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On Exact Solutions to the Kolmogorov–Feller Equation

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Abstract—The integrodifferential Kolmogorov–Feller equation describing the stochastic dynamics of a system subjected to a regular "force" and a random external disturbance in the form of short pulses with random "amplitudes" and occurrence times is considered. The equation is written in differential form. A method for finding the regular force from a given stationary probability distribution is described. The method is illustrated by examples.

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The Kolmogorov–Feller (KF) equation is an integrodifferential equation governing the transition probability density of Markov stochastic processes with jump changes in the state of the described system. The KF equation is used in various fields of physics and engineering, such as control theory and Brownian motion [1], wave theory in random inhomogeneous scattering media [2], the modeling of tropical cyclones [3], the motion of loads over a bridge [4], buffeting [5], ground acceleration due to strong earthquakes or shock waves [6], the behavior of trains on a uneven track [7], and the simulation of processes in neurodynamics [8], economics, ecology [9], and medicine. Among the fundamental applications, we note the use of the KF equation in quantum mechanics models [10].

To be more specific, we consider a first-order nonlinear system described by the stochastic differential equation

$$\dot{x} = f(x) + \xi(t). \tag{1}$$

Here, $\xi(t)$ is an external noise perturbation representing a sequence of delta pulses:

$$\xi(t) = \sum_k a_k \delta(t - t_k). \tag{2}$$

The time intervals $\tau_k = t_k - t_{k-1}$ between neighboring pulses and the pulse "amplitudes" a_k are assumed to be statistically independent and are described by probability density functions $w(\tau)$ and $W(a)$.

If the time intervals are exponentially distributed, i.e.,

$$w(\tau) = \nu \exp(-\nu\tau), \tag{3}$$

then $\xi(t)$ becomes a white shot noise with Poisson-distributed pulse occurrence times. Accordingly, $x(t)$ becomes a Markov stochastic process and a closed KF equation can be obtained for it [11]:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}[f(x)P] + \nu\int_{-\infty}^{\infty} W(z)[P(x-z,t) - P(x,t)]\,dz. \tag{4}$$

Here, $P(x,t)$ is the probability density of the stochastic process $x(t)$, which satisfies the stochastic equation (1). Equation (4) is a special case of the generalized Kolmogorov equation derived in [12] for an arbitrary non-Gaussian white noise $\xi(t)$. Integrating both sides of the KF equation (4) with respect to x in the infinite limits and taking into account the normalization condition for the probability density, we obtain

$$\lim_{x\to\pm\infty} f(x)P(x,t) = 0. \tag{5}$$

As far as we know, equations of type (4) have been solved basically by applying numerical methods. However, in certain cases, analytical solutions can be found.

Using the shift operator, we write part of the integrand as

$$P(x-z,t) = \exp\left(-z\frac{\partial}{\partial x}\right)P(x,t). \tag{6}$$

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Substituting (6) into (4) and performing integration, we obtain the KF equation in differential form:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}[f(x)P] + \nu\left[\Theta\left(i\frac{\partial}{\partial x}\right) - 1\right]P. \quad (7)$$

Here, $\Theta(u) = \langle \exp(iua) \rangle$ is the characteristic function of the random pulse amplitudes and the angle brackets denote the statistical average with respect to a with the probability distribution $W(a)$.

Now we can show that, for some particular forms of $W(a)$, we can find exact solutions of the KF equation written in the form (7). Specifically, for positive amplitudes with the exponential distribution

$$W(a) = \lambda \exp(-\lambda a), \quad (8)$$

Eq. (7) becomes

$$\left(\lambda + \frac{\partial}{\partial x}\right)\left[\frac{\partial P}{\partial t} + \frac{\partial}{\partial x}(f(x)P)\right] = -\nu\frac{\partial P}{\partial x}. \quad (9)$$

