ISSN 1064-5624, Doklady Mathematics, 2016, Vol. 94, No. 2, pp. 594–597. © Pleiades Publishing, Ltd., 2016.
Original Russian Text © S.V. Sokolov, 2016, published in Doklady Akademii Nauk, 2016, Vol. 470, No. 4, pp. 393–396.

MATHEMATICAL PHYSICS

On The Problem of Falling Motion of a Circular Cylinder and a Vortex Pair in a Perfect Fluid[1]

S. V. Sokolov

Presented by Academician of the RAS V.V. Kozlov January 11, 2016

Received April 25, 2016

Abstract—We consider a system consisting of a heavy circular cylinder in the field of gravity interacting dynamically with a vortex pair in a perfect fluid. The circulation about the cylinder is assumed to be zero. It is shown that, unlike the famous Föppl configuration, the vortices cannot be in a relative equilibrium. An asymptotic system and a suitable regularization are considered.

DOI: 10.1134/S1064562416050173

1. INTRODUCTION AND SETTING OF THE PROBLEM

The study of the dynamics of a heavy body immersed in a perfect fluid containing vortical structures is a fundamental problem in modern mathematical physics [1–9].

Föppl's well-known solution is organized as follows [10]: Consider the flow over a circular cylinder. The flow is symmetric about the line passing through the cylinder's center parallel to the flow direction. In the flow, there are two point vortices that are at rest relative to the cylinder. Their circulations are of the same magnitude and opposite in sign.

Consider the fall of a rigid body, a circular cylinder in our case, in an unbounded volume of perfect fluid. The flow is plane parallel, with the fluid at infinity being at rest, and orthogonal to the cylinder's generator. There are two straight vortex filaments with circulations Γ_1 and Γ_2 parallel to the cylinder's generator.

In this paper we extend the analysis due to Föppl by adding gravity. Given that the gravity force is directed downward, the vortices are assumed to be located symmetrically about the vertical through the cylinder's center. Their circulations satisfy the equation $\Gamma_1 = -\Gamma_2$, and the circulation is $\Gamma = 0$. Thus, the system is essentially two-dimensional.

[1] The article was translated by the author.

Blagonravov Institute of Machines Science, Russian Academy of Sciences, Moscow, 101990 Russia
e-mail: sokolovsv72@mail.ru

2. GOVERNING EQUATIONS AND INTEGRALS OF MOTION

According to [5], the equations of motion for a cylinder and N point vortices in the presence of gravity can be written as

$$\dot{\mathbf{r}}_i = -\mathbf{v} + \operatorname{grad}\varphi_i(\mathbf{r})\big|_{\mathbf{r}=\mathbf{r}_i}, \quad \dot{\mathbf{r}}_c = \mathbf{v}, \tag{1}$$

$$a\dot{v}_1 = \lambda v_2 - \sum_{i=1}^{N} \lambda_i(\dot{y}_i^* - \dot{y}_i),$$

$$a\dot{v}_2 = -\lambda v_1 + \sum_{i=1}^{N} \lambda_i(\dot{x}_i^* - \dot{x}_i) - ag.$$

Here, $\mathbf{r}_c = (x_c, y_c)$ is the radius vector from the origin of a laboratory frame of reference Oxy to the cylinder's center, $\mathbf{v} = (v_1, v_2)$ is the velocity of the cylinder, $\mathbf{r}_i = (x_i, y_i)$ is the radius vector from the center of the cylinder to the ith vortex, $\mathbf{r}_i^* = (x_i^*, y_i^*) = \dfrac{R^2\mathbf{r}_i}{r_i^2}$ is the radius vector from the center of the cylinder to the image of the ith vortex, R is the radius of the cylinder, the constant a is the mass plus the added mass of the cylinder, ag is the gravity force, $\lambda = \dfrac{\Gamma}{2\pi}$, and $\lambda_i = \dfrac{\Gamma_i}{2\pi}$. The density of the fluid is assumed to be 2π. The function $\varphi_i(\mathbf{r})$ represents that portion of the velocity potential $\varphi(\mathbf{r})$ of the fluid that does not have a singularity at the point $\mathbf{r} = \mathbf{r}_i$:

$$\varphi(\mathbf{r}) = -\frac{R^2}{r^2}(\mathbf{r}, \mathbf{v}) - \lambda \arctan\frac{y}{x}$$
$$+ \sum_{i=1}^{N} \lambda_i \left(\arctan\left(\frac{y - y_i^*}{x - x_i^*}\right) - \arctan\left(\frac{y - y_i}{x - x_i}\right) \right).$$

