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MATHEMATICS

Infinite Quantum Graphs

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Abstract—Infinite quantum graphs with δ-interactions at vertices are studied without any assumptions on the lengths of edges of the underlying metric graphs. A connection between spectral properties of a quantum graph and a certain discrete Laplacian given on a graph with infinitely many vertices and edges is established. In particular, it is shown that these operators are self-adjoint, lower semibounded, nonnegative, discrete, etc. only simultaneously.

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Consider a metric graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_n\}$ is a countable vertex set and $\mathcal{E} = \{e_k\}$ is a countable edge set. Each edge $e \in \mathcal{E}$ has length $|e| \in (0, \infty)$. We write $v \in e$ if v is a vertex of the edge e. Given a vertex $v \in \mathcal{V}$, we use $\mathcal{E}_v$ to denote the set of edges going from v. The number

$$\deg(v) := \#\{e: \; e \in \mathcal{E}_v\}$$

is called the degree of the vertex $v \in \mathcal{V}$. Given two vertices $v, u \in \mathcal{V}$ we write $v \sim u$ and say that the vertices u and v are adjacent if there exists an edge $e_{u,v} \in \mathcal{E}$ connecting v with u. Throughout the paper, we assume that the graph $\mathcal{G}$ is connected and has no isolated vertices, loops, and multiple edges. We also assume that $\mathcal{G}$ is directed, that is, each edge $e \in \mathcal{E}$ has a direction, an origin e_o, and a terminal vertex e_i.

The main object of study in this paper is a quantum graph with δ-interactions at vertices (see Definition 1). Our main result is a connection between the spectral properties of a quantum graph and a certain discrete Laplacian given on $\mathcal{G}$ (see (10)).

The duality between continuous and discrete operators on graphs was first noticed by physicists; the first mathematical results were obtained about 30 years ago. We mention related papers [2, 4, 10] (and the references therein).

Quantum graphs model various nanostructures arising in experiments; they also serve as a tool for studying properties of various quantum systems. The spectral theory of such graphs is comparatively new; it has been extensively developed during the past two decades. The mathematical interest in this theory is caused, in particular, by the interaction of geometric and topological methods with those of ODE theory. Quantum graphs have an extensive literature; we only mention monograph [1] and collection of papers [5] for further reference. However, it should be mentioned that most statements concerning quantum graphs assume either the finiteness of the edge set ($\#\mathcal{E} < \infty$) or the existence of a positive lower bound for the edge lengths ($\inf_{e \in \mathcal{E}}|e| > 0$). Our main purpose is to study the spectral properties of quantum graphs without these constraints on the geometry of the metric graph $\mathcal{G}$.

Notation. $E_T(\cdot)$ denotes the resolution of identity of a self-adjoint operator $T = T^*$ on a Hilbert space $\mathfrak{H}$; $T^-: = E_T((-\infty, 0))T$ and $\kappa_-(T) = \dim \operatorname{ran}(T^-)$ is the total multiplicity of the negative spectrum of T. In particular, if T^- is compact, then $\kappa_-(T)$ is the number of negative eigenvalues of the operator T (counting multiplicities); $\mathfrak{S}_p(\mathfrak{H})$, $p \in (0, \infty]$ are the von Neumann–Schatten ideals in $\mathfrak{H}$.

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1. QUANTUM GRAPHS WITH δ-INTERACTIONS

Consider the Hilbert space $L^2(\mathcal{G}) = \oplus_{e\in\mathcal{E}} L^2(e)$ on the graph $\mathcal{G}$. We define the maximal operator on this space by

$$\mathbf{H}_{\max} = \bigoplus_{e\in\mathcal{E}} H_{e,\max}, \quad H_{e,\max} = -\frac{d^2}{dx_e^2}, \\ \operatorname{dom}(H_{e,\max}) = W^{2,2}(e), \tag{1}$$

where $W^{2,2}(e)$ is the Sobolev space on the edge e. It is easy to show that, for each function $f_e \in W^{2,2}(e)$, the quantities

$$f_e(e_0) = \lim_{x\to e_0} f_e(x), \quad f_e(e_i) = \lim_{x\to e_i} f_e(x), \tag{2}$$

and

$$f_e'(e_0) = \lim_{x\to e_0} \frac{f_e(x) - f_e(e_0)}{|x - e_0|}, \\ f_e'(e_i) = \lim_{x\to e_i} \frac{f_e(x) - f_e(e_i)}{|x - e_i|} \tag{3}$$

are well defined.

