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MATHEMATICS

On Coincidences of Families of Mappings on Ordered Sets

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Abstract—New results on fixed points and coincidences of families of set-valued mappings of partially ordered sets obtained without commutativity assumptions are presented. These results develop theorems on fixed points of an isotone self-mapping of an ordered set (for families of set-valued mappings) and theorems about coincidences of two set-valued mappings one of which is isotone and the other is covering (for finite families of set-valued mappings).

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The theory of fixed points and coincidences of mappings of ordered sets goes back to the well-known Knaster–Tarski theorem [1]. There exist various versions and generalizations of this theorem. It is extensively used in applications. Its well-known generalization to set-valued mappings was proposed in 1971 by Smithson [2]. The problem of coincidences of two mappings of ordered sets was first considered in [3–6]. In [3, 4], it was studied for two single-valued mappings, and in [5, 6], the obtained results were extended to the case of two set-valued mappings.

As is known, results of the theory of fixed points and coincidences in ordered sets can be applied to a similar theory in metric spaces. Given a metric space (X,ρ), we can specify a partial order on $X\times\mathbb{R}_+$ determined by the metric ρ using a method proposed in [7, 8]. Namely, for any $x,y\in X$ and $r_1,r_2\in\mathbb{R}_+$, we set $(x,r_1)\preceq(y,r_2)\Leftrightarrow\rho(x,y)\le r_1-r_2$. There exist modifications of this method. Such a passage from a metric space to an ordered set makes it possible to derive theorems on fixed points and coincidences in a metric space from the corresponding results in an ordered set. For example, in [1], such a derivation of Nadler's theorem about fixed points of a set-valued mapping [9] from Smithson's theorem [2] was presented. In [5, 6], a similar reduction was performed for results about coincidences of two mappings of ordered sets.

In this paper, we present theorems on common fixed points of an (infinite) family of set-valued mappings of an ordered set and new theorems on coincidences of a finite set of n ($n\ge 2$) set-valued mappings of ordered sets, which generalize Smithson's theorem [2] and results of [3–6].

By $\rightrightarrows$ we denote a set-valued mapping. Suppose that $(X,\preceq)$ is an ordered set, $A\neq\varnothing$, and $\mathcal{F}=\{F_\alpha\}_{\alpha\in A}$ is a family of set-valued mappings $F_\alpha: X\rightrightarrows X$, $\alpha\in A$. Following [3–6], for each element $x\in X$, we set $\mathbb{O}_X(x)=\{x'\in X|x'\preceq x\}$. The set of $\mathcal{F}$-values at a point $x\in X$ is defined as the set $\{y_\alpha\}_{\alpha\in A}\subseteq X$, where $y_\alpha\in F_\alpha(x)$, $\alpha\in A$. The set of common fixed points of a family $\mathcal{F}$ is denoted by $\mathrm{Comfix}(\mathcal{F}) := \{x\in X|x\in\bigcap_{\alpha\in A}F_\alpha(x)\}$. A family $\mathcal{F}$ is said to be concordantly isotone if, given any $x\in X$, any set $\{y_\alpha\}_{\alpha\in A}$ of $\mathcal{F}$-values at x, and any $x'\in X$, $x'\prec x$, there exists a set $\{z_\alpha\}_{\alpha\in A}$ of $\mathcal{F}$-values at x' such that $z_\alpha\preceq y_\beta$, $\forall\alpha,\beta\in A$.

Let $x_0\in X$. We use $\mathscr{C}_1(x_0;\mathcal{F})$ to denote the set of all pairs of the form (S,f), where $S\subseteq\mathbb{O}_X(x_0)$ is a chain in X and f is a special $\mathcal{F}$-selector on S, i.e., $f=\{f_\alpha\}_{\alpha\in A}$, $f_\alpha: S\to X$, $f_\alpha(x)\in F_\alpha(x)$, and $x\succeq f_\alpha(x)$ $\forall x\in S$, and $v\prec u\Rightarrow v\preceq f_\alpha(u)$, $\alpha\in A$, $\forall u,v\in S$. The following theorem is valid.