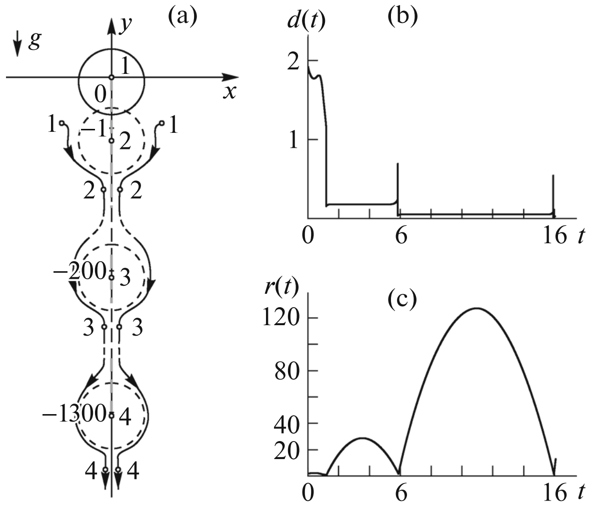

Fig. 1. Circular cylinder and a pair of point vortices in the gravity field. (a) Repeatedly scattered, the pair moves ahead of the cylinder; the first three acts of scattering; (b) the vortex to vortex distance $|x_1 - x_2|$; and (c) the distance from the vortices to the cylinder, $(x_1^2 + y_1^2)^{(1/2)}$.

System (1), which governs the motion of the cylinder and the vortices in the gravity field, preserves the standard measure and can be represented in Hamiltonian form with the Hamiltonian function

$$H = \frac{1}{2}av^2 + \frac{1}{2}\sum_{i=1}^{N}\left(\lambda_i^2 \ln(r_i^2 - R^2) - \lambda_i\lambda \ln r_i^2\right) + \frac{1}{2}\sum_{i<j}\lambda_i\lambda_j \ln\frac{R^4 - 2R^2(\mathbf{r}_i, \mathbf{r}_j) + r_i^2 r_j^2}{|\mathbf{r}_i - \mathbf{r}_j|^2} + agy_c.$$

The nonzero components of the tensor of the Poisson structure read

$$\{v_1, x_i\} = \frac{1}{a}\frac{r_i^4 - R^2(x_i^2 - y_i^2)}{r_i^4},$$

$$\{v_1, y_i\} = \{v_2, x_i\} = -\frac{1}{a}\frac{2R^2 x_i y_i}{r_i^4},$$

$$\{v_2, y_i\} = \frac{1}{a}\frac{r_i^4 + R^2(x_i^2 - y_i^2)}{r_i^4},$$

$$\{v_1, v_2\} = \frac{\lambda}{a^2} - \sum_{i=1}^{N}\frac{\lambda_i}{a^2}\frac{r_i^4 - R^4}{r_i^4},$$

$$\{x_i, y_i\} = -\frac{1}{\lambda_i}, \quad \{x_c, v_1\} = \{y_c, v_2\} = \frac{1}{a}.$$

The system has two integrals of motion: an autonomous integral P, namely, the projection of the system's linear momentum onto the horizontal axis, and a nonautonomous Q, the projection of the system's linear momentum onto the vertical axis:

$$Q = a(v_2 + gt) + \lambda x_c - \sum_{i=1}^{N}\lambda_i(x_i^* - x_i),$$

$$P = av_1 - \lambda y_c + \sum_{i=1}^{N}\lambda_i(y_i^* - y_i).$$

3. FÖPPL'S SYSTEM IN THE GRAVITY FIELD

The motion of the cylinder and the vortex pair is governed by Eqs. (1) restricted, for symmetry reasons, to the following manifold:

$$\lambda = 0, \quad x_2 = -x_1, \quad y_2 = y_1, \quad \lambda_2 = -\lambda_1, \quad v_1 = 0. \tag{2}$$

