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On the Closeness of Trajectories for Model Quasi-Gasdynamic Equations[1]

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Abstract—On a model example of a linear hyperbolic equation with small parameter multiplying the highest time derivative it is shown that the closeness of individual trajectories to the dynamics of the limiting parabolic equation essentially depends on the Fourier spectra of the initial data. The trajectories stay close if the higher modes decay sufficiently fast. If the initial data are irregular and there are relatively high modes, then the convergence of the trajectories becomes non-uniform. Namely, the boundary layer is formed and there exist small moments of time such that the difference of the solutions reaches in the mean a finite value as the coefficient of the highest time derivative tends to zero. These results reflect the difficulties that may arise in the analysis of the systems of non-linear quasi-gasdynamic equations.

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1. INTRODUCTION

In the mathematical modeling of complex gasdynamic processes it is often useful to use a quasi-gasdynamic (QGD) system of equations and the corresponding kinetic difference schemes. A new modification of QGD is described in [1], [2] and contains the second derivative with respect to time along with the second spatial derivatives.

The main systems of equations of hydro- and gas dynamics, the Euler and Navier–Stokes equations, are known to be deduced in the past by using the methods of the kinetic theory. These equations have proven themselves in practice. However, there remain a lot of questions which are useful to be looked at from the point of view of the kinetic theory and the application of this theory leads to additional terms in the traditional systems of equations.

From the computational point of view the usefulness of hyperbolization of the Navier–Stokes equations can be described as follows (for illustrative purposes, it suffices to consider the case of one spatial variable). If the system of equations has the form

$$\partial_t \mathbf{u} + A\partial_x \mathbf{u} = \nu\partial_{xx}^2 \mathbf{u}, \tag{1}$$

then the use the explicit scheme (which can turn out to be the most adequate for flows with complex structure) requires the time step $\tau \sim h^2$, where h is the typical size of the spatial grid. The hyperbolized system is as follows

$$\varepsilon\partial_{tt}^2 \mathbf{u} + \partial_t \mathbf{u} + A\partial_x \mathbf{u} = \nu\partial_{xx}^2 \mathbf{u}, \tag{2}$$

where ε is small, and then the time step is $\tau \sim h\sqrt{\varepsilon}$. If we take ε of the order of h, then there is a significant gain in the computation time.

The question naturally arises, how close are the solutions of the original parabolic equation (1) and its QGD modification (2). In this paper it is shown that even in the simplest case of a linear equation this question is not trivial. Roughly speaking, the solutions remain close if the higher modes of the initial data for (1) decay sufficiently fast. If the initial data for (1) contain high frequency modes of finite amplitude, then the convergence of the trajectories becomes non-uniform, and a boundary layer is formed in the limiting dynamics, when there are small values of the time t such that the difference between the solutions of systems (1) and (2), generally speaking, is of the order $O(1)$.

Thus, strictly speaking, when using the hyperbolization of the basic system of equations in gas dynamics problems we need to make sure that we are dealing with a sufficiently smooth background flow. On the other hand, in practice the discretization can be carried out up to a certain limit and the grid size necessarily bounds the number of the resolved frequencies (a similar situation, for example, takes place

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in the calculation of multiphase flows based on the concept of multi-velocity continuum).

The physical motivation of the existence of the limits of the resolution in the models of continua was proposed in the seminal work [3]. With this paradigm the results obtained in this note provide a certain theoretical justification for the possibility to use the QGD system with a small parameter in the highest time derivative in the case of regions with large gradients, which are shocks in the case of the Euler equation, and in the case of the Navier–Stokes equations describe the development of certain transients, for example, the development of turbulence.

In this note we study the closeness of the solutions of linear model equations of the type (1) and (2) in terms of the conditions on the initial data only. The passage to a model linear system of equations is naturally carried out by means of splitting the system into a set of the corresponding equations. Similar estimates for individual modes arising from the dispersion relation were obtained in [4, 5]. In [6] an estimate of the closeness of the solutions in the norm $L_2(0,T;H^1)\cap L_\infty(0,T;L_2)$ was obtained for linear equations with variable coefficients, provided that the L_2-norm of the second time derivative of the solution of the analogue of (2) is bounded. In [7] a result on the closeness of the solutions in the norms L_2 and C was obtained under the condition that the second time derivative of the solutions of the analog of (1) be bounded in L_2 and C, respectively.

