ISSN 1064-5624, Doklady Mathematics, 2016, Vol. 94, No. 1, pp. 393–395. © Pleiades Publishing, Ltd., 2016.
Original Russian Text © A.Yu. Savin, B.Yu. Sternin, 2016, published in Doklady Akademii Nauk, 2016, Vol. 469, No. 2, pp. 154–156.

MATHEMATICS

Elliptic Expansion–Contraction Problems on Manifolds with Boundary

A. Yu. Savin* and B. Yu. Sternin**

Presented by Academician of the RAS A.T. Fomenko February 29, 2016

Received March 9, 2016

Abstract—Expansion–contraction boundary value problems on manifolds with boundary are studied. The trajectory symbols of such problems are calculated, an analogue of the Shapiro–Lopatinskii condition is obtained, and the corresponding finiteness theorem is given.

DOI: 10.1134/S1064562416040104

This paper studies the new class of boundary value problems

$$(D, B) : H^s(M) \to H^{s-m}(M) \oplus H^{s-b}(X), \quad m = \operatorname{ord} D, \quad b = \operatorname{ord} B, \tag{1}$$

on a smooth compact manifold M with boundary $X = \partial M$ in which the main operator D and the boundary operator B are nonlocal and associated with smooth self-mappings of the manifold M. Problems of the form (1) include some known classes of problems as special cases, including problems for invertible mappings (i.e., diffeomorphisms) (see [1]) and problems with homotheties in $\mathbb{R}^n$ (see [2]); finally, problems of the form (1) include, as a special case, the Bitsadze–Samarskii problem [3] in which the values of the function on the boundary are associated with its values on a submanifold contained inside the domain.

In this paper, we obtain condition for problems (1) to be Fredholm in the case where the problem is associated with a contraction, i.e., a mapping of a manifold with boundary strictly into itself. Of most interest is finding an analogue of the Shapiro–Lopatinskii condition in this situation. We show that, in the case of nonlocal problems associated with contractions, the ellipticity condition is of a completely new form in comparison with the classical situation. The point is that, in the presence of contractions, it is necessary to freeze the coefficients at once on the entire orbit of a boundary point under the action of the contraction and its iterations. The generalized Shapiro–Lopatinskii condition which we obtain in this way consists in the requirement of the unique solvability of the infinite matrix system of ordinary differential equations corresponding to the trajectories of boundary points.

RUDN University, ul. Miklukho-Maklaya 6, Moscow, 117198 Russia
Leibniz Hanover University, Welfengarten 1, 30167 Hannover, Germany
e-mail: *antonsavin@mail.ru, **sternin@mail.ru

1. STATEMENT OF THE PROBLEM

Let M be a compact smooth manifold with boundary $X = \partial M$.

Definition 1. A smooth mapping $g : M \to M$ is called a contraction if it is a diffeomorphism of the manifold M and a submanifold $g(M) \subset M \setminus X$ lying strictly inside M.

The contraction g induces the contraction and expansion operators

$$Tu(x) = \begin{cases} u(g^{-1}x), & \text{if} \quad x \in g(M), \\ 0, & \text{otherwise} \end{cases}$$

and $T^{-1}u(x) = u(gx)$, which act on the spaces $H^s(M)$ with $|s| < 1/2$.

We consider operators being finite sums of the form

$$D = \sum_k T^k D_k : H^s(M) \to H^{s-m}(M), \quad m = \operatorname{ord} D, \tag{2}$$

where the D_k are differential operators of order m; in the presence of summands with positive k, it is assumed that $|s - m| < 1/2$. This paper studies the Fredholm solvability of boundary value problems of the form

$$(D, B) : H^s(M) \to H^{s-m}(M) \oplus H^{s-b}(X), \tag{3}$$

in which the boundary operator B has order b and equals

$$B = Ai^*C : H^s(M) \to H^{s-b}(X),$$

where A is a pseudodifferential operator on X; $i^* : H^s(M) \to H^{s-1/2}(X)$ is the restriction operator to the submanifold, and $C = \sum_{k \le 0} C_k T^k$ is the operator with expansions T^{-1}, where the C_k are differential operators on M.

2. TRAJECTORY SYMBOL

We obtain an ellipticity condition for problem (3) by using the localization method, i.e., freezing the coefficients of the operators on the orbits of the iterations of the diffeomorphism g.

It turns out that the points of the manifold are divided into three types according to the behavior of their orbits:

(1) points for which all inverse iterations of the mapping g are defined;

(2) points which are taken to the boundary by a finite number of inverse iterations;

(3) points which go beyond the boundary of the domain after a finite number of inverse iterations.

Let us construct the trajectory symbols of our problem for orbits of all the three types.