Let $\alpha\colon \mathcal{V} \to \mathbb{R}$. We specify boundary conditions at the vertices by

$$f \quad \text{continuous at} \quad v, \\ \sum_{e\in\mathcal{E}_v} f_e'(v) = \alpha(v) f(v). \tag{4}$$

If $\alpha(v) = 0$, then (4) is the standard Kirchhoff condition. Otherwise, condition (4) is called a δ-interaction at the vertex ν of the force $\alpha(v)$ (see [1]). The family of symmetric boundary conditions at the vertices is significantly larger (see [1]); however, the requirement of continuity at the vertices leads to conditions of the form (4). We define an operator $\mathbf{H}_\alpha$ as the closure of the preminimal operator $\mathbf{H}_\alpha^0$:

$$\mathbf{H}_\alpha^0 = \mathbf{H}_{\max} \restriction \operatorname{dom}(\mathbf{H}_\alpha^0), \\ \operatorname{dom}(\mathbf{H}_\alpha^0) = \{f \in \operatorname{dom}(\mathbf{H}_{\max}) \cap L_c^2(\mathcal{G})\colon \\ f \text{ satisfies } (4), v \in \mathcal{V}\}. \tag{5}$$

Here, $L_c^2(\mathcal{G})$ is the linear submanifold in $L^2(\mathcal{G})$ consisting of functions not identically vanishing only on a finite set of edges. Note that the operator $\mathbf{H}_\alpha$ is symmetric. Simple examples show that it may be non-self-adjoint.

Example 1. Consider the half-axis $\mathbb{R}_+$. Suppose given a strictly increasing sequence of points $\{x_k\}_{k\in\mathbb{N}_0}$ such that $x_0 = 0$ and $x_k \uparrow +\infty$. Taking the points x_k for vertices and the intervals $e_k = (x_{k-1}, x_k)$ for edges, we obtain the simplest infinite metric graph. For this graph, by virtue of definitions (3), conditions (4) take the form $f'(0) = 0$,

$$f(x_k-) = f(x_k+) =: f(x_k), \\ f'(x_k+) - f'(x_k-) = \alpha_k f(x_k), \quad k \in \mathbb{N}, \tag{6}$$

where $\alpha = \{\alpha_k\}_{k\in\mathbb{N}_0}$ is a real sequence with $\alpha_0 = 0$.

The operator $\mathbf{H}_\alpha$ is known as the one-dimensional Schrödinger operator with δ-interactions at the points x_k, $k \in \mathbb{N}$:

$$H_\alpha = -\frac{d^2}{dx^2} + \sum_{k\in\mathbb{N}} \alpha_k \delta(x - x_k). \tag{7}$$

It was shown in [13] that the operator H_α is self-adjoint if $\sum_k |e_k|^2 = \infty$; in the case $\sum_k |e_k|^2 < \infty$, this operator may be non-self-adjoint (see [13, Proposition 5.9]).

Definition 1. The self-adjoint extensions of the operator $\mathbf{H}_\alpha$ in the space $L^2(\mathcal{G})$ are called quantum graphs with δ-interactions at vertices.

Remark 1. If $\deg(v_0) = \infty$, then, according to [9, Theorem 5.2], condition (4) leads to a nonclosed operator, whose closure is generated by the Dirichlet condition $f(v_0) = 0$ at the vertex v_0. Thus, hereafter, we assume without the loss of generality that $\deg(v) < \infty$ for all $v \in \mathcal{V}$.

