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MATHEMATICS

# Estimates for the Norms of Monotone Operators on Weighted Orlicz–Lorentz Classes

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**Abstract**—A monotone operator $P$ mapping the Orlicz–Lorentz class to an ideal space is considered. The Orlicz–Lorentz class is the cone of measurable functions on $R_+ = (0, \infty)$ whose decreasing rearrangements with respect to the Lebesgue measure on $R_+$ belong to the weighted Orlicz space $L_{\Phi,\nu}$. Reduction theorems are proved, which make it possible to reduce estimates of the norm of the operator $P$: $\Lambda_{\Phi,\nu} \to Y$ to those of the norm of its restriction to the cone of nonnegative step functions in $L_{\Phi,\nu}$. The application of these results to the identity operator from $\Lambda_{\Phi,\nu}$ to the weighted Lebesgue space $Y = L_1(R_+; g)$ gives exact descriptions of associated norms for $\Lambda_{\Phi,\nu}$.

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## 1. WEIGHTED ORLICZ SPACE

This section briefly describes the necessary (mostly known) general properties of weighted Orlicz spaces (see, e.g., [1–4]).

**Definition 1.** *The notation* $\Theta$ *is used for the class of functions* $\Phi : [0, \infty) \to [0, \infty]$, *such that*

$$\Phi(0) = 0;$$

$\Phi$ *increases and is left continuous on* $R_+$;

$$\Phi(+\infty) = \infty;$$

$\Phi$ *is not identically equal to zero or infinity on* $R_+$.

For $\Phi \in \Theta$, we set

$$t_0 = \sup\{t \in [0,\infty) : \Phi(t) = 0\}; \quad t_\infty = \inf\{t \in R_+ : \Phi(t) = \infty\}. \tag{1}$$

We have $t_0 \in [0,\infty); t_\infty \in (0,\infty]; t_0 \le t_\infty$,

$$\Phi(t) = 0, \quad t \in [0, t_0], \quad \Phi(t) = \infty, \quad t > t_\infty$$

(the last relation holds if $t_\infty < \infty$). Throughout this paper, we assume that

$$\Phi \in \Theta, \quad t_0 t_\infty^{-1} = 0; \quad \nu \in M, \quad \nu > 0 \text{ almost everywhere on } R_+. \tag{2}$$

Here, $M = M(R_+)$ denotes the set of Lebesgue measurable functions on $R_+$. Given $\lambda > 0$, $f \in M$, we set

$$J_\lambda(f) := \int_0^\infty \Phi(\lambda^{-1}|f(x)|)\nu(x)\,dx, \tag{3}$$

$$\|f\|_{\Phi,\nu} = \inf\{\lambda > 0: J_\lambda(f) \le 1\} < \infty, \tag{4}$$

$$L_{\Phi,\nu} = L_{\Phi,\nu}(R^+) = \{f \in M: \|f\|_{\Phi,\nu} < \infty\}.$$

**Example 1.** Suppose that condition (2) holds, $p \in (0,1]$, and $\Phi$ is $p$-convex on $[t_0, t_\infty)$, i.e., for $\alpha, \beta \in (0,1]$, $\alpha^p + \beta^p = 1$, we have

$$\Phi(\alpha t + \beta\tau) \le \alpha^p \Phi(t) + \beta^p \Phi(\tau), \quad t, \tau \in [t_0, t_\infty). \tag{5}$$

For example, the function $\Phi(t) = t^p$ is $p$-convex for $p \in (0,1]$ and 1-convex for $p > 1$.

**Example 2** (Young function). Suppose that $\Phi: [0,\infty) \to [0,\infty]$ is a Young function, i.e.,

$$\Phi(t) = \int_0^t \varphi(\tau)\,d\tau, \tag{6}$$

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$0 \le \varphi$ increases and is left continuous on $R_+, \varphi(0) = 0$; and $\varphi$ is not identically equal to zero or infinity on $R_+$.