Finally, we note that the inclusion of nonlinear effects is likely to give an even more complex picture of the behavior of the trajectories of a dynamical system with singular perturbation by the second time derivative.

2. ON THE CLOSENESS OF INDIVIDUAL TRAJECTORIES ON A FINITE INTERVAL

We consider the problem of closeness of individual trajectories for the linear systems with constant coefficients (1) and (2).

Let in (1), (2) $\mathbf{u}=\{u_1,\ldots,u_n\}$, and let ε, ν be diagonal positive-definite $(n\times n)$-matrices, and let A be a $(n\times n)$-matrix with complete set of eigenvalues and eigenvectors. Then multiplication of the systems (1) and (2) by the corresponding left eigenvectors gives a family of linear equations of the type

$$\partial_t\bar{u}+a\partial_x\bar{u}=\nu\partial_{xx}^2\bar{u}, \tag{3}$$

and

$$\varepsilon\partial_{tt}^2u+\partial_tu+a\partial_xu=\nu\partial_{xx}^2u, \tag{4}$$

where ε and ν are positive (small) numbers and a is also number of arbitrary sign. Therefore it suffices to study the closeness of the individual trajectories for equations (3) and (4).

We thus consider one-dimensional equations (3), (4) in the class of functions periodic with respect to $x\in[0,2\pi]$ and supplemented with initial data

$$u(0,x)=\bar{u}(0,x)=u_0(x);\quad \partial_tu(0,x)=\dot{u}_0(x). \tag{5}$$

We use the Fourier series to represent the solutions

$$v(t,x)=\sum_{k\in\mathbb{Z}}v_k(t)e^{ikx},$$

where $$v_k(t)=\frac{1}{2\pi}\int_0^{2\pi}v(t,x)e^{-ikx}dx.$$

Let $\bar{u}(t,x)$ and $u(t,x)$ be the solutions of (3), (5) and (4), (5), respectively, with $\bar{u}_0^k(t)$ and $u_0^k(t)$ being the corresponding Fourier coefficients. Then the following result holds.

Theorem 1. *Led $T>0$ be fixed. Suppose that the following series converges*

$$\sum_{k\in}(|u_k^0|+|\dot{u}_k^0|)<+\infty \quad \textit{and} \quad \varepsilon\le\frac{\nu}{a^2}. \tag{6}$$

Then $\Lambda\equiv\sup\limits_{(t,x)\in[0,T]\times[0,2\pi]}|u(t,x)-\bar{u}(t,x)|\to0$ *as* $\varepsilon\to0$.

Proof. For the problem (3), (5) we clearly have

$$\bar{u}_k(t)=u_0^ke^{-(iak+\nu k^2)t},\quad \text{where}\quad u_0^k\equiv u_0^k(0). \tag{7}$$

For the problem (4), (5) we obtain

$$\begin{gathered}\varepsilon\ddot{u}_k+\dot{u}_k+(iak+\nu k^2)u_k=0,\\ u_k(0)=u_0^k,\quad \dot{u}_k(0)=\dot{u}_0^k,\end{gathered} \tag{8}$$

and from the corresponding characteristic equation we find

$$\begin{gathered}\lambda_{1,2}(k,\varepsilon)=\frac{1}{2\varepsilon}(\mp\sqrt{1-4\varepsilon\kappa(k)}-1),\\ \kappa(k)\equiv iak+\nu k^2\end{gathered} \tag{9}$$

and

$$\begin{gathered}u_k(t)=u_0^k\frac{\lambda_2e^{\lambda_1t}-\lambda_1e^{\lambda_2t}}{\lambda_2-\lambda_1}+\dot{u}_0^k\frac{e^{\lambda_2t}-e^{\lambda_1t}}{\lambda_2-\lambda_1},\\ \text{where}\quad u_0^k\equiv u_0^k(0),\quad \dot{u}_0^k\equiv\dot{u}_0^k(0).\end{gathered} \tag{10}$$