Theorem 1. *Suppose that, for an ordered set $(X,\preceq)$ and a family $\mathcal{F}=\{F_\alpha\}_{\alpha\in A}$ of set-valued mappings on X, the following conditions hold:*

(i) *the family $\mathcal{F}$ is concordantly isotone*;

(ii) *for some $x_0\in X$, there exists a set $\{y_\alpha^0\}_{\alpha\in A}$ of $\mathcal{F}$-values at x_0, such that $x_0\succeq y_\alpha^0$ $\forall\alpha\in A$*;

(iii) *for any pair $(S,f)\in\mathscr{C}_1(x_0;\mathcal{F})$, there exists a common lower bound $w\in X$ for the chains $f_\alpha(S)$*

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$\forall\alpha \in A$ *and a set* $\{w_\alpha\}_{\alpha\in A}$ *of* $\mathcal{F}$*-values at the point* w *such that* $w \succeq w_\alpha$ $\forall\alpha \in A$.

In the case where the family $\mathrm{Comfix}(\mathcal{F})$ consists of one set-valued mapping, Theorem 1 generalizes Smithson's theorem [2] and coincides with a special case of Theorem 1 in [5] (see also [6]), where the covering mapping is the identity. In [5] (see also [6]), it was mentioned that, under the conditions of Theorem 1 of [5], the set of coincidences may contain no minimal elements, but it was not mentioned that, in this special case, the fixed point set of an isotone mapping always contains a minimal element.

Suppose that a set A is endowed with a linear order $\preceq_1$. The set $\{y_\alpha\}_{\alpha\in A}$ of $\mathcal{F}$-values at a point $x \in X$ is called a nonincreasing chain of $\mathcal{F}$-values (with respect to the order $\preceq_1$) at the point x if $\alpha \preceq_1 \beta \Rightarrow y_\beta \preceq y_\alpha$ $\forall\alpha,\beta \in A$. We say that a family $\mathcal{F} = \{F_\alpha\}_{\alpha\in A}$ of set-valued mappings is concordantly chain-isotone on X if, for any $x \in X$, any nonincreasing chain $\{y_\alpha\}_{\alpha\in A}$ of $\mathcal{F}$-values at x, and any $x' \in X$, $x' \prec x$, there exists a nonincreasing chain $\{z_\alpha\}_{\alpha\in A}$ of $\mathcal{F}$-values at x' such that $z_\alpha \preceq y_\beta$ $\forall\alpha,\beta \in A$.

We refer to a special $\mathcal{F}$-selector $f = \{f_\alpha\}_{\alpha\in A}, f_\alpha : S \to X$, $\alpha \in A$, on a chain $S \subseteq X$ as a special chain $\mathcal{F}$-selector if $\forall x \in S$ $\{f_\alpha(x)\}_{\alpha\in A}$ is a nonincreasing chain of $\mathcal{F}$-values at x.

Take a point $x_0 \in X$ and let $\mathcal{C}_2(x_0;\mathcal{F})$ denote the set of all pairs of the form (S, f), where $S \subseteq \mathbb{O}_X(x_0)$ is a chain, $f = \{f_\alpha\}_{\alpha\in A}$ is a special chain $\mathcal{F}$-selector on S, and, in addition, the chain S satisfies the following condition:

$$\begin{gathered}\forall x \in S \quad \exists x' \in \mathbb{O}_X(x_0),\\ \exists\{y_\alpha\}_{\alpha\in A} \text{ is a nonincreasing chain}\\ \text{of } \mathcal{F}\text{-values at } x'\\ \text{such that } x = \inf\{y_\alpha\}_{\alpha\in A}.\end{gathered} \tag{$*$}$$

Theorem 2. *Suppose given a partially ordered set* $(X,\preceq)$, *a linearly ordered set* $(A,\preceq_1)$, *and a family* $\mathcal{F} = \{F_\alpha\}_{\alpha\in A}$, F_α: $X \rightrightarrows X$; *suppose also that the following conditions hold:*

(i) *the family* $\mathcal{F}$ is concordantly chain-isotone;

(ii) *for each point* $x \in X$, *any nonincreasing chain of* $\mathcal{F}$*-values at x has an infimum*;

(iii) *for some* $x_0 \in X$, *there exists a nonincreasing chain* $\{y^0_\alpha\}_{\alpha\in A}$ *of* $\mathcal{F}$*-values at* x_0, *such that* $x_0 \succeq y^0_\alpha$ $\forall\alpha \in A$;

(iv) *for any pair* $(S, f) \in \mathcal{C}_2(x_0;\mathcal{F})$, *where* $f = \{f_\alpha\}_{\alpha\in A}$, *there exists a common lower bound* $w \in X$ *of all chains* $f_\alpha(S), \alpha \in A$, *and a nonincreasing chain* $\{w_\alpha\}_{\alpha\in A}$ *of* $\mathcal{F}$*-values at w such that* $w \succeq w_\alpha, \alpha \in A$.