2. THE MAIN RESULT

First, we introduce our second main object of study, which is a difference operator on a graph. For this purpose, we define a function $m\colon \mathcal{V} \to (0, \infty)$ by

$$m\colon v \mapsto \sum_{e\in\mathcal{E}_v} |e|, \quad v \in \mathcal{V}, \tag{8}$$

$$b\colon \mathcal{V} \times \mathcal{V} \mapsto [0, \infty): \\ b(v,u) = \begin{cases} |e_{v,u}|^{-1}, & v \sim u, \\ 0, & v \nsim u. \end{cases} \tag{9}$$

On the weight Hilbert space $\ell^2(\mathcal{V}; m)$, consider the minimal operator h_α generated by the difference expression

$$(\tau_{\mathcal{G},\alpha} f)(v) \\ := \frac{1}{m(v)} \left(\sum_{u\in\mathcal{V}} b(v,u)(f(v) - f(u)) + \alpha(v) f(v) \right), \tag{10} \\ v \in \mathcal{V}.$$

Namely, the operator h_α on $\ell^2(\mathcal{V}; m)$ is defined as the closure of the preminimal operator

$$h_\alpha^0\colon \operatorname{dom}(h_\alpha^0) \to \ell^2(\mathcal{V}), \\ f \mapsto \tau_{\mathcal{G},\alpha} f, \tag{11}$$

where $\operatorname{dom}(h_\alpha^0) := \ell_c^2(\mathcal{V}; m)$ is the set of functions taking nonzero values only at finitely many vertices. It readily follows from the constraints imposed on the metric graph $\mathcal{G}$ that h_α^0 is a densely defined symmetric operator; therefore, h_α is well defined. Discrete operators of the form (10) play an important role in the theories of electric networks and of Markov processes, and their spectral properties have an extensive literature (see, e.g., [7, 8]).

Remark 2. Sometimes, instead of h_α, it is useful to consider the operator $\tilde{h}_\alpha$ defined on the space $\ell^2(\mathcal{V})$ by the difference expression

$$(\tilde{\tau}_{\mathcal{G},\alpha} f)(v) = \frac{1}{\sqrt{m(v)}}$$
$$\times\left(\sum_{u\in\mathcal{V}} b(v,u)\left(\frac{f(v)}{\sqrt{m(v)}} - \frac{f(u)}{\sqrt{m(u)}}\right) + \frac{\alpha(v)}{\sqrt{m(v)}} f(v)\right), \quad (12)$$
$$v \in \mathcal{V}.$$

It is easy to show that the operators h_α and $\tilde{h}_\alpha$ are unitarily equivalent.

It turns out that the spectral properties of the operators $\mathbf{H}_\alpha$ and h_α are closely related. The following theorem is the main result of this paper.

Theorem 1. *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *be a metric graph such that* $\sup_{e\in\mathcal{E}}|e| < \infty$. *Suppose that* $\alpha: \mathcal{V} \to \mathbb{R}$ *and* $\mathbf{H}_\alpha$ *is a closed symmetric operator on the graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *with* δ*-interactions* (4) *at the vertices. Suppose also that* h_α *is the discrete Laplacian given by* (10), (11) *on* $\ell^2(\mathcal{V}; m)$ *and* $m: \mathcal{V} \to (0, \infty)$ *and* $b: \mathcal{V} \times \mathcal{V} \to [0, \infty)$ *are functions of the forms* (8) *and* (9), *respectively. Then*

(i) *the deficiency indices of the operators* $\mathbf{H}_\alpha$ *and* h_α *are equal:*

$$n_\pm(\mathbf{H}_\alpha) = n_\pm(h_\alpha) \le \infty \quad (13)$$

(*in particular, the operator* $\mathbf{H}_\alpha$ *is self-adjoint if and only if so is* h_α);

(ii) *the operator* $\mathbf{H}_\alpha$ *is lower semibounded if and only if so is* h_α.

