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## MATHEMATICS

# On Extensions of Some Block Designs

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**Abstract**—There are some results concerning $t$-designs in which the number of points in the intersection of two blocks takes less than $t$ values. For example, if $t = 2$, then the design is symmetric (in such a design, $v = b$ or, equivalently, $k = r$). In 1974, B. Gross described $t$-($v$, $k$, $l$) designs that, for some integer $s$, $0 < s < t$, do not contain two blocks intersecting at exactly s points. Below, it is proved that potentially infinite series of designs from the claim of Gross' theorem are finite. Gross' theorem is substantially sharpened.

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An incidence system ($X$, **B**), where $X$ is a set of points and **B** is a set of blocks, is called a $t$-($v$, $k$, $\lambda$) design if $|X| = v$, each block contains precisely $k$ points, and any $t$ points lie in precisely $\lambda$ blocks. Given a $t$-($v$, $k$, $\lambda$) design $\mathbf{D} = (X, \mathbf{B})$, its derivative design $\mathbf{D}_x$ at a point $x \in X$ is one with the set of points $X_x = X - \{x\}$ and the set of blocks $\mathbf{B}_x = \{B - \{x\} \mid x \in B \in \mathbf{B}\}$. A design **E** is said to be an extension of a design **D** if the derivative design of **E** at some point is isomorphic to **D**.

Let $b$ be the number of blocks and $r$ be the number of blocks containing a given point. For any 2-($v$, $k$, $\lambda$) design, it is true that $vr = bk$ and $(v - 1)\lambda = r(k - 1)$. A design is called symmetric if $b = v$. A 2-design is called quasi-symmetric with intersection numbers $x$, $y$ (assuming that $x < y$) if, for any two blocks $B$, $C \in \mathbf{B}$, we have $|B \cap C| \in \{x, y\}$. The block graph of a quasi-symmetric design ($X$, **B**) has the blocks of the latter as vertices, and two blocks $B$, $C \in \mathbf{B}$ are adjacent if $|B \cap C| = y$.

A $t$-design with $\lambda = 1$ is called a Steiner system. Below is a well-known conjecture concerning Steiner systems.

**Conjecture**. *Let a $t$-($v$, $k$, $\lambda$) design exists and $\lambda = 1$. Then $t \le 5$, and, if $t = 5$, then we have one of two Witt designs, namely, with $v = 12$, $k = 6$ or $v = 24$, $k = 8$.*

This conjecture is folklore, and an obstacle to its publication was the following 1974 result of B. Gross [1, Theorem 1.55].

**Proposition**. *Let* **D** *be a $t$-($v$, $k$, 1) design and s be an integer such that $0 \le s < t$. If the design does not have two blocks intersecting at exactly s points, then one of the following assertions holds*:

(1) $s = t - 2$, *and* **D** *is a projective plane or its extension*.

(2) $s = 0$, $k = t + 1$, $v = 2t + 3$.

(3) $s = 1$, $k = t + 1$, $v = 2t + 2$.

(4) $s = 0, 2$, $t = 4$, $k = 7$, *and* $v = 23$.

(5) $s = 1, 3$, $t = 5$, $k = 8$, *and* $v = 24$.

The projective plane of order $q$ is defined as a 2-($q^2 + q + 1$, $q + 1$, 1) design. The existence of projective planes is known only when $q$ is a prime power. D. Hughes proved in 1961 [1, Proposition 1.34] that a projective plane can be extended only for $q = 2, 4, 10$. Later, it was proved that there is no projective plane of order 10 [1, p. 9].

Below, we prove that the infinite series of designs from items (2) and (3) exist only for small $t$.

**Theorem**. *Let* **D** *be a $t$-($2t + 2$, $t + 1$, 1) design. Then there are no two blocks in the design intersecting at exactly one point, the complement of any block is a block, and $t = 3, 5$.*

**Remark**. By the extension theorem in [2], any $t$-($2t + 3$, $t + 1$, 1) design is uniquely extended to a ($t + 1$)-($2t + 4$, $t + 2$, 1) design in which the complement of any block is a block. Clearly, in the extended design, there are no two blocks intersecting at a single point. Therefore, the initial $t$-($2t + 3$, $t + 1$, 1) design does not have two disjoint blocks and $t = 2, 4$.