The stationary (time-independent) probability distribution for system (1) excited by pulses (2) of positive polarity can be found by solving Eq. (9) with any function $f(x)$, i.e., with an arbitrary nonlinearity in Eq. (1) for which it exists. The steady-state solution of Eq. (9) has the form (see [13])

$$P_{\mathrm{ST}}(x) = \frac{\exp(-\lambda x)}{f(x)}\exp\left[-\nu\int\frac{dx}{f(x)}\right]. \quad (10)$$

The case of an arbitrary polarity of pulses (2), when the pulse amplitudes can be positive and negative, is of more interest in terms of physics. For this problem, instead of (8), we use the Laplace distribution

$$W(a) = \frac{\beta}{2}\exp(-\beta|a|). \quad (11)$$

Then, instead of (9), we obtain the partial differential equation

$$\left(\frac{\partial^2}{\partial x^2} - \beta^2\right)\left[\frac{\partial P}{\partial t} + \frac{\partial}{\partial x}(f(x)P)\right] = -\nu\frac{\partial^2 P}{\partial x^2}. \quad (12)$$

The stationary distribution is found by solving the ordinary differential equation

$$\frac{d^2}{dx^2}(f(x)P_{\mathrm{ST}}) + \nu\frac{dP_{\mathrm{ST}}}{dx} - \beta^2(f(x)P_{\mathrm{ST}}) = 0. \quad (13)$$

Specifically, for the linearized system (1) with $f(x) = -\nu x$, the solution of Eq. (13) is expressed in terms of the modified Bessel function of the second kind of zero order:

$$P_{\mathrm{ST}}(x) = \frac{\beta}{\pi}K_0(\beta|x|). \quad (14)$$

The probability density (14) is similar in shape to the Gaussian distribution, but it has an integrable singularity at the origin.

Unfortunately, there are few exact solutions of Eq. (13) that correspond to various functions $f(x)$. Nearly all of them violate the necessary boundary conditions and symmetry properties.

The task of finding distributions $P_{\mathrm{ST}}(x)$ by means of Eq. (13) with a given function $f(x)$ can be pulse a direct problem. An inverse problem is to find the "force" $f(x)$ or the corresponding potential $U(x) = -\int f(x)dx$ from a given distribution $P_{\mathrm{ST}}(x)$. In some cases, the inverse problem is simpler.

In Eq. (13), we set

$$f(x)P_{\mathrm{ST}} = \frac{dQ}{dx}. \quad (15)$$

In this case, the function $Q(x)$ is defined up to a constant. Substituting (15) into (13) and integrating the result with respect to x, we obtain the relation

$$\frac{d^2Q}{dx^2} - \beta^2 Q = -\nu(P_{\mathrm{ST}} + P_0).$$

In view of the obvious condition $P_{\mathrm{ST}}(x) \to 0$ as $x \to \pm\infty$ and KF equation (4), we have $P_0 = \frac{\beta^2 Q(\infty)}{\nu}$. Now, setting $Q(\infty) = 0$ yields the following relations to determine the functions $P_{\mathrm{ST}}(x)$ and $f(x)$:

$$P_{\mathrm{ST}}(x) = \frac{1}{\nu}\left(\beta^2 Q - \frac{d^2Q}{dx^2}\right), \quad f(x) = \frac{1}{P_{\mathrm{ST}}}\frac{dQ}{dx}. \quad (16)$$

In view of (5) and (16), the function $Q(x)$ must satisfy the "normalization condition" and the inequality

$$\int_{-\infty}^{\infty} Q(x)dx = \frac{\nu}{\beta^2}, \quad \beta^2 Q - Q'' \geq 0.$$

It is convenient, using Eq. (13), to express the force f directly in terms of P_{ST}:

$$f(x) = -\frac{\nu}{P_{\mathrm{ST}}(x)}\int_0^x P_{\mathrm{ST}}(y)\cosh\beta(x-y)dy + \frac{C_1\cosh\beta x + C_2\sinh\beta x}{P_{\mathrm{ST}}(x)}. \quad (17)$$

Here, C_1 and C_2 are unknown constants determined by conditions (5).

Note that, if $P_{\mathrm{ST}}(x)$ is an even function, then the force $f(x)$ in Eq. (17) has to be an odd function. It follows that $C_1 = 0$ and, in the evaluation of integral (17), it is sufficient to consider only the domain $x > 0$.