Suppose that $\mathbf{H}_\alpha$ (*and, hence,* h_α) *is self-adjoint. Then*

(iii) *the operator* $\mathbf{H}_\alpha$ *is nonnegative (positive definite) if and only if so is* h_α;

(iv) *the total multiplicities of negative spectra of the operators* $\mathbf{H}_\alpha$ *and* h_α *coincide:*

$$\kappa_-(\mathbf{H}_\alpha) = \kappa_-(h_\alpha); \quad (14)$$

(v) *for each* $p \in (0, \infty]$,

$$\mathbf{H}_\alpha^- \in \mathfrak{S}_p(L^2(\mathcal{G})) \Leftrightarrow h_\alpha^- \in \mathfrak{S}_p(\ell^2(\mathcal{V}; m)) \quad (15)$$

(*in particular, the negative spectra of the operators* $\mathbf{H}_\alpha$ *and* h_α *are discrete simultaneously*);

(vi) *if* $h_\alpha^- \in \mathfrak{S}_\infty(\ell^2(\mathcal{V}; m))$, *then, for each* $p \in (0, \infty]$, *the absolute values of the negative eigenvalues of* $\mathbf{H}_\alpha$ *and* h_α *numbered in decreasing order are related by*

$$|\lambda_j(\mathbf{H}_\alpha)| = j^{-p}(a + o(1))$$
$$\Leftrightarrow |\lambda_j(h_\alpha)| = j^{-p}(b + o(1)) \quad (16)$$

as $j \to \infty$; *moreover, either* $ab \neq 0$ *or* $a = b = 0$;

(vii) $\sigma_{ess}(\mathbf{H}_\alpha) \subset (0, \infty)$ $(\sigma_{ess}(\mathbf{H}_\alpha) \subset [0, \infty))$ *if and only if* $\sigma_{ess}(h_\alpha) \subset (0, \infty)$ *and* $(\sigma_{ess}(h_\alpha) \subset [0, \infty))$;

(viii) *the spectrum of the operator* $\mathbf{H}_\alpha$ *is discrete if and only if* $\#\{e \in \mathcal{E}: |e| > \varepsilon\}$ *is finite for all* $\varepsilon > 0$ *and the spectrum of* h_α *is discrete;*

(ix) *for each* $p \in (0, \infty]$,

$$(\mathbf{H}_\alpha - \mathrm{i})^{-1} - (\mathbf{H}_{\tilde{\alpha}} - \mathrm{i})^{-1} \in \mathfrak{S}_p(L^2(\mathcal{G}))$$
$$\Leftrightarrow (h_\alpha - \mathrm{i})^{-1} - (h_{\tilde{\alpha}} - \mathrm{i})^{-1} \in \mathfrak{S}_p(\ell^2(\mathcal{V}; m)). \quad (17)$$

Remark 3. The condition $\sup_{e\in\mathcal{E}}|e| < \infty$ is not essentially restrictive, because the graph $\mathcal{G}$ can be modified by adding virtual vertices, which does not affect the operator $\mathbf{H}_\alpha$. However, this changes the difference operator h_α.

Theorem 1 is proved by a method proposed in [13, 14] for the case of Hamiltonians with δ-interactions (7). Namely, we use the apparatus of boundary triplets and the corresponding Weyl functions (see definitions in Appendix A). One of the key difficulties in this approach is the construction of an adequate boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for the maximal operator $\mathbf{H}_{\max}$ in the case where $\inf_{e\in\mathcal{E}}|e| = 0$. It turns out that, under an appropriate choice of the boundary triplet, h_α is a boundary operator parameterizing the extension $\mathbf{H}_\alpha$ of the minimal operator $\mathbf{H}_{\min} := \mathbf{H}_{\max}^*$:

$$\mathbf{H}_\alpha = \mathbf{H}_{\max} \upharpoonright \mathrm{dom}(\mathbf{H}_\alpha),$$
$$\mathrm{dom}(\mathbf{H}_\alpha) = \ker(\Gamma_1 - h_\alpha\Gamma_0).$$

Let us illustrate Theorem 1 for the example of operator (7) in Example 1.