**Corollary**. *Let* **D** *be a $t$-($v$, $k$, 1) design and s be an integer such that $0 \le s < t$. If the design does not have two blocks intersecting at exactly s points, then one of the following assertions holds*:

(1) $s = t - 2$, *and* **D** *is a projective plane or its extension*.

(2) $s = 0$, $k = t + 1$, $v = 2t + 3$, *and* $t = 4$.

(3) $s = 1$, $k = t + 1$, $v = 2t + 2$, *and* $t = 5$.

(4) $s = 0, 2$, $t = 4$, $k = 7$, *and* $v = 23$.

(5) $s = 1, 3$, $t = 5$, $k = 8$, *and* $v = 24$.

First, we state some auxiliary results.

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**Lemma 1** [1, Proposition 1.4]. *Any t-(v, k, λ) design is an s-design for any* $0 \le s \le t$ *and*

$$\lambda_s \binom{k-s}{t-s} = \lambda \binom{v-s}{t-s}.$$

**Lemma 2** [3, Proposition 0]. *Let* **B** *be the family of k-subsets of a v-set X,* $|\mathbf{B}| = b$, *and the maximum number of points in the intersection of two blocks be equal to d. Then, for any integer i such that* $1 \le i \le k - d$, *we have the* $\beta(i)$*-bound*

$$\binom{v}{d+2i-1} \ge b \sum_{0 \le j < i} \binom{k}{d+2i-1-j}\binom{v}{j},$$

*which holds as an equality if and only if, for any* $(d + 2i - 1)$*-subset Y of X, there is* $B \in$ *B with* $|Y \cap B| \ge d + i$.

Note that, for the Steiner $t$-$(v, k, 1)$-system, an equality in $\beta(1)$ holds with $d = t - 1$. Hauck [4] proved that, if equalities in $\beta(1)$ and $\beta(2)$ hold under the conditions of Lemma 2, then $(X, \mathbf{B})$ is the 5-(24, 8, 1) design or a $t$-$(2t + 2, t + 1, 1)$ design. The theorem implies that an equality in $\beta(1)$ and $\beta(2)$ holds only for three designs: the Hadamard 3-(8, 4, 1) one and two Witt designs, namely, 5-(12, 6, 1) and 5-(24, 8, 1) ones.

Let us prove the theorem.

**Lemma 3.** *Let* $\mathbf{D} = (X, \mathbf{B})$ *be a* $t$-$(2t + 3, t + 1, 1)$ *design. Then the following assertions hold*:

(1) *t is an even number that is not divided by* 3, *and* $t + 3$ *is a prime number.*

(2) *If S is a* $(t - 2)$*-subset of X, then* $\mathbf{D}_S$ *is a quasi-symmetric* $2$-$(t + 5, 3, 1)$ *design and its block graph* Γ *is strongly regular with parameters* $((t + 5)(t + 4)/6, 3(t + 2)/2, (t + 8)/2, 9)$.

(3) *If B is a block and* $\mathbf{B}_i$ *is the set of blocks intersecting B at exactly i points, then any* $(t - 1)$*-subset of* $X - B$ *lies in three blocks from* $\mathbf{B}_1$ *and in* $(t - 2)/2$ *blocks from* $\mathbf{B}_2$, *specifically,* $|\mathbf{B}_2| = \binom{t+2}{3}(t-2)/2$.

**Proof**. By Lemma 1, we have $\lambda_{t-1} = (t + 2)/2$ and $\lambda_{t-2} = (t + 5)(t + 4)/6$, so $t$ is an even number that is not divided by 3. Theorem 1 from [5] implies that $t + 3$ is a prime number.