Then the set $\mathrm{Comfix}(\mathcal{F}) \neq \varnothing$ *contains a minimal element.*

In the special case where $A = \{1, 2, \dots, n\}$ and $\preceq_1$ is the usual order on the set $\mathbb{N}$ of positive integers, any nonincreasing chain of the form $x_n \preceq \dots \preceq x_1$ automatically has the infimum X in x_n; therefore, condition (ii) in Theorem 2 is automatically satisfied. Condition ($*$) for each pair $(S, f) \in \mathcal{C}_2(x_0;\mathcal{F})$, takes the form

$$\begin{gathered}\forall x \in S \quad \exists x' \in \mathbb{O}_X(x_0), \quad \exists\{y_i\}_{1\le i\le n},\\ y_n \preceq \dots \preceq y_1, \text{ is a nonincreasing chain}\\ \text{of } \mathcal{F}\text{-values at } x' \text{ such that } x = y_n.\end{gathered} \tag{$**$}$$

Examples show that Theorems 1 and 2 do not follow from each other. Mappings in a concordant chain-isotone family consisting of at least two mappings may be nonisotone. In the case $n = 1$, Theorem 2 coincides with Theorem 1 of [5, 6] under the assumption that the ordered covering mapping is the identity.

Now, consider the question about the existence of coincidences for a set of n set-valued mappings ($n \ge 2$). Suppose given ordered sets $(X, \preceq)$ and $(Y, \preceq)$ and a set $\mathcal{F} = \{F_1, \dots, F_n\}$ of set-valued mappings $F_i : X \rightrightarrows Y$, $i = 1, 2, \dots, n$, $n \ge 2$. Take a point $x_0 \in X$.

We say that mappings $F_1, \dots, F_{n-1}$ concordantly cover the mappings $F_2, \dots, F_n$ on $\mathbb{O}_X(x_0)$ if, for any $x \in \mathbb{O}_X(x_0)$ and any nonincreasing chain $\{y_i\}_{1\le i\le n}$ of $\mathcal{F}$-values at x, i.e., $y_1 \succeq y_2 \dots \succeq y_n$, $y_i \in F_i(x)$, there exists an $x' \in \mathbb{O}_X(x_0)$, $x' \preceq x$, such that $y_{i+1} \in F_i(x')$, $i = 1, 2, \dots, n-1$.

Remark 1. It follows from the definitions of concordantly covering mappings and a set-valued mapping $F : X \rightrightarrows Y$ orderly covering a set $Q \subset Y$ [5, Definition 1] that if $n = 2$ and a mapping F_1 orderly covers $F_2(\mathbb{O}_X(x_0))$, then F_1 concordantly covers F_2. The converse is not generally true, because, in the definition of concordantly covering mappings for $n = 2$, given any $x \in \mathbb{O}_X(x_0)$ and any $y_1 \succeq y_2$, $y_i \in F_i(x)$, $i = 1, 2$, the existence of a point $x' \preceq x$ such that $F_1(x') = y_2$ is required only for points $y_2 \in F_2(x)$, while in the definition of orderly covering mappings, for a map F_1 to orderly cover $F_2(\mathbb{O}_X(x_0))$, the existence of such points x' is required for any point $y_2 \in F_2(\mathbb{O}_X(x_0))$, $y_2 \preceq y_1$.

Let $\mathcal{C}_3(x_0;\mathcal{F})$ denote the set of all pairs of the form (S, f), where $S \subseteq \mathbb{O}_X(x_0)$ is a chain, $f = \{f_i\}_{1\le i\le n}, f_i : S \to Y$,

$$f_i(x) \in \left(\bigcap_{j=i+1}^{n} F_j(\mathbb{O}_X(x_0))\right) \bigcap F_i(x),$$

$i = 1, 2, \dots, n-1$, $f_n(x) \in F_n(x)$, $f_1(x) \succeq \dots \succeq f_n(x)$ for each $x \in S$, and, moreover, if $x, z \in S$ and $x \prec z$, then $f_1(x) \preceq f_n(z)$.