Let us find the fixed points of the phase flow of system (1). We have

$$x_1 = \text{const}, \quad y_1 = \text{const}. \tag{3}$$

Using (2) in (1), we get

$$\frac{R^2 x_1 y_1 v_2}{(x_1^2 + y_1^2)^2} - \frac{2\lambda_1(y_1 - y_1^*)x_1 x_1^*}{((x_1 - x_1^*)^2 + (y_1 - y_1^*)^2)((x_1 + x_1^*)^2 + (y_1 - y_1^*)^2)} = 0,$$

$$v_2 = v_2(0) - gt, \tag{4}$$

$$x_c = x_c(0),$$

$$y_c = y_c(0) + v_2(0)t - \frac{1}{2}gt^2,$$

where $v_2(0)$, $x_c(0)$, and $y_c(0)$ are the initial conditions. Obviously, conditions (3) and (4) are incompatible, which means that Föppl configurations never exist in the presence of gravity.

A numerical analysis of (1) on manifold (2) suggests the following hypotheses.

Hypothesis 1. Depending on the circulations of the vortices, one of the following two scenarios occurs:

(a) After a single act of scattering, the cylinder leaves the vortices behind and eventually moves alone guided by the gravity.

(b) The vortices moving ahead of the cylinder are scattered by the latter many times. They move with an increasing velocity and a decreasing amplitude in a series of similar patterns.

Hypothesis 2. In the case of multiple scattering, due to the interaction with the vortex pair, the average acceleration of the cylinder is less than the acceleration of gravity.

The most intriguing case of multiple scattering is shown in the figure (Fig. 1). The parameters of the system are $R = 0.5$, $a = 10$, $\lambda_1 = 7$, and $g = 10$. The time of motion is $t_0 = 16.05$. Initially, the coordinates of the vortices are $(-1,-1)$ and $(1,-1)$ and the coordinates of the cylinder's center are $(0,0)$.

4. RESTRICTED FÖPPL PROBLEM, ASYMPTOTIC SYSTEM, AND REGULARIZATION

Suppose that the cylinder is so massive that it is guided only by the force of gravity, but is unaffected by the vortices. This means that system (1) has to be further restricted to an invariant manifold that is a submanifold of the manifold from the previous section. This invariant manifold is given by the relations

$$\lambda = 0, x_2 = -x_1, \quad y_2 = y_1, \quad \lambda_2 = -\lambda_1, \quad v_1 = 0,$$
$$v_2 = v_2(0) - gt, \quad x_c = x_c(0),$$
$$y_c = y_c(0) + v_2(0)t - \frac{1}{2}gt^2.$$

From (1), it follows that

$$\begin{aligned}\dot{x}_1 &= \frac{2R^2 x_1 y_1 v_2}{(x_1^2 + y_1^2)^2} - \frac{4\lambda_1 (y_1 - y_1^*) x_1 x_1^*}{((x_1 - x_1^*)^2 + (y_1 - y_1^*)^2)((x_1 + x_1^*)^2 + (y_1 - y_1^*)^2)},\\ \dot{y}_1 &= -v_2 - \frac{R^2(x_1^2 - y_1^2)v_2}{(x_1^2 + y_1^2)^2} + \lambda_1\left(\frac{1}{2x_1} + \frac{2x_1^*(x_1^2 - x_1^{*2} - (y_1 - y_1^*)^2)}{((x_1 - x_1^*)^2 + (y_1 - y_1^*)^2)((x_1 + x_1^*)^2 + (y_1 - y_1^*)^2)}\right).\end{aligned} \tag{5}$$

Simulations of (5) demonstrate a good qualitative agreement with the behavior of the original unrestricted problem.