We estimate $|\Lambda(k,\varepsilon,t)|\equiv|u_k(t)-\bar{u}_k(t)|$. We set $D\equiv\sqrt{1-4\varepsilon\kappa(k)}=\sqrt{R}e^{-i\varphi/2}$, $F\equiv(1-4\varepsilon\nu k^2)^2+16\varepsilon^2a^2k^2$, $R\equiv\sqrt{F}$, $\cos\varphi=\dfrac{1-4\varepsilon\nu k^2}{R}$, $\sin\varphi=\dfrac{4\varepsilon ak}{R}$. Then

$$\Lambda(k,\varepsilon,t) = \frac{\varepsilon}{D}\left\{u_0^k\left[\frac{D-1}{2\varepsilon}e^{-\frac{D+1}{2\varepsilon}t}+\frac{D+1}{2\varepsilon}e^{\frac{D-1}{2\varepsilon}t}-\frac{D}{\varepsilon}e^{-\kappa(k)t}\right] + \dot{u}_0^k\left[e^{\frac{D-1}{2\varepsilon}t}-e^{-\frac{D+1}{2\varepsilon}t}\right]\right\}. \quad (11)$$

To estimate $\Lambda(k,\varepsilon,t)$ we first estimate $I \equiv \dfrac{1-\sqrt{R}\left|\cos\left(\frac{\varphi}{2}\right)\right|}{\varepsilon}$, since if I is negative and unbounded with respect to ε and k, then with appropriate choice of the sign of $\cos\left(\frac{\varphi}{2}\right)$ there are unbounded terms in the exponentials in (11). We have

$$I = \frac{1}{\varepsilon}\left[1-\frac{1}{\sqrt{2}}(R+1-4\varepsilon\nu k^2)^{1/2}\right] = \frac{1}{\varepsilon}\frac{1-(R+1-4\varepsilon\nu k^2)/2}{1+(R+1-4\varepsilon\nu k^2)^{1/2}/\sqrt{2}},$$

if we multiply and divide the expression in brackets by $1+(R+1-4\varepsilon\nu k^2)^{1/2}/\sqrt{2}$. Acting similarly once again we obtain $I = \dfrac{8k^2(\nu-\varepsilon a^2)}{Q}$ for some $Q > 0$. Since $\varepsilon \le \dfrac{\nu}{a^2}$, it follows that $I > 0$. Therefore the exponents in (11) have negative real parts.

We consider in (11) the coefficient of u_k^0:

$$u_k^0\left[\frac{1}{2}e^{-\frac{D+1}{2\varepsilon}t}+\frac{1}{2}e^{\frac{D-1}{2\varepsilon}t}-e^{-\kappa(k)t}+\frac{1}{D}\left(e^{\frac{D-1}{2\varepsilon}t}-e^{-\frac{D+1}{2\varepsilon}t}\right)\right].$$

The absolute value of the sum of the first three terms is not greater than 2. Consider the last term

$$\frac{1}{D}\left(e^{\frac{D-1}{2\varepsilon}t}-e^{-\frac{D+1}{2\varepsilon}t}\right) = 2be^{-b}\frac{\sinh Db}{Db},$$

where $b = \dfrac{t}{2\varepsilon}$, $D = \sqrt{1-4\varepsilon\nu k^2 - i4\varepsilon ak}$.

Suppose first that $a = 0$. When k varies from 0 to $\dfrac{1}{\sqrt{4\varepsilon\nu}}$ the function D varies from 1 to 0 (and is real). Next, when k varies from $\dfrac{1}{\sqrt{4\varepsilon\nu}}$ to $+\infty$ D varies from $i0$ to $i\infty$ and is purely imaginary. We fix $b \ge 0$. For real $x \in [0,c]$ the function $\dfrac{\sinh x}{x}$ increases from 1 to $\dfrac{\sinh c}{c}$. Therefore for $k \in \left(0,\dfrac{1}{\sqrt{4\varepsilon\nu}}\right)$, $D \in [0, 1]$ and

$$2be^{-b}\frac{\sinh Db}{Db} \le 2be^{-b}\frac{\sinh b}{b} = 2e^{-b}\sinh b = 1-e^{-2b} < 1.$$