2.1. The Trajectory Symbol for Orbits Lying in the Manifold

Let $x \in M_\infty = \bigcap_{n\geq 0} g^n M$. Then the orbit $\{g^n x\}$, $n \in \mathbb{Z}$, is entirely contained in the manifold. Therefore, the coefficients of our operator on such an orbit are frozen in the same way as for $\mathbb{Z}$-operators on closed manifolds (see [4, 5]). The symbol thus obtained is an operator on the function space on the orbit (identified with $\mathbb{Z}$) and is written in the form

$$\sigma(D)(x,\xi)\colon\ l^2(\mathbb{Z},\mu_{x,\xi,s}) \to l^2(\mathbb{Z},\mu_{x,\xi,s-m}),$$
$$x \in M_\infty, \quad \xi \in T_x^* M,$$
$$\sigma(D)(x,\xi) = \sum_k \mathcal{T}^k[(\partial g)^{n*}\sigma(D_k)](x,\xi). \tag{4}$$

Let us define the objects involved in this formula. First, the shift operator on sequences acts by $\mathcal{T}w(n) = w(n-1)$. Secondly, the expression $[(\partial g)^{n*}\sigma(D_k)](x,\xi)$ acts as the operator of multiplication of sequences, where $\partial g = (dg^t)^{-1} : T^*M \to T^*M$ is the codifferential of the diffeomorphism $g : M \to M$. Finally, symbol (4) acts on the space of sequences square summable with respect to the weight

$$\mu_{x,\xi,s}(n) = \frac{\partial g^{n*}[\mu\sigma(\Delta^s)](x,\xi)}{\mu(x)}, \tag{5}$$

where μ is a smooth volume form on M and Δ is the nonnegative Laplace operator on M.

2.2. The Trajectory Symbol for Orbits Going from the Manifold and not Reaching the Boundary

For definiteness, suppose that $x \in U = M \setminus (X \cup g(M))$. The orbit $g^n x$, $n \geq 0$, of this point is isomorphic to the set $\mathbb{Z}_+ = \{n \in \mathbb{Z} | n \geq 0\}$. Localization on this orbit is performed in the same way as in the preceding section with taking into account the fact that the symbol

$$\sigma(D)(x,\xi)\colon\ l^2(\mathbb{Z}_+,\mu_{x,\xi,s}) \to l^2(\mathbb{Z}_+,\mu_{x,\xi,s-m}), \quad x \in U,$$

acts on spaces of one-sided sequences. As previously, the symbol and the corresponding weights are determined by (4) and (5).

2.3. The Trajectory Symbol for Orbits Reaching the Boundary

For definiteness, we assume that $x \in X$. A straightforward calculation shows that freezing the coefficients of the problem (D, B) at the points of the orbit $g^n x$, $n \geq 0$, and the Fourier transform $x \mapsto \xi$ yield an operator acting on the spaces

$$\begin{matrix} \Pi_+ L^2(T_x^*M,\mu\sigma(\Delta^s)) \\ \oplus \\ \bigoplus_{n\geq 1} L^2(T_{g^n x}^*M,\mu\sigma(\Delta^s)) \end{matrix} \to \begin{matrix} \Pi_+ L^2(T_x^*M,\mu\sigma(\Delta^{s-m})) \\ \oplus \\ \bigoplus_{n\geq 1} L^2(T_{g^n x}^*M,\mu\sigma(\Delta^{s-m})) \\ \oplus \\ L^2(T_x^*X,\mu_X\sigma(\Delta_X^{s-b})). \end{matrix} \tag{6}$$

Let us define the objects involved in this formula. In (6), μ_X is the volume form on X, Δ_X is the nonnegative Laplace operator on X, and L^2 denotes L^2-spaces of square integrable functions on the corresponding cotangent spaces with respect to the specified measures. It remains to define the projection Π_+. In a neighborhood of a point $x \in X$, we choose coordinates (x', t) so that the manifold M is determined by the inequality $t \geq 0$; by

$$\Pi_+\colon\ L^2(T_x^*M,\mu\sigma(\Delta^s)) \to L^2(T_x^*M,\mu\sigma(\Delta^s))$$

we denote the Fourier image of the projection onto the subspace of functions identically vanishing at $t < 0$. Next, we transform operator (6) by applying the isometric isomorphisms

$$\begin{aligned} L^2(T_{g^n x}^*M,\mu\sigma(\Delta^s)) &\overset{(\partial g^n)^*}{\simeq} L^2(T_x^*M,(\partial g^n)^*(\mu\sigma(\Delta^s))) \\ &\simeq L^2(T_x^*X, L^2(\mathbb{R},\mu_{x,\eta,s})), \end{aligned} \tag{7}$$

where $\mu_{x,\eta,s}$ denotes the weight

$$\mu_{x,\eta,s}(\tau,n) = \frac{(\partial g^n)^*(\mu\sigma(\Delta^s))}{\mu},$$

and $L^2(T_x^*X, L^2(\mathbb{R},\mu_{x,\eta,s}))$ denotes the L^2 space of functions on T_x^*X taking values in the space $L^2(\mathbb{R},\mu_{x,\eta,s})$. In the last isomorphism, we have used the decomposition $T_x^*M \simeq T_x^*X \times \mathbb{R}$. Taking into account isomorphism (7), we see that operator (6) can be represented in terms of the family of operators of multiplication by the operator function