Then $\Phi \in \Theta$ and, since $0 \le \varphi \uparrow$, $\Phi$ is 1-convex on $[t_0, t_\infty)$.

**Theorem 1.** *Suppose that conditions* (2) *holds and* $\Phi$ *is p-convex on* $[t_0, t_\infty)$ *for* $p \in (0,1]$. *Then, in* $L_{\Phi,\nu}$, *the triangle inequality holds: if* $f, g \in L_{\Phi,\nu}$, *then* $f + g \in L_{\Phi,\nu}$ *and*

$$\|f + g\|_{\Phi,\nu} \le (\|f\|_{\Phi,\nu}^p + \|g\|_{\Phi,\nu}^p)^{1/p}.$$

*The quantity* $\|f\|_{\Phi,\nu}$ *is a quasi-norm* (*a norm for* $p = 1$); *moreover, it is monotone, i.e.,*

$$f \in M, \quad |f| \le g \in L_{\Phi,\nu} \Rightarrow f \in L_{\Phi,\nu},$$
$$\|f\|_{\Phi,\nu} \le \|g\|_{\Phi,\nu}$$

*and has the Fatou property:*

$$f_n \in M, \quad 0 \le f_n \uparrow f \Rightarrow \|f\|_{\Phi,\nu} = \lim_{n\to\infty} \|f_n\|_{\Phi,\nu}.$$

**Conclusion.** Under the conditions of Theorem 1, the set $L_{\Phi,\nu}$ forms a quasi-Banach ideal space (it is Banach for $p = 1$, in particular, for a Young function $\Phi$) with Fatou property.

**Remark 1.** For $\Phi(t) = t^p, p \in R_+$, the Orlicz space $L_{\Phi,\nu}$ coincides with the weighted Lebesgue space

$$L_p(R_+;\nu) = \left\{ f \in M : \|f\|_{L_p(\nu)} = \left( \int_0^\infty |f(x)|^p \nu(x) dx \right)^{1/p} < \infty \right\}.$$

**Remark 2.** Suppose that, in (2), the condition $t_0 t_\infty^{-1} = 0$ is violated, i.e., $0 < t_0 \le t_\infty < \infty$. Then, for any function $f \in M$,

$$t_0 \|f\|_{\Phi,\nu} \le \|f\|_{L_\infty} \le t_\infty \|f\|_{\Phi,\nu},$$

so that $L_{\Phi,\nu} = L_\infty$ c and the norms on these spaces are equivalent. Here, $L_\infty = L_\infty(R_+)$is the space of essentially bounded functions.

## 2. ESTIMATES OF THE NORM OF A MONOTONE OPERATOR ON A CONE

Below we describe a discretization procedure consistent with the properties of a weight function. We assume here that the weight $\nu$ satisfies the conditions

$$0 < V(t) := \int_0^t \nu \, d\tau < \infty, \quad t \in R_+, \tag{7}$$

where $V$ strictly increases and

$$V(+\infty) = \infty. \tag{8}$$

We fix $b > 1$ and define a sequence $\{\mu_m\}$ by

$$V(\mu_m) = b^m, \quad m \in Z = \{0, \pm 1, \pm 2, \ldots\}. \tag{9}$$

We have

$$0 < \mu_m \uparrow;$$
$$\sum_m \Delta_m = R_+; \ \ \Delta_m = [\mu_m, \mu_{m+1}), \ \ m \in Z. \tag{10}$$

Consider the cone

$$\Omega \equiv \{ f \in L_{\Phi,\nu} : 0 \le f \downarrow \} \tag{11}$$

of nonnegative decreasing functions in the Orlicz space. We also associate procedure (9), (10) with the cones

$$S \equiv \left\{ f \in L_{\Phi,\nu} : f = \sum_m \gamma_m \chi_{\Delta_m}; \gamma_m \ge 0, m \in Z \right\} \tag{12}$$

of nonnegative step functions and

$$\tilde{\Omega} \equiv \Omega \cap S = \left\{ f \in L_{\varphi,\nu} : f = \sum_m \alpha_m \chi_{\Delta_m}; 0 \le \alpha_m \downarrow \right\}. \tag{13}$$

of nonnegative decreasing step functions.