Assume that $t \ge 8$. Consider a $(t - 2)$-subset $S$ of $X$. Let $\mathbf{D}_S = (X_S, \mathbf{B}_S)$, where $X_S = X - S$ and $\mathbf{B}_S$ is the set of all blocks containing $S$. Then $\mathbf{D}_S$ is a quasi-symmetric 2-$(t + 5, 3, 1)$ design with a strongly regular block graph Γ. By Theorem 5.3 from [1], Γ has the eigenvalues $3(t + 2)/2$, $(t - 4)/2$, and $-3$. Therefore, Γ has the parameters $((t + 5)(t + 4)/6, 3(t + 2)/2, (t + 8)/2, 9)$.

Let $B$ be a block, $S$ be a $(t - 1)$-subset of $X - B$, and $C$ be a block containing $S$. Since $s = 0$, we conclude that $C$ intersects $B$ at a single point and the number of blocks intersecting $B$ at a single point is $\binom{t+2}{t} = \binom{t+2}{2}$. Therefore, a point from $B$ lies in $(t + 2)/2$ blocks intersecting $B$ at a single point.

Let $\mathbf{B}_i$ be the set of blocks intersecting $B$ at exactly $i$ points. Then any $(t - 1)$-subset of $X - B$ lies in three blocks from $\mathbf{B}_1$ and in $(t - 2)/2$ blocks from $\mathbf{B}_2$. From this, $|\mathbf{B}_2| = \binom{t+2}{3}(t-2)/2$.

**Lemma 4.** *Let the conditions of the theorem be satisfied. Then the following assertions hold*:

(1) *The complement of a block is a block*, *and* $t + 2$ *is a prime number.*

(2) *For this design, an equality holds in the bound* $\beta(2)$ *in Lemma* 2; *specifically, any* $(t + 2)$*-subset of X contains a unique block*.

(3) *If B is a block and* $\mathbf{D}_2$ *is the set of blocks intersecting B at* $t - 1$ *points, then* $(B, \mathbf{D}_2)$ *is a* $2$-$(t + 1, t - 1, t(t^2 - 1)/4)$ *design*.

**Proof**. Let the conditions of Theorem 1 hold. By Lemma 3, $t + 2$ is a prime number. By Lemma 1, $\lambda_{t-1} = (t + 3)/2$ and $\lambda_{t-2} = (t + 4)(t + 3)/6$.

The remark implies that the complement of any block is a block.

By Lemma 2, we have $\binom{v}{d+3} \ge b\left(\binom{k}{d+3} + \binom{k}{d+2}(v-k)\right)$, where $d$ is the maximum number of points in the intersection of two blocks. Since $b = \binom{v}{l}\Big/\binom{k}{l}$ and $d = t - 1$, we obtain an equality in $\beta(2)$ if $v = 2t + 2$ and $k = t + 1$. By Lemma 2, any $(t + 2)$-subset of $X$ contains a block, which is obviously unique.

Let $B$ be a block, $C = X - B$, $\Lambda_2$ be the number of blocks intersecting $B$ in a given 2-subset, and $\mu_2$ be the number of blocks of $\mathbf{D}_2$ containing this 2-subset of $B$. Then any $(t - 1)$-subset S of $X - B$ lies in the block $C$ and in $(t + 1)/2$ blocks $D_1, \ldots, D_{(t+1)/2}$ intersecting $B$ at two points; moreover, $\{B \cap D_i \mid i = 1, 2, \ldots, (i + 1)/2\}$ forms a partition of $B$. Furthermore, any block intersecting $C$ in $C - S$ is obtained by deleting $D_i$ from $B$, and $\Lambda_2 = (t + 1)/2$.

Fix a 2-subset $R$ of $B$. Then, for any 2-subset $T$ of $B - R$, the complement of any block intersecting $B$ in $R$ belongs to $\mathbf{D}_2$. Therefore, $\mu_2 = \binom{t-1}{2}(t+1)/2$ and $(B, \mathbf{D}_2)$ is a 2-$(t + 1, t - 1, t(t^2 - 1)/4)$ design. The lemma is proved.

Let us complete the proof of the theorem. Any point of the design $(B, \mathbf{D}_2)$ lies in $r$ blocks, where $r(t -$

$2) = t^2(t^2 - 1)$. Since $t$ is an odd number, $t - 2$ divides $t + 1$ and $t = 3, 5$. The theorem is proved.

The corollary follows from our theorem in view of the remark and the proposition.

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