Example 2. Let H_α be the Schrödinger operator (7) with δ-interactions on the half-axis $\mathbb{R}_+$. In this case, we have

$$m(x_k) = |e_k| + |e_{k+1}|,$$
$$b(x_k, x_n) = \begin{cases} |e_{\max(n,k)}|^{-1}, & |n-k| = 1, \\ 0, & |n-k| \neq 1, \end{cases}$$

and for $k \ge 1$, the difference expression (12) takes the form of a three-term recurrence relation:

$$(\tilde{\tau}_\alpha f)(x_k) = -\frac{|e_k|^{-1} f(x_{k-1})}{\sqrt{m(x_{k-1})m(x_k)}}$$
$$+ \frac{\alpha_k + |e_k|^{-1} + |e_{k+1}|^{-1}}{m(x_k)} f(x_k) - \frac{|e_{k+1}|^{-1} f(x_{k+1})}{\sqrt{m(x_k)m(x_{k+1})}}.$$

Thus, the corresponding difference operator h_α is the minimal operator generated on $\ell^2(\mathbb{N})$ by the Jacobi matrix

$$J = \begin{pmatrix} a_0 & b_1 & 0 & 0 & \dots \\ b_1 & a_1 & b_2 & 0 & \dots \\ 0 & b_2 & a_2 & b_3 & \dots \\ 0 & 0 & b_3 & a_3 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \qquad \begin{aligned} a_k &= \frac{\alpha_k + |e_k|^{-1} + |e_{k+1}|^{-1}}{m(x_k)}, \\ b_k &= -\frac{|e_k|^{-1}}{\sqrt{m(x_{k-1})m(x_k)}}. \end{aligned} \quad (18)$$

Theorem 1 was proved for this case in [13]; in the recent paper [15], it was extended to matrix Schrödinger operators with δ-interactions.

Remark 4. Let us explain the relationship between the operators $\mathbf{H}_\alpha$ and h_α from the elementary point of view. The kernel $\mathcal{L} = \ker(\mathbf{H}_{\max})$ of the maximal operator $\mathbf{H}_{\max}$ consists of piecewise-linear functions on $\mathcal{G}$. Each function $f \in \mathcal{L}$ is uniquely determined by its values at the vertices $\{f(e_i), f(e_0)\}_{e \in \mathcal{E}}$. Therefore, for all $f \in \mathcal{L}_c = \mathcal{L} \cap L^2_c(\mathcal{G})$, we have

$$\|f\|^2_{L^2(\mathcal{G})} = \sum_{e \in \mathcal{E}} |e| \frac{|f(e_i)|^2 + \operatorname{Re}(f(e_i)f(e_0)) + |f(e_0)|^2}{3}. \quad (19)$$

Obviously,

$$\sum_{e \in \mathcal{E}} |e|(|f(e_i)|^2 + |f(e_0)|^2)$$

$$= \sum_{v \in \mathcal{V}} |f(v)|^2 \sum_{e \in \mathcal{E}_v} |e| = \|f\|^2_{\ell^2(\mathcal{V};m)}$$

determines an equivalent norm on $\mathcal{L}_c$. On the other hand, for each $f \in \mathcal{L}_c$, we have

$$(\mathbf{H}_\alpha f, f) = \sum_{e \in \mathcal{E}} |f'(x)|^2 dx + \sum_{v \in \mathcal{V}} \alpha(v)|f(v)|^2$$

$$= \sum_{e \in \mathcal{E}} \frac{|f(e_0) - f(e_i)|^2}{|e|} + \sum_{v \in \mathcal{V}} \alpha(v)|f(v)|^2$$

$$= \frac{1}{2} \sum_{u,v \in \mathcal{V}} b(v,u)|f(v) - f(u)|^2$$

$$+ \sum_{v \in \mathcal{V}} \alpha(v)|f(v)|^2 =: \mathfrak{t}_{\mathcal{G},\alpha}[f].$$

It remains to note that,

$$(h_\alpha, f, f)_{\ell^2(\mathcal{G};m)} = \mathfrak{t}_{\mathcal{G},\alpha}[f], \quad f \in \ell^2_c(\mathcal{G};m). \quad (20)$$

This relation sheds some light on the relationship between the operators $\mathbf{H}_\alpha$ and h_α.

