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MATHEMATICS

On the Chromatic Number of a Space with a Forbidden Regular Simplex

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Abstract—Explicit exponential lower bounds for the chromatic numbers of spaces with forbidden monochromatic regular simplexes are found for the first time.

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1. INTRODUCTION AND FORMULATION OF THE RESULT

In 1950, E. Nelson asked for the minimum number of colors required for coloring the plane so that any two points at a distance of 1 from each other have different colors (see [1]). Although this question sounds simple, it still remains open. The Nelson problem is a central one in combinatorial geometry (see [2]). The development of Euclidean Ramsey theory began from the Nelson problem. In this theory, a set $S \subset \mathbb{R}^d$ is called a Ramsey set if the minimum number $\chi_S(\mathbb{R}^n)$ of colors required for coloring $\mathbb{R}^n$ so that no congruent set S is completely monochromatic tends to infinity with growing n. Similarly, S is called an exponentially Ramsey set if, for some constant $c > 1$, it is true that $\chi_S(\mathbb{R}^n) > (c + o(1))^n$. It is well known that any Ramsey set lies on some sphere (see [3]). The exponential Ramsey property has been proved for many sets. Specifically, this was shown for the vertex set of an arbitrary simplex and for Cartesian products of exponentially Ramsey sets (see [3–6]). However, almost every time, this is proved implicitly, so an estimate of the form $\chi_S(\mathbb{R}^n) \geq (c + o(1))^n$ for a particular c cannot be written within the framework of the method used. In this paper, we are interested in lower bounds for $\chi_{S_k}(\mathbb{R}^n)$, where S_k are the vertices of a regular k-dimensional simplex. Since $\mathbb{R}^n$ is homothetic to itself, $\chi_{S_k}(\mathbb{R}^n)$ is independent of the side length of S_k. For brevity, we denote it by $\chi_k(\mathbb{R}^n)$. In the case $k = 1$, we deal with the classical Nelson problem and the best of the proved lower bounds is $\chi_1(\mathbb{R}^n) \geq (1.239\ldots + o(1))^n$ (see [7]). In the case $k = 2$, it was recently proved that $\chi_2(\mathbb{R}^n) \geq (1.052\ldots + o(1))^n$ (see [8]). As was mentioned above, for every $k \geq 3$, it was proved that, for some $c_k > 1$, it holds that $\chi_k(\mathbb{R}^n) \geq (c_k + o(1))^n$, but no explicit expression for c_k was presented. We have proved the following result.

Theorem 1. *Let k be a positive integer. Then*

$$\chi_k(\mathbb{R}^n) \geq \left(1 + \frac{1}{3^{5^k}} + o(1)\right)^n.$$

2. PROOF SKETCH OF THEOREM 1

Let $v(n) = \left[\frac{n}{2}\right]$. Let $v'(n)$ be the largest positive integer smaller than $\frac{n}{4} - 1$ such that $v(n) - v'(n)$ is a prime number. From number theory, it is known that $\lim\limits_{n\to\infty} \frac{v'(n)}{n} = \frac{1}{4}$ (see [9]).

Define the graph $G_n = (V_n, E_n)$ as follows: its vertices are all points in $\mathbb{R}^n$ such that exactly $v(n)$ of their coordinates are equal to unity, while the remaining $n - v(n)$ coordinates are zero; an edge joins two vertices of G_n if and only if the scalar product of their position vectors is equal to $v'(n)$. Assume that the side

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length of a simplex S_k whose congruent copies in $\mathbb{R}^n$ cannot be monochromatic is equal to $\sqrt{2(v(n)-v'(n))}$, i.e., to the length of any edge of G_n. The Stirling formula implies that $|V_n| = (2+o(1))^n$. In the classical work [5], it was, in fact, shown that any sufficiently large subset of V_n contains a vertex of almost maximum degree, i.e., for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that any subset $W \subset V_n$ of cardinality at least $(2-\delta(\varepsilon)+o(1))^n$ contains a vertex of degree at least $(2-\varepsilon+o(1))^n$. We have managed to specify this assertion as follows.

Lemma 1. *If* $w \le 5\times 10^{-53}$, *then any subset* W *of vertices of* G_n *consisting of at least* $(2-w+o(1))^n$ *vertices contains a vertex of degree at least* $\left(2-\sqrt[5]{w}+o(1)\right)^n$.

We have also proved the following result.

Lemma 2. *Any subset* W *of vertices of* G_n *consisting of at least* $(2-5\times 10^{-53}+o(1))^n$ *vertices contains four pairwise adjacent vertices.*

Now let $k \ge 3$ be a fixed positive integer. Let

$$w = (5\cdot 10^{-53})^{5^{k-3}}.$$

Consider an arbitrary subset W of vertices of G_n consisting of at least $(2-w+o(1))^n$ vertices. It is easy to see that Lemma 1 can be applied $k-3$ times to W and then Lemma 2 can be applied once to W. Thus, we have proved the following result.

Lemma 3. *Let* $k \ge 3$. *Then any subset* W *of vertices of* G_n *consisting of at least* $(2-(5\times 10^{-53})^{5^{k-3}}+o(1))^n$ *vertices contains* $k+1$ *pairwise adjacent vertices.*

This lemma implies that, for every $k \ge 3$,

$$\chi_k(\mathbb{R}^n) \ge \left(\frac{2}{2-(5\cdot 10^{-53})^{5^{k-3}}}+o(1)\right)^n = (c_k+o(1))^n. \quad (1)$$

Indeed, assume that, for some k, inequality (1) does not hold for arbitrarily large n; hence, its negation holds true. Consider a coloring of $\mathbb{R}^n$ in $(c_k+o(1))^n$ colors, whose existence is guaranteed by the negation of inequality (1). By the Dirichlet principle, among the vertices $V_n \subset \mathbb{R}^n$, there is a monochromatic subset consisting of

$$(2-(5\cdot 10^{-53})^{5^{k-3}}+o(1))^n$$

vertices. Applying Lemma 3, we find $k+1$ pairwise adjacent vertices in this subset. Thus, we find a monochromatic set of vertices in a regular simplex with the required side length, although, by the definition of $\chi_k(\mathbb{R}^n)$, this is not possible. This contradiction completes the proof of (1).

Note that

$$\frac{2}{2-(5\cdot 10^{-53})^{5^{k-3}}} = \frac{2}{2-x} = 1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+\ldots$$
$$> 1+\frac{x}{2} = 1+\frac{(5\cdot 10^{-53})^{5^{k-3}}}{2}$$
$$> 1+10^{-53\cdot 5^{k-3}} = 1+(10^{-53/125})^{5^k} > 1+\left(\frac{1}{3}\right)^{5^k} = 1+\frac{1}{3^{5^k}},$$

i.e., inequality (1) is stronger than that in Theorem 1. Thus, we have proved Theorem 1, assuming that $k \ge 3$. To complete the proof of Theorem 1, it remains to be noted that the above-mentioned results $\chi_1(\mathbb{R}^n) \ge (1.239\ldots+o(1))^n$ and $\chi_2(\mathbb{R}^n) \ge (1.052\ldots+o(1))^n$ are much stronger than the inequalities given in Theorem 1 for k equal to 1 or 2. Therefore, Theorem 1 is completely proved.

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