Now let $k \in \left[\dfrac{1}{\sqrt{4\varepsilon\nu}},\infty\right)$. Then $D = iy$, $y \in [0, \infty)$ and

$$2be^{-b}\frac{\sinh Db}{Db} = 2be^{-b}\frac{\sinh iyb}{iyb} = 2be^{-b}\frac{\sin yb}{yb} \le 2be^{-b} \le 2e^{-1} < 1.$$

The coefficient of $\dot{u}_k^0$ in (11) is equal to $\varepsilon 2be^{-b}\dfrac{\sinh Db}{Db}$, therefore

$$|\Lambda(k,\varepsilon,t)| \le 3|u_k^0| + \varepsilon|\dot{u}_k^0|. \quad (12)$$

If $a \ne 0$, then for $\varepsilon \le \dfrac{\nu}{a^2}$ somewhat more lengthy calculations show that estimate (12) still holds.

Next, since $\Lambda \le \sum_{k\in\mathbb{Z}}|\Lambda(k, \varepsilon, t)|$, taking into account the hypotheses of Theorem 1 and (12), for any $\delta > 0$ there exists a $k_0 > 0$, such that $\sum_{|k|\ge k_0\in\mathbb{Z}}|\Lambda(k, \varepsilon, t)| < \dfrac{\delta}{2}$. In the case when $|k| < k_0$ we have $\lambda_1 \sim -\dfrac{1}{\varepsilon} + \kappa(k)$, $\lambda_2 \sim -\kappa(k)$ as $\varepsilon \to 0$ and $|\Lambda(k, \varepsilon, t)| \le \text{const}\,\varepsilon$.

This directly gives the assertion of Theorem 1.

3. ON THE EXISTENCE OF BOUNDARY LAYER

If the initial data contain sufficiently high modes with finite amplitude, then unlike in the case of the limiting parabolic operator, they do not decay as $\varepsilon \to 0$ and preserve a finite amplitude forming thereby a boundary layer.

Theorem 2. *Consider the sequence of single mode initial data*

$$u_k^0 = u_{k(\varepsilon)}^0 = \begin{cases} 0, & k \ne k(\varepsilon), \\ u_{k(\varepsilon)}^0, & k = k(\varepsilon), \end{cases}$$

where $k(\varepsilon) = \varepsilon^{-m}$, $m > \dfrac{1}{2}$. *Let* $u^\varepsilon(x, t)$ *and* $\bar{u}^\varepsilon(t, x)$ *be the corresponding solutions of* (3), (4). *Then*

$$\limsup_{\varepsilon\to 0}\frac{\|(u^\varepsilon(t,\cdot)-\bar{u}^\varepsilon(t,\cdot))|_{t=\varepsilon}\|_{C[0,2\pi]}}{|u_{k(\varepsilon)}^{(0)}|} \ge \text{const}, \quad (13)$$

where const > 0 *is independent of* ε. *Furthermore, this estimate holds for any consistent choice of the second initial condition* $\dot{u}_{k(\varepsilon)}^{(0)}$.

Proof. To simplify notation we set in this example $\nu = 1$, $a = 0$. Then the solution of (3) contains only one non-zero Foureir coefficient $\bar{u}_k(t) = u_k^0 e^{-k^2 t}$, $k = k(\varepsilon)$ and for $t = \varepsilon$ this coefficient is exponentially small.