Below are examples of solutions found by means of (16) and (17).

Example 1. Let us substitute in formula (17) the hyperbolic distributions

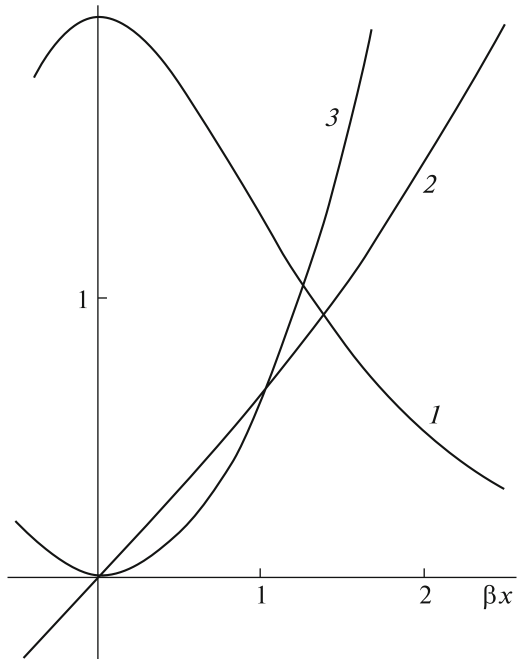

Fig. 1. Behavior of the stationary probability distribution, force, and potential function (curves *1–3*, respectively).

$$P_{\mathrm{ST}}(x) = \frac{\beta}{\pi \cosh \beta x}, \tag{18}$$

which is infinitely divisible [14]. Calculating the corresponding force and determining the unknown constant C_2 from condition (5), we find

$$f(x) = -\frac{\nu}{2\beta}[\beta x(\cosh 2\beta x + 1) - \ln(2\cosh \beta x)\sinh 2\beta x]. \tag{19}$$

The potential corresponding to force (19) is given by

$$U(x) = \frac{\nu}{4\beta^2}\Big[\beta^2 x^2 + \beta x \sinh 2\beta x - \ln(2\cosh\beta x)\cosh 2\beta x - \ln\Big(\frac{\cosh\beta x}{2}\Big)\Big]. \tag{20}$$

Three functions are shown in Fig. 1:

$$Y_1 = \frac{2\pi}{\beta}P_{\mathrm{ST}}(x), \quad Y_2 = -\frac{2\beta}{\nu}f(x), \quad Y_3 = \frac{4\beta^2}{\nu}U(x). \tag{21}$$

Curve *1* depicts the stationary probability distribution (18); curve *2,* force (19); and curve *3,* the corresponding potential (20). As follows from (20), the "tails" of the potential function increase according to a parabolic law, so curve *3* resembles in form the harmonic potential corresponding to linear system (1).

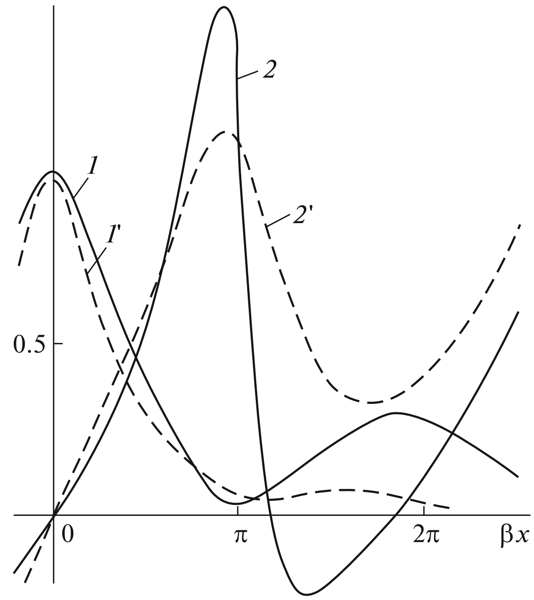

Fig. 2. Stationary probability distribution (26) (curves *1* and *1'*) and the corresponding force (27) (curves *2* and *2'*). The solid curves were constructed for the parameter values $a = 1.2$, $\frac{k}{\beta} = 1$, and $\frac{\gamma}{\beta} = 0.2$, while the dashed curves, for $a = 2$, $\frac{k}{\beta} = 1$, and $\frac{\gamma}{\beta} = 0.5$.