Let $(\mathrm{N}, \eta)$ be a space with nonnegative complete $\sigma$-finite measure $\eta$; we use $L = L(\mathrm{N}, \eta)$ to denote the set of all $\eta$-measurable functions and set $L^+ = \{u \in L : u \ge 0\}$ and $M^+ = \{f \in M : f \ge 0\}$. Let $P : M^+ \to L^+$ be a monotone operator, i.e.,

$$f, h \in M^+, \quad f \le h \quad \mu \text{ almost everywhere}$$
$$\Rightarrow Pf \le Ph \quad \eta \text{ almost everywhere}.$$

For the cone $\mathrm{A} \subset L_{\Phi,\nu}$ of nonnegative functions and an ideal space $Y = Y(\mathrm{N}, \eta) \subset L$, we define the norm of the restriction of $P$ to this cone as

$$\|P\|_{\mathrm{A}\to Y} = \sup\{ \|Pf\|_Y : f \in \mathrm{A}, \|f\|_{\Phi,\nu} \le 1 \}. \tag{14}$$

**Theorem 2.** *Suppose that conditions* (7) *and* (8) *hold, a function* $\Phi \in \Theta$ *is p-convex on* $[t_0, t_\infty)$ *for* $p \in (0,1]$ (*see* (5)), *and the discretization procedure* (9), (10) *is performed. Then the norms of the restrictions of a monotone operator to cones* (11)–(13) *satisfy the inequalities*

$$\|P\|_{\tilde{\Omega}\to Y} \le \|P\|_{\Omega\to Y} \le b^{1/p} \|P\|_{\tilde{\Omega}\to Y}; \tag{15}$$

$$(1 - b^{-1})^{1/p} \|P\|_{S\to Y} \le \|P\|_{\Omega\to Y} \le b^{1/p} \|P\|_{S\to Y}. \tag{16}$$

**Remark 3.** Relations (16) reduce estimating the norm of the restriction of a monotone operator to the cone $\Omega$ to estimating the norm of this operator on the cone of nonnegative step functions. In some cases, such a reduction makes it possible to use known results concerning step functions (or their purely discrete analogues) for obtaining required estimates on the cone $\Omega$. This approach is realized in Section 3 for the example of a description of associated norms.

## 3. ASSOCIATED NORMS FOR THE CONE $\Omega$ IN THE ORLICZ SPACE

We keep the notation of Sections 1 and 2 and apply results of Section 2 in the special case where the ideal space $Y$ coincides with the weighted Lebesgue space $L_1(R_+;g)$, $g \in M^+$, and the monotone operator $P$ is the identity operator. In this case, the norm $\|P\|_{\Omega\to Y}$ coincides with the associated norm for the cone $\Omega$:

$$\|P\|_{\Omega\to Y} = \sup\left\{\int_0^\infty fg dt\colon f\in\Omega; \|f\|_{\varphi,\nu}\le 1\right\} \equiv \|g\|'_\Omega. \quad (17)$$

According to Theorem 2, we have

$$\|P\|_{\Omega\to Y} \cong \|P\|_{S\to Y}. \quad (18)$$

In the case under consideration,

$$\|P\|_{S\to Y} = \sup\left\{\sum_{m\in Z}\alpha_m g_m\colon \alpha_m \ge 0; \sum_{m\in Z}\Phi(\alpha_m)\beta_m \le 1\right\} \quad (19)$$

for

$$g_m = \int_{\Delta_m} g dt \ge 0;$$
$$\beta_m = \int_{\Delta_m} \nu dt = b^m(b-1), \quad m\in Z. \quad (20)$$

When $\Phi$ is a Young function (see Example 2), using known properties of discrete norms, we obtain an explicit description of norm (19) in terms of the complementary Young function $\Psi$, that is,

$$\Psi(t) = \int_0^t \psi(\tau)d\tau, \quad t\in[0,\infty];$$
$$\psi(\tau) = \inf\{\sigma\colon \varphi(\sigma)\ge\tau\}, \quad \tau\in[0,\infty]. \quad (21)$$

The discrete description thus obtained can be reduced to the integral form given below by the “anti-discretization method.”