3. SPECTRAL PROPERTIES OF QUANTUM GRAPHS

The relationship between operators (7) and Jacobi matrices (18) discovered in [13] has made it possible to obtain new results about the spectral properties of the Schrödinger operator with δ-interactions. Similarly, applying the fairly well developed spectral theory of difference operators on graphs, we can obtain new results about quantum graphs. Because of space limitations, we present only a few statements on self-adjointness, semiboundedness, and negative spectrum.

In the rest of the paper, we assume that

$$\sup_{e \in \mathcal{E}} |e| < \infty. \quad (21)$$

3.1. Self-Adjointness

It is well known (see, e.g., [8]) that the operator h_α with potential $\alpha \equiv 0$ is bounded if and only if

$$C_{\mathcal{G}} := \sup_{u \in \mathcal{V}} \frac{1}{m(v)} \sum_{v \in \mathcal{V}} b(u,v) = \sup_{v \in \mathcal{V}} \frac{\sum_{e \in \mathcal{E}_v} |e|^{-1}}{\sum_{e \in \mathcal{E}_v} |e|} < \infty, \quad (22)$$

and $C_{\mathcal{G}} \le \|h_0\|_{\ell^2(\mathcal{V};m)} \le 2C_{\mathcal{G}}$. This fact and Theorem 1 (i) imply the following result.

Lemma 1. *If condition* (22) *holds, then the operator* $\mathbf{H}_\alpha$ *is self-adjoint for all* $\alpha\colon \mathcal{V} \to \mathbb{R}$.

Corollary 1. If $\inf_{e \in \mathcal{E}} |e| > 0$, then the operator $\mathbf{H}_\alpha$ is self-adjoint for all $\alpha\colon \mathcal{V} \to \mathbb{R}$.

Corollary 1 follows from Lemma 1, because condition (21) and the inequality $\inf_{e \in \mathcal{E}} |e| > 0$ imply (22). Another proof uses [9, Theorem 3.2]. We also mention that (22) is equivalent to the condition $\inf_{e \in \mathcal{E}} |e| > 0$ only if $\sup_{v \in \mathcal{V}} \deg(v) < \infty$.

3.2. Semiboundedness

Following [8], we introduce the following condition.

Condition 1. *Given any sequence* $\{v_n\}_{n \in \mathbb{N}}$ *of vertices such that* $b(v_n, v_{n+1}) \neq 0$ *for all* $n \in \mathbb{N}$, *the series*

$$\sum_{n \in \mathbb{N}} m(v_n) = \sum_{n \in \mathbb{N}} \sum_{e \in \mathcal{E}_{v_n}} |e| = \infty. \quad (23)$$

Our next result is as follows.

Proposition 1. *If Condition* 1 *holds and* $\alpha\colon \mathcal{V} \to \mathbb{R}$ *is such that*

$$C_- := \inf_{v \in \mathcal{V}} \frac{\alpha(v)}{m(v)} = \inf_{v \in \mathcal{V}} \frac{\alpha(v)}{\sum_{e \in \mathcal{E}_v} |e|} > -\infty, \quad (24)$$

then the operator $\mathbf{H}_\alpha$ *is self-adjoint and lower semibounded.*

Simple examples of operators with δ-interactions on $\mathbb{R}_+$ show that both conditions (23) and (24) are essential for the validity of Proposition 1 (see [13]).