$$\sigma(D,B)(x,\eta):\quad \begin{matrix}\pi_+L^2(\mathbb{R},(1+|\tau|^2)^s)\\ \oplus\\ L^2(\mathbb{R}\times\mathbb{Z}_{>0},\mu_{x,\eta,s})\end{matrix}\to\begin{matrix}\pi_+L^2(\mathbb{R},(1+|\tau|^2)^{s-m})\\ \oplus\\ L^2(\mathbb{R}\times\mathbb{Z}_{>0},\mu_{x,\eta,s-m})\\ \oplus\\ \mathbb{C},\end{matrix}\tag{8}$$

which depends on the variables η and ranges in the space of operators acting on the variables τ. Here,

$$\pi_+:\ L^2(\mathbb{R},(1+|\tau|^2)^s)\to L^2(\mathbb{R},(1+|\tau|^2)^s)$$

is the projection onto the Fourier image of the space of functions identically vanishing at $t<0$ and $\mathbb{Z}_{>0}=\{n\in\mathbb{Z}\mid n\ge 1\}$. Applying the inverse Fourier transform $\tau\mapsto t$ to (8), we obtain the operator function

$$\sigma(D,B)(x,\eta):\quad \begin{matrix}H^s(\mathbb{R}_+)\\ \oplus\\ H^s(\mathbb{R}\times\mathbb{Z}_{>0},\mu_{x,\eta,s})\end{matrix}\longrightarrow\begin{matrix}H^{s-m}(\mathbb{R}_+)\\ \oplus\\ H^{s-m}(\mathbb{R}\times\mathbb{Z}_{>0},\mu_{x,\eta,s-m})\\ \oplus\\ \mathbb{C},\end{matrix}\tag{9}$$

which is called the trajectory symbol of the boundary value problem and denoted by $\sigma(D,B)(x,\eta)$. A straightforward calculation shows that symbol (9) acts as

$$(u_0,u_1,u_2,\ldots)\overset{\sigma(D,B)(x,\eta)}{\longmapsto}(v_0,v_1,v_2,\ldots,w),$$

$$v_n=\sum_{k\le n}[(\partial g^{n-k})^*\sigma(D_k)]\left(x,\eta,-i\frac{d}{dt}\right)u_{n-k},\quad n\ge 0,$$

$$w=\sigma(A)(x,\eta)$$

$$\times\left(\pi_+\sum_{k\le 0}[(\partial g^{-k})^*\sigma(C_k)]\left(x,\eta,-i\frac{d}{dt}\right)u_{-k}\right)\Big|_{t=0}.$$

3. THE FINITENESS THEOREM

Definition 2. We say that the problem (D,B) is elliptic if the trajectory symbol $\sigma(D)(x,\xi)$ is invertible at any $(x,\xi)\in T_0^*M$ (see Sections 2.1 and 2.2) and the trajectory symbol $\sigma(D,B)(x,\eta)$ is invertible at any $(x,\eta)\in T_0^*X$ (see Section 2).

The following theorem is the main result of this paper.

Theorem 1. *An elliptic problem is Fredholm in the corresponding Sobolev spaces.*

Remark 1. If the shift operators are absent in (3), i.e., $D_k=C_k=0$ for $k\ne 0$, then the ellipticity condition in Definition 2 is equivalent to the requirement that the main operator D is elliptic and the Shapiro–Lopatinskii condition holds on the boundary of the manifold.

4. EXAMPLE

Consider the boundary value problem

$$(D,B)=((aT+b+cT^{-1})\Delta i^*):H^s(M)\to\to H^{s-2}(M)\oplus H^{s-1/2}(X),\tag{10}$$

where M is the unit ball in R^k centered at the origin, $X=\partial M$ is the unit sphere, and the contraction and expansion operators T and T^{-1} correspond to the contraction $x\mapsto qx$, $0<q<1$. Finally, a,b,c are some numbers. The boundary value problem (10) is considered for $|s-2|<1/2$.

Applying Theorem 1, we shows that problem (10) is Fredholm if precisely one root of the equation $az^2+bz+c=0$ lies inside the disk $|z|=q^{\dim M-2(s-2)}$.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research, project nos. 15-01-08392 and 16-01-00373.

REFERENCES

1. A. Antonevich and A. Lebedev, *Functional-Differential Equations, 1: C*-Theory* (Longman, Harlow, 1994).
2. L. E. Rossovskii, Tr. Mosk. Mat. O-va **62**, 199–228 (2001).
3. A. V. Bitsadze and A. A. Samarskii, Dokl. Akad. Nauk SSSR **185** (4), 739–740 (1969).
4. A. Yu. Savin, Differ. Equations **47** (6), 890–893 (2011).
5. A. Savin and B. Sternin, in *Pseudo-Differential Operators, Generalized Functions and Asymptotics* (Birkhäuser, Basel, 2013), pp. 1–26.