,p) = \cos\left(\frac{t|p|}{2I_1}\right),$$

$$S(t,p) = \frac{p_3}{|p|}\sin\left(\frac{t|p|}{2I_1}\right), \quad \text{for} \quad \mathrm{Kil}(p) > 0,$$

$$C(t,p) = \cosh\left(\frac{t|p|}{2I_1}\right),$$

$$S(t,p) = \frac{p_3}{|p|}\sinh\left(\frac{t|p|}{2I_1}\right), \quad \text{for} \quad \mathrm{Kil}(p) < 0,$$

$$C(t,p) = 1, \quad S(t,p) = \frac{tp_3}{2I_1}, \quad \text{for} \quad \mathrm{Kil}(p) = 0.$$

If this equation have no positive roots, put $t_0(p) = +\infty$. Notice that if $G = SO_3$, then the Killing form is positively defined, and if $G = PSL_2(\mathbb{R})$, then the Killing form is alternating.

Define the cone $\mathscr{C} = \{p \in \mathfrak{g}^* \mid p_3^2 - K(\eta)(p_1^2 + p_2^2) = 0\} \subset \mathfrak{g}^*$, where

$$K(\eta) = \begin{cases} \dfrac{9}{9 - 4\eta^2}, & \text{when } \eta < -1, \\ \dfrac{1}{4\eta^2 - 1}, & \text{when } \eta > -1. \end{cases}$$

Theorem 2. (1) *When* $2I_1 \geq I_3$, *the cut time equals* $t_0(p)$.

(2) *When* $2I_1 < I_3$, *for p inside* $\mathscr{C}$ *the cut time equals* $\frac{2I_1\pi}{|p|}$, *and for p outside of* $\mathscr{C}$ *the cut time equals* $t_0(p)$. *On the cone* $\mathscr{C}$ *these values coincide one another.*

(3) *A geodesic with an initial momentum p has no cut points only if* $G = PSL_2(\mathbb{R})$ *and* $p_3 = 0$ (*then* $t_0(p) = +\infty$).

There is a following sub-Riemannian structure on G. Consider the plane $\mathrm{span}\{e_1, e_2\} \subset \mathfrak{g}$ and the restriction of the Killing form to this plane. Produce by the left shifts the distribution of two-dimensional planes in the tangent bundle TG and the quadratic form on this distribution. The sub-Riemannian cut locus for $PSL_2(\mathbb{R})$ was found by Berestovskii [1]. For SO_3 the answer was given by Boscain and Rossi [2], and the full proof was achieved by Berestovskii and Zubareva [4]. Note that the bundle of unit tangent vectors on X is a configuration space of a mobile robot. So, the sub-Riemannian shortest arcs model energy-optimal motions of a mobile robot that is able to go forward and rotate.

It turns that the Riemannian problem approximate the sub-Riemannian one when $I_3 \to +\infty$ (equivalent to $\eta \to -1$) in the following sense.

Proposition 2. *The parametrization of geodesics, the conjugate time, the conjugate locus, the cut time and the cut locus in the sub-Riemannian problem on* $PSL_2(\mathbb{R})$ *or* SO_3 *are produced of the same objects of the Riemannian problem passing to the limit* $\eta \to -1-0$ *or* $\eta \to -1+0$, *respectively.*

When $\eta \to -1$ the intervals J_η converge to the circle of all rotations around the fixed point of X. This circle is the "local" component of the sub-Riemannian cut locus. The "global" component is the same for the Riemannian and the sub-Riemannian cases (the set P of the central symmetries of X).

ACKNOWLEDGMENTS

This work was performed under financial support from the Government, represented by the Ministry of Education and Science of the Russian Federation, within the scope of Project no. RFMEFI60414X0138.

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