Theorem 3. *Suppose given ordered sets* $(X, \preceq)$ *and* $(Y, \preceq)$ *and a set* $\mathcal{F} = \{F_1, \dots, F_n\}$ *of set-valued mappings* $F_i : X \rightrightarrows Y$, $i = 1, 2, \dots, n$, *satisfying the following conditions:*

(i) *the mappings* $F_1, \dots, F_{n-1}$ *concordantly cover the mappings* $F_2, \dots, F_n$ *on* $\mathbb{O}_X(x_0)$;

(ii) *for some point* $x_0 \in X$, *there exists a nonincreasing chain* $y_0 = \{y_{0,j}\}_{1\le j\le n} \subseteq Y$ *of* $\mathcal{F}$*-values at* x_0, $y_{0,1} \succeq \dots \succeq y_{0,n}$;

(iii) *the mapping* F_n *is isotone*;

(iv) *for each pair* $(S, f) \in \mathcal{C}_3(x_0; \mathcal{F})$, *the chain* S *has a lower bound* $w \in X$, *there exists a nonincreasing chain* $z_0 = \{z_{0,j}\}_{1\le j\le n} \subseteq Y$ *of* $\mathcal{F}$*-values at* w, $z_{0,1} \succeq \dots \succeq z_{0,n}$, *and each* $z_{0,j}$ *is a lower bound of* $\{f_j(x) | x \in S\}$, $j = 1, 2, \dots, n$.

Then the coincidence set $\mathrm{Coin}(F_1, \dots, F_n) := \{x \in X | \bigcap_{i=1}^{n} F_i(x) \neq \varnothing\}$ *of the mappings* $F_1, \dots, F_n$ *is nonempty.*

By virtue of Remark 1, for $n = 2$, Theorem 3 strengthens Theorem 2 of [5].

There arises the important question on sufficient conditions for the existence of minimal and least elements in coincidence sets. Consider the following additional conditions.

Condition A. For any $u, v \in \mathbb{O}_X(x_0)$, $u \preceq v$, and any $y \in F_n(u)$, $z \in F_n(v)$, the set $F_n(u) \cap \Omega_Y(y, z)$, where $\Omega_Y(y, z) := \mathbb{O}_Y(y) \cap \mathbb{O}_Y(z)$, is nonempty.

Condition B. For $\forall x_1, x_2 \in \mathbb{O}_X(x_0)$, the set $\Omega_X(x_1, x_2)$ is nonempty, and there exists a point $\omega(x_1, x_2) \in \Omega_X(x_1, x_2)$ such that $\forall v_1 \in F_n(x_1)$, $\forall v_2 \in F_n(x_2)$, $\exists v \in F_n(\omega(x_1, x_2)) \cap \Omega_Y(v_1, v_2)$, and, for any sets $\{y_k\}_{k=1,\dots,n-1}$ and $\{u_k\}_{k=1,\dots,n-1}$ of values (at the points x_1 and, x_2, respectively) such that $y_{n-1} \preceq \dots \preceq y_1, y_k \in F_k(x_1)$ and $u_{n-1} \preceq \dots \preceq u_1$, $u_k \in F_k(x_2)$, $k = 1, 2, \dots, n-1$, and any set $\{v_k\}_{k=1,\dots,n-1}$, $v_k \in \Omega_Y(y_k, u_k)$, $k = 1, 2, \dots, n-1$, $v_1 \succeq \dots \succeq v_{n-1}$, there exists a point $x, x \preceq \omega(x_1, x_2)$ such that $v_k \in F_k(x)$, $k = 1, 2, \dots, n-1$.

Theorem 4. *Suppose that all conditions of Theorem* 3 *and the additional Condition* A *hold.*

Then the set $\mathrm{Coin}(F_1, \dots, F_n)$ *is nonempty, and it contains a minimal element.*

Theorem 5. *Suppose that all conditions of Theorem* 3 *and the additional Condition B hold.*

Then the set $\mathrm{Coin}(F_1, \dots, F_n) \cap \mathbb{O}_X(x_0)$ *is nonempty and contains a minimal element.*

Note that, in the case $n = 2$, Theorems 4 and 5 imply, respectively, Theorems 2 and 3 of [5] (see also [6]).

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