Therefore it suffices to show that the corresponding Fourier coefficient of the solution of equation (4) satisfies

$$\limsup_{\varepsilon\to 0}\left|\frac{u_{k(\varepsilon)}(t)|_{t=\varepsilon}}{u^{(0)}_{k(\varepsilon)}}\right| = \text{const} \neq 0.$$

In view of (9) we have

$$\lambda_{1,2} = -\frac{1}{2\varepsilon} \mp \frac{i}{\varepsilon}G(\varepsilon), \quad G(\varepsilon) = \varepsilon^{-(2m-1)/2}\sqrt{1-\frac{\varepsilon^{2m-1}}{4}},$$

where $G(\varepsilon) \to \infty$ as $\varepsilon \to 0$. We fix an arbitrary second initial condition setting $\dot{u}^{(0)}_{k(\varepsilon)} = \alpha u^{(0)}_{k(\varepsilon)}$, where α is arbitrary (and can depend on k; this is, for instance, the case when the second initial condition is found from equation (4) by setting $t = 0$ and $\varepsilon = 0$ there, then $\alpha = -k(\varepsilon)^2$). The solution of (4) (see (10)) for $t = \varepsilon$ takes the form

$$u_k(t)|_{t=\varepsilon} = \frac{u^0_k}{4}\left[\left(2+\frac{i(1+2\alpha\varepsilon)}{G(\varepsilon)}\right)e^{\lambda_1\varepsilon} + \left(2-\frac{i(1+2\alpha\varepsilon)}{G(\varepsilon)}\right)e^{\lambda_2\varepsilon}\right]$$
$$= \frac{u^0_k e^{-1/2}}{4}\left[\left(2+\frac{i(1+2\alpha\varepsilon)}{G(\varepsilon)}\right)e^{-iG(\varepsilon)} + \left(2-\frac{i(1+2\alpha\varepsilon)}{G(\varepsilon)}\right)e^{iG(\varepsilon)}\right]$$
$$= \frac{u^0_k e^{-1/2}}{2}\left[2\cos G(\varepsilon) + \frac{1+2\alpha\varepsilon}{G(\varepsilon)}\sin G(\varepsilon)\right].$$

Since $G(\varepsilon) \to \infty$ as $\varepsilon \to 0$, it follows that there exists a sequence $\varepsilon_n \to 0$ as $n \to \infty$, for which $G(\varepsilon_n) = 2\pi n$. This gives that

$$\limsup_{\varepsilon\to 0}\left|\frac{u_{k(\varepsilon)}(\varepsilon)}{u^{(0)}_{k(\varepsilon)}}\right| \geq e^{-1/2},$$

and completes the proof of Theorem 2.

Thus, by limiting the spatial resolution the choice of the time step must also take into account the formation of the boundary layer. Probably, this effect will be more important in a non-linear version of the equations of the type (3), (4).

In conclusion, we observe that the solutions of equations (3) and (4) tend to zero as $t \to \infty$. If the equations have a right-hand side and a nonlineariry, then clearly there can be no closeness of the solutions on the infinite time interval, and the behavior of the solutions as is commonly described by the global attractors [8]. In this case a general result holds on the upper semicontinuity on ε as $\varepsilon \to 0$ of the attractors of the singularly perturbed hyperbolic equation. Namely, every μ-neighborhood of the attractor of the limiting equation contains all the attractors of the perturbed equation for $\varepsilon < \varepsilon_0(\mu)$ [4, 8].

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REFERENCES

1. A. A. Zlotnik and B. N. Chetverushkin, Comput. Math. Math. Phys. **48** (3), 420–446 (2008).
2. T. G. Elizarova, *Quasi-Gas Dynamic Equations* (Nauchnyi Mir, Moscow, 2007; Springer-Verlag, Berlin, 2009).
3. B. N. Chetverushkin, Math. Models Comput. Simul. **5** (3), 266–279 (2013).
4. A. A. Ilyin and Yu. G. Rykov, Preprint No. 12, IPM RAN (Keldysh Inst. of Applied Mathematics, Russian Academy of Sciences, Moscow, 2012).
5. A. A. Ilyin and Yu. G. Rykov, Preprint No. 75, IPM RAN (Keldysh Inst. of Applied Mathematics, Russian Academy of Sciences, Moscow, 2012).
6. S. I. Repin and B. N. Chetverushkin, Dokl. Math. **88** (1) 417–420 (2013).
7. E. E. Myshetskaya and V. F. Tishkin, Comput. Math. Math. Phys. **55** (8), 1270–1275 (2015).
8. A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations* (Nauka, Moscow, 1989; North-Holland, Amsterdam, 1992).