Example 2. In (17), we substitute the Laplace distribution

$$P_{\mathrm{ST}}(x) = \frac{\alpha}{2}\exp(-\alpha|x|). \tag{22}$$

Evaluating the integral yields

$$f(x) = -\frac{\nu\alpha}{\beta^2 - \alpha^2}\operatorname{sgn}(x), \quad U(x) = \frac{\nu\alpha|x|}{\beta^2 - \alpha^2}. \tag{23}$$

The restoring force (23) describes a homogeneous force field changing its sign on passing through the point $x = 0$. The corresponding potential is confining for $\beta > \alpha$ and has a "modulus" form. Some dynamical and statistical problems for systems with such a potential and their physical interpretations were discussed in [15].

Example 3. We put in formula (17)

$$P_{\mathrm{ST}}(x) = C(1 + q|x|)\exp(-k|x|). \tag{24}$$

For $k < q$, the probability distribution (24) has two maxima, so it should correspond to a two-well potential. Calculating the force associated with the symmetric probability density (24), we obtain

$$f(x) = -\nu\left\{\frac{e^{(k-\beta)|x|}[k(k^2+\beta^2) + q(k^2-\beta^2)] - q(k^2+\beta^2)}{(1+q|x|)(k^2-\beta^2)^2} - \frac{k}{k^2-\beta^2}\right\}\operatorname{sgn} x. \tag{25}$$

By using this force, for $k > \beta$, we can find a confining potential, which is expressed in terms of the incomplete gamma function.

Example 4. In formula (17), we set

$$P_{ST} = C(a + \cos kx)\exp(-\gamma|x|). \tag{26}$$

The stationary probability distribution (26) is shown in Fig. 2 by curves *1* and *1'*. It can be seen that curve (26) can have one, two, or more maxima, depending on the parameter values. This means that the potential of the force $f(x)$ involves one, two, or several wells.

Calculating the integral and taking into account the condition (5) at infinity, we obtain the following expression for the force:

$$\begin{gathered} f(x) = -\frac{\nu}{a + \cos kx}[E(x) + F(x)], \\ E(x) = \frac{a\gamma}{\beta^2 - \gamma^2}[1 - \exp(-(\beta - \gamma)x)], \\ F(x) = \frac{k(k^2 + \beta^2 - \gamma^2)\sin kx + \gamma(k^2 - \beta^2 + \gamma^2)[\exp(-(\beta - \gamma)x) - \cos kx]}{[k^2 + (\beta + \gamma)^2][k^2 + (\beta - \gamma)^2]}. \end{gathered} \tag{27}$$

Formula (27) is valid for $x \geq 0$. To the domain $x \leq 0$, $f(x)$ is continued as an odd function. Assume the parameter values $a > 1$ and $\beta > \gamma$, for which the function (27) is limited.

In Fig. 2, the function $-\frac{f(x)\beta}{\nu}$ is depicted for $a = 1.2$, $\frac{k}{\beta} = 1$, and $\frac{\gamma}{\beta} = 0.2$ (solid curve *2*) and for $a = 2$, $\frac{k}{\beta} = 1$, and $\frac{\gamma}{\beta} = 0.5$ (dashed curve *2'*). Curve *2* corresponds to the stationary distribution shown by solid curve *1*, while curve *2'*, to the distribution shown by dashed curve *1'*.

In the case of a single potential well, the force is negative for $x > 0$ (curve *2'*) and positive for $x < 0$. In other words, the force tends to return the system to the equilibrium position $x = 0$.

In the case of two wells (curve *2*), the function $f(x)$ changes its sign in the domain $x > 0$, i.e., there appears a second "center of attraction."

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