Following [11], we explore the behavior of solutions of system (5) at long times. On the basis of numerical integration of the restricted problem, we seek solutions such that $x_1 \to 0$ and y_1 is bounded as $t \to \infty$. In view of (4) and the initial conditions, we introduce the new variables $x_1 = \frac{x_1}{t}$ and $t = \frac{1}{2}t^2$. Equations (5) now become

$$\begin{aligned}\dot{x}_1 &= -\frac{2R^2 x_1 g}{y_1^3},\\ \dot{y}_1 &= g\left(1 - \frac{R^2}{y_1^2}\right) + \frac{\lambda_1}{2x_1}.\end{aligned} \tag{6}$$

Notice that the asymptotic system (6) is Hamiltonian with the Hamiltonian function

$$H = gR^2 \frac{x_1}{y_1^2} - g x_1 - \frac{\lambda_1}{2}\ln|x_1|.$$

Since x_1 and y_1 are separated from zero, we see that the right-hand side of the asymptotic system has no singularities and, therefore, the solutions of the original system are smooth at long times.

Similar to the case of the unrestricted problem, simulations of the restricted problem reveal, among other things, the existence of a scenario in which the vortices repeatedly get close to the surface of the cylinder. When this is the case, each vortex almost merges with its inverse image, thereby making the right-hand side of (5) grow infinitely. As a result, the numerical procedures become unstable, so the closeness of the numerical solution to the exact one must be discussed. Consider a regularization procedure. Suppose that a vector field $v(x)$ in a domain M gives rise to the differential equation

$$\dot{x} = v(x). \tag{7}$$

Let $f(x)$ be a positive function on M. Then the integral curves of the equation

$$\dot{x} = f(x)v(x),$$

coincide with those of (7). This is equivalent to the change of time $d\tau = \frac{dt}{f(x)}$.

In our case, we can put in (5)

$$f(x_1, y_1) = ((x_1 - x_1^*)^2 + (y_1 - y_1^*)^2)((x_1 + x_1^*)^2 + (y_1 - y_1^*)^2).$$

Integration of the equations that ensue reveals a smooth behavior of the integral curves at the points where the vortices get close to the surface of the cylinder. This indicates that the numerical solutions of the original (unrestricted) system converge to the exact solutions and thus completes the analysis of the restricted Föppl problem.

ACKNOWLEDGMENTS

I express my gratitude to A.V. Borisov, I.S. Mamaev, and D.V. Treshchev for fruitful discussions.

This work was supported by the Russian Foundation for Basic Research, project nos. 16-01-00170, 16-01-00809.

REFERENCES

1. A. V. Borisov and I. S. Mamaev, Reg. Chaot. Dyn. **8** (2), 163–166 (2003).

2. A. V. Borisov, I. S. Mamaev, and S. M. Ramodanov, Reg. Chaot. Dyn. **8** (4), 449–462 (2003).
3. A. V. Borisov, I. S. Mamaev, and S. M. Ramodanov, J. Math. Phys. **48** (6), 065403 (2007).
4. B. N. Shashikanth, Phys. Fluids **18**, 127103 (2006).
5. S. V. Sokolov and S. M. Ramodanov, Reg. Chaot. Dyn. **18** (1–2), 184–193 (2013).
6. I. A. Bizyaev, A. V. Borisov, and I. S. Mamaev, Reg. Chaot. Dyn. **19** (6), 694–701 (2014).
7. K. A. O'Neil and N. Cox-Steib, Reg. Chaot. Dyn. **19** (5), 523–532 (2014).
8. A. Carvalho and H. E. Cabral, Reg. Chaot. Dyn. **19** (3), 348–362 (2014).
9. S. P. Kuznetsov, Reg. Chaot. Dyn. **20** (3), 345–382 (2015).
10. L. Föppl, Sitz. K. Bäyr Akad. Wiss. I **7** (1913).
11. A. V. Borisov, V. V. Kozlov, and I. S. Mamaev, Reg. Chaot. Dyn. **12** (5), 531–565 (2007).