**Theorem 3.** *Suppose that* $\Phi, \Psi$ *are complementary Young functions and conditions* (7) *and* (8) *hold. Then, for any fixed number* $a\in(0,1)$, *the following two-sided estimate of the associated norm* (17) *is valid:*

$$\|g\|'_\Omega \cong \|\rho_a(g)\|_{\Psi,\nu}$$
$$= \inf\left\{\lambda > 0\colon \int_0^\infty \Psi(\lambda^{-1}\rho_a(g;t))\nu(t)dt \le 1\right\}, \quad (22)$$

*where*

$$\rho_a(g;t) := V(t)^{-1}\int_{\delta_a(t)}^t |g(\tau)|d\tau,$$
$$\delta_a(t) := V^{-1}(aV(t)), t\in R_+. \quad (23)$$

*Norms* (22) *with different* $a\in(0,1)$ *are equivalent.*

If condition (8) is violated, the answer is somewhat different from (22).

**Theorem 4.** *Suppose that the assumptions of Theorem* 3 *hold but condition* (8) *is replaced by*

$$V(+\infty) < \infty. \quad (24)$$

*Let*

$$b = \frac{V(+\infty)}{V(1)} > 1, \quad a = b^{-2}. \quad (25)$$

*Then*

$$\|g\|'_\Omega \cong \|\rho_a(g)\chi_{(0,1)}\|_{\Psi,\nu} + \int_{V^{-1}(aV(+\infty))}^\infty |g|dt. \quad (26)$$

**Remark 4.** *Suppose that the assumptions of Theorem* 3 *hold and, in addition, the function* $\Phi$ *satisfies the* $\Delta_2$*-condition, i.e.,*

$$\exists C\in(1,\infty)\colon \Phi(2t)\le C\Phi(t), \quad t\in R_+.$$

*Then*

$$\|g\|'_\Omega \cong \left\|V(t)^{-1}\int_0^t |g(\tau)|d\tau\right\|_{\Psi,\nu}. \quad (27)$$

## 4. APPLICATIONS TO THE WEIGHTED ORLICZ–LORENTZ CLASSES

Let $f\in M(R_+)$ be such that its distribution function $\lambda_f$ is not identically equal to infinity, where

$$\lambda_f(y) = \mu\{x\in R_+\colon |f(x)| > y\}, \quad y\in R_+.$$

Let $f^*$ be the decreasing rearrangement of the function $f$, i.e.,

$$f^*(t) = \inf\{y\in R_+\colon \lambda_f(y)\le t\}, \quad t\in R_+.$$

The weighted Orlicz–Lorentz class $\Lambda_{\Phi,\nu}$ consists of functions $f\in M(R_+)$ such that $f^*\in L_{\Phi,\nu}$. It is endowed with a functional $\|f^*\|_{\Phi,\nu}$ (see (3) and (4)) taking equal values at $f \in M(R_+)$ and $|f| \in M^+(R_+)$. Let $\Lambda^+_{\Phi,\nu} = M^+ \cap \Lambda_{\Phi,\nu}$.

Suppose that $(\mathrm{N},\eta)$ is a space with a nonnegative complete $\sigma$-finite measure $\eta$ and $L = L(\mathrm{N},\eta)$ is the set of all $\eta$-measurable functions $u$: $\mathrm{N} \to R$; let $L^+ = \{u \in L: u \geq 0\}$.