3.3. Negative Spectrum

In what follows, we assume that the function α is nonnegative, i.e., $\alpha: \mathcal{V} \to [0, \infty)$, and Condition 1 holds. Our main purpose in this section is to estimate the number of negative eigenvalues of the operator $\mathbf{H}_{-\alpha}$ in terms of the interactions α. First, we note that the operator h_0 is nonnegative. Moreover, by virtue of the first Beurling–Deny theorem, it is a generator of a symmetric Markov semigroup. Let $\mathfrak{t}_{\mathcal{G}} := \mathfrak{t}_{\mathcal{G},0}$ be the quadratic form (20) in Remark 4 with $\alpha \equiv 0$. In [11, 12] (see also [6]), it was mentioned that the key role in CLR-type estimates is played by the Sobolev-type inequalities

$$\|f\|^2_{\ell^q(\mathcal{V};m)} := \left(\sum_{v\in\mathcal{V}} |f(v)|^q m(v)\right)^{2/q} \le K\mathfrak{t}_{\mathcal{G}}[f], \qquad q = \frac{2D}{D-2}, \tag{25}$$

in which $D > 2$ and $K > 0$ are constants not depending on f. In turn, according to Varopoulos' theorem (see, e.g., [11]), (25) is equivalent to the following estimate for the semigroup $T(t) = \exp(-h_0 t)$ generated by the difference operator h_0:

$$\mathcal{M}_T(t) := \|T(t)\|_{1\to\infty} \le Ct^{-D/2}, \quad t > 0. \tag{26}$$

According to Theorem 1 (iv), we have $\kappa_-(\mathbf{H}_{-\lambda\alpha}) = \kappa_-(h_{-\lambda\alpha})$. Combining this equality with Theorem 14 of [11] (see also [6, Theorem 2.1]), we obtain the following result.

Proposition 2. *There exist $D > 2$ and $K > 0$ such that* (25) *holds if and only if*

$$\kappa_-(\mathbf{H}_{-\lambda\alpha}) \le C\lambda^{D/2}\sum_{v\in\mathcal{V}} \alpha(v)^{D/2} \sum_{e\in\mathcal{E}_v} |e|, \qquad C = eK, \tag{27}$$

for all $\alpha \in \ell^{D/2}(\mathcal{V}; m)$ and $\lambda > 0$.

The most difficult point in estimating the number $\kappa_-(\mathbf{H}_{-\lambda\alpha})$ is to verify estimate (25) or the equivalent condition (26). However, such a verification has already been performed in many cases (see, e.g., [11, 12]).

Corollary 4. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a metric graph in which $\mathcal{V}$ is a group of polynomial growth $D \ge 3$. If α: $\mathcal{V} \to [0, \infty)$ belongs to the space $\ell^{D/2}(\mathcal{V})$, then*

$$\kappa_-(\mathbf{H}_{-\lambda\alpha}) \le C(\mathcal{G})\lambda^{D/2}\sum_{v\in\mathcal{V}} \alpha(v)^{D/2}\deg(v), \tag{28}$$

where $C(\mathcal{G})$ is a constant depending only on the graph $\mathcal{G}$.

In particular, in the case $\mathcal{G} = \mathbb{Z}^D$, we obtain the following estimate (cf. [12]).

Corollary 5. *Let $\mathcal{G} = \mathbb{Z}^N$, $N \ge 3$. If $\alpha: \mathcal{V} \to [0, \infty)$ belongs to the space $\ell^{N/2}(\mathbb{Z}^N)$, then*

$$\kappa_-(\mathbf{H}_{-\lambda\alpha}) \le C_N\lambda^{N/2}\sum_{v\in\mathcal{V}} \alpha(v)^{N/2}, \tag{29}$$

where C_N is a constant depending only on N.

APPENDIX A

BOUNDARY TRIPLETS

Consider a densely defined closed symmetric operator A on $\mathfrak{H}$. Let $\mathfrak{N}_z = \mathfrak{N}_z(A) := \mathfrak{H} \ominus \operatorname{ran}(A - z^*) = \ker(A^* - z)$, $z \in \mathbb{C}_\pm$, be its defect subspaces, and let $n_\pm(A) := \dim\mathfrak{N}_{\pm i}(A)$ be its deficiency indices. Suppose that $n_+(A) = n_-(A) \le \infty$.