**Theorem 5.** *Let $Y \subset L$ be an ideal space with quasi-norm $\|\cdot\|_Y$, and let $P$: $M^+ \to L^+$ be a monotone operator such that*

$$\exists C \in [1,\infty): \|Pf\|_Y \leq C\|Pf^*\|_Y, \quad f \in M^+(R_+). \tag{28}$$

*Then*

$$\|P\|_{\Omega\to Y} \leq \|P\|_{\Lambda^+_{\Phi,\nu}\to Y} \leq C\|P\|_{\Omega\to Y}; \tag{29}$$

*moreover, if $C = 1$ in* (28), *then the norms in* (29) *are equal.*

**Corollary.** *Under the assumptions of Theorem* 5,

$$\|P\|_{\Lambda^+_{\Phi,\nu}\to Y} \cong \|P\|_{S\to Y}. \tag{30}$$

**Example 3.** Theorem 5 covers all operators of the form

$$(Pf)(x) = \int_0^\infty k(x,\tau)f(\tau)\,d\tau, \quad x \in \mathrm{N}, \quad f \in M^+(R_+), \tag{31}$$

where $k$ is a nonnegative measurable function on $\mathrm{N} \times R_+$ such that $k(x,\tau)$ decreases and is right continuous as a function of $\tau \in R_+$. For such operators, property (28) with $C = 1$ follows from the space $Y \subset L$ being ideal and Hardy's well-known lemma

$$(Pf)(x) = \int_0^\infty k(x,\tau)f(\tau)\,d\tau \leq \int_0^\infty k(x,\tau)f^*(\tau)\,d\tau = (Pf^*)(x).$$

**Example 4.** If $Y = Y(R_+)$ is a rearrangement-invariant ideal space, then inequality (28) holds for the Hardy–Littlewood maximal operator $\mathrm{M}$: $M^+(R_+) \to M^+(R_+)$, where

$$(\mathrm{M}f)(x) = \sup\left\{|\Delta|^{-1}\int_\Delta f(\tau)\,d\tau: \Delta \subset R_+; x \in \Delta\right\}$$

(the supremum is taken over all intervals $\Delta$ containing the point $x \in R_+$). Therefore, in this case, Theorem 5 can be applied, too.

Now, suppose that the ideal space $Y$ coincides with the weighted Lebesgue space $L_1(R_+;g)$, $g \in M^+$, and the monotone operator $P$ is the identity operator. Then $\|P\|_{\Lambda^+_{\Phi,\nu}\to Y}$ coincides with the associated norm for the function $g \in M^+$ on the Orlicz–Lorentz class. It reduces to the associated norm (17) on the cone $\Omega$ for the decreasing rearrangement $g^*$:

$$\|g\|'_* := \sup\left\{\int_0^\infty fg\,dt: f \in M^+; \|f^*\|_{\Phi,\nu} \leq 1\right\} = \|g^*\|'_\Omega.$$

Thus, Theorems 3 and 4 imply the following result.

**Theorem 6.** *If the assumptions of Theorem* 3 *hold, then*

$$\|g\|'_* \cong \|\rho_a(g^*)\|_{\Psi,\nu} = \inf\left\{\lambda > 0: \int_0^\infty \Psi(\lambda^{-1}\rho_a(g^*;t))\nu(t)\,dt \leq 1\right\}, \tag{32}$$

*where $\rho_a$ is defined by* (23). *Under the assumptions of Theorem* 4,

$$\|g\|'_* \cong \|\rho_a(g^*)\chi_{(0,1)}\|_{\Psi,\nu} + \int_{V^{-1}(aV(+\infty))}^\infty g^*\,dt. \tag{33}$$

**Remark 5.** Suppose that the assumptions of Theorem 3 and the function $\Phi$ in this theorem satisfies the $\Delta_2$-condition. Then

$$\|g\|'_* \cong \left\|V(t)^{-1}\int_0^t g^*(\tau)\,d\tau\right\|_{\Psi,\nu}. \tag{34}$$

**Remark 6.** Relations (32)–(34) are modifications of results of paper [8], which develop previous results of [9] (in [9], it was assumed that both functions $\Phi$, $\Psi$ satisfy the $\Delta_2$-condition). Duality for Orlicz, Lorentz, and Orlicz–Lorentz spaces was studied in [2, 4, 5, 10–13].


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