Definition 2. A set $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H}$ is a Hilbert space and Γ_0 and Γ_1 are linear mappings from $\operatorname{dom}(A^*)$ to $\mathcal{H}$, is called a boundary triplet for the operator A^* if

(i) the abstract Green's identity holds

$$(A^*f, g) - (f, A^*g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \operatorname{dom}(A^*); \tag{A.1}$$

(ii) the mapping $\Gamma: f \mapsto \{\Gamma_0 f, \Gamma_1 f\}$ from $\operatorname{dom}(A^*)$ to $\mathcal{H} \times \mathcal{H}$ is surjective.

A boundary triplet for an operator A^* exists only if $n_+(A) = n_-(A)$. In this case, $n_\pm(A) = \dim\mathcal{H}$ and $\ker(\Gamma) = \ker(\Gamma_0) \cap \ker(\Gamma_1) = \operatorname{dom}(A)$.

An extension $\tilde{A}$ of an operator A is said to be proper if $A \subset \tilde{A} \subset A^*$. The set of all proper (not necessarily closed) extensions of A is denoted by Ext_A. Recall that a linear relation in $\mathfrak{H}$ is defined as a linear subspace in $\mathfrak{H} \times \mathfrak{H}$. We denote the set of closed linear relations in $\mathfrak{H}$ by $\tilde{\mathcal{C}}(\mathfrak{H})$. Each linear operator T on $\mathfrak{H}$ is identified with its graph $\operatorname{gr}(T)$; therefore, the set $\mathcal{C}(\mathfrak{H})$ of closed linear operators on $\mathfrak{H}$ is identified with the subset of $\tilde{\mathcal{C}}(\mathfrak{H})$.

Proposition 3 [3]. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then the mapping $\Gamma = \{\Gamma_0, \Gamma_1\}$: $\operatorname{dom}(A^*) \to \mathcal{H} \times \mathcal{H}$ determines the following one-to-one correspondence between* Ext_A *and the set of all linear relations in $\mathcal{H}$*

$$\Theta \mapsto A_\Theta := A^* \upharpoonright \{f \in \operatorname{dom}(A^*): \{\Gamma_0 f, \Gamma_1 f\} \in \Theta\}. \tag{A.2}$$

Moreover, $A_\Theta^* = A_{\Theta^*}$; in particular, A_Θ is a symmetric (self-adjoint) extension if and only if the linear relation Θ is symmetric (self-adjoint), and $n_\pm(A_\Theta) = n_\pm(\Theta)$.

The relation Θ is called the boundary relation of the operator $\tilde{A} = A_\Theta$. If Θ is the graph of a linear operator B, i.e., $\Theta = \mathrm{gr}(B)$, then (A.2) takes the form

$$A_B = A^* \restriction \ker(\Gamma_1 - B\Gamma_0),\quad \mathrm{dom}(A_B) = \{f \in \mathrm{dom}(A^*)\colon\ \Gamma_1 f = B\Gamma_0 f\}. \tag{A.3}$$

In this case, B is called the boundary operator of the extension $\tilde{A} = A_B$. An important role in the study of the spectral properties of proper extensions is played by Krein's formula for resolvents [3]:

$$(A_\Theta - z)^{-1} = (A_0 - z)^{-1} + \gamma(z)(\Theta - M(z))^{-1}\gamma^*(\bar{z}),\quad z \in \rho(A_\Theta) \cap \rho(A_0), \tag{A.4}$$

where $A_0 = A^* \restriction \ker(\Gamma_0)$, $\gamma(z) = (\Gamma_0 \restriction \mathfrak{N}_z)^{-1}$ is the gamma-field of the operator A_0, and $M(z) = \Gamma_1\gamma(z)$ is the Weyl function of A_0. We emphasize that it is formula (A.4) which has made it possible to reveal deep relations (not covered by Proposition 3) between the spectral properties of the operator $\tilde{A} = A_\Theta$ and the corresponding boundary relation Θ (see details in [3]).

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