ISSN 1064-5624, Doklady Mathematics, 2016, Vol. 94, No. 1, pp. 365–368. © Pleiades Publishing, Ltd., 2016.
Original Russian Text © A.P. Krishchenko, 2016, published in Doklady Akademii Nauk, 2016, Vol. 469, No. 1, pp. 17–20.

MATHEMATICS

Asymptotic Stability Analysis of Autonomous Systems by Applying the Method of Localization of Compact Invariant Sets

Corresponding Member of the RAS A. P. Krishchenko

Received March 10, 2016

Abstract—The asymptotic stability and global asymptotic stability of equilibria in autonomous systems of differential equations are analyzed. Conditions for asymptotic stability and global asymptotic stability in terms of compact invariant sets and positively invariant sets are proved. The functional method of localization of compact invariant sets is proposed for verifying the fulfillment of these conditions. Illustrative examples are given.

DOI: 10.1134/S1064562416040025

1. INTRODUCTION

Stability analysis is an important problem in the theory of differential equations. Beginning with Lyapunov's work [1], this problem has been extensively studied in various aspects [2–4] and still remains topical (see [5, 6], etc.) One of the causes is that the Lyapunov function method is usually invoked when techniques based on the first approximation fail. However, in that case, a Lyapunov function has to be constructed, which is a complicated problem [3].

It is well known that an important role in stability theory is played by the invariance principle [7] and positively invariant sets are frequently used in the study of nonlinear systems. In Sections 2 and 5 below, it is shown that necessary and sufficient conditions for the asymptotic stability and global asymptotic stability of equilibria of an autonomous system can be formulated in terms of compact invariant sets and positively invariant sets. In Section 3, the fulfillment of these conditions is verified by applying the functional method of localization of compact invariant sets. An overview of this method is also presented in Section 3. Examples are given in Section 4.

2. CONDITIONS FOR ASYMPTOTIC STABILITY

Given an autonomous system

$$\dot{x} = f(x), \quad f \in C^1(\mathbf{R}^n), \tag{1}$$

the following necessary and sufficient conditions for the asymptotic stability and global asymptotic stability of its equilibria are valid.

Theorem 1. *An equilibrium is asymptotically stable if and only if the following conditions hold:*

(i) *In some of its neighborhoods, the equilibrium is a unique compact invariant set of the system.*

(ii) *Any neighborhood of the equilibrium contains a bounded positively invariant neighborhood of this equilibrium.*

Theorem 2. *An equilibrium is globally asymptotically stable if and only if the following conditions hold:*

(i) *The equilibrium is asymptotically stable.*

(ii) *The equilibrium is a unique compact invariant set of the system.*

(iii) *As $t \to +\infty$, any trajectory of the system reaches a bounded positively invariant set.*

The proofs of Theorems 1 and 2 are presented in Section 4 and are based on inversion theorems and the properties of ω-limit sets of trajectories [8] and localizing sets for compact invariant sets [9] of system (1).

Below is a consequence of Theorems 1 and 2.

Theorem 3. *An equilibrium is globally asymptotically stable if and only if the following conditions hold:*

(i) *The equilibrium is a unique compact invariant set of the system.*

(ii) *Any neighborhood of the equilibrium contains a bounded positively invariant neighborhood of this equilibrium.*

(iii) *As $t \to +\infty$, any trajectory of the system reaches a bounded positively invariant set.*

Bauman Moscow State Technical University,
Vtoraya Baumanskaya ul. 5, Moscow, 105005 Russia
e-mail: apkri@bmstu.ru

3. FUNCTIONAL METHOD OF LOCALIZATION

To analyze the stability of an equilibrium with the help of Theorems 1–3, we need the following results concerning the functional method of localization of compact invariant sets of system (1) [9–11].

Any function $\varphi \in C^1(\mathbf{R}^n)$ is associated with the set

$$S(\varphi) = \{x \in \mathbf{R}^n\colon \dot{\varphi}(x) = 0\},$$

which is called a universal cross section. Let

$$\varphi_{\inf} = \inf_{S(\varphi)} \varphi(x), \quad \varphi_{\sup} = \sup_{S(\varphi)} \varphi(x).$$

Theorem 4 [9, 10]. *All compact invariant sets of an autonomous system are contained in the set*

$$\Omega(\varphi) = \{x \in \mathbf{R}^n\colon \varphi_{\inf} \le \varphi(x) \le \varphi_{\sup}\}.$$

The following result holds for compact invariant sets of system (1) contained in a subset $Q \subset \mathbf{R}^n$.

Theorem 5 [9, 10]. *All compact invariant sets of the system contained in a subset $Q \subset \mathbf{R}^n$ are contained in the set*

$$\Omega(\varphi, Q) = \{x \in Q\colon \varphi_{\inf}(Q) \le \varphi(x) \le \varphi_{\sup}(Q)\}, \quad (2)$$

where

$$\varphi_{\inf}(Q) = \inf_{S(\varphi,Q)} \varphi(x), \quad \varphi_{\sup}(Q) = \sup_{S(\varphi,Q)} \varphi(x),$$

and

$$S(\varphi, Q) = \{x \in Q\colon \dot{\varphi}(x) = 0\}.$$

If, for example, $\varphi_{\inf}(Q) = \inf_{S(\varphi,\, Q)}\varphi(x) = \inf_Q \varphi(x)$, then the left inequality in (2) holds for all $x \in Q$, so it can be dropped.

The sets $\Omega(\varphi)$ and $\Omega(\varphi, Q)$ are called the localizing sets corresponding to the function φ, while any sets containing all compact invariant sets of the system are referred to merely as localizing sets. Note that, if $Q = \mathbf{R}^n$, then, for any localizing function φ, we have $\Omega(\varphi, \mathbf{R}^n) = \Omega(\varphi)$.

Localizing sets indicate subsets of the state space in which the system has trajectories with complex, in particular, chaotic behavior, while outside localizing sets, the behavior of any trajectory is described relatively easily [12]. Among the properties of localizing sets and universal sections, we consider those that are important for analyzing the asymptotic stability and global asymptotic stability of equilibria.

Let a localizing set $\Omega(\varphi)$ be nontrivial, i.e., $\Omega(\varphi) \ne \mathbf{R}^n$, $x_0 \notin \Omega(\varphi)$, and $c = \varphi(x_0)$. The set $\varphi_c = \{x \in \mathbf{R}^n : \varphi(x) = c\}$ is a semipermeable C^1-manifold of dimension $n - 1$. The trajectories of the system intersect this manifold transversally, and it is the interface between the sets $\varphi_c^+ = \{x \in \mathbf{R}^n\colon \varphi(x) > c\}$ and $\varphi_c^- = \{x \in \mathbf{R}^n\colon \varphi(x) < c\}$.

If $c > \varphi_{\sup}$ and $\dot{\varphi} > 0$ on the set φ_c, then φ_c^+ is positively invariant, while, if $c > \varphi_{\sup}$ и $\dot{\varphi} < 0$ on φ_c, then φ_c^- is positively invariant and contains $\Omega(\varphi)$.

For $c < \varphi_{\inf}$, the properties of the sets $\varphi_c^{\pm}$ are similar.

Thus, each component of the boundary $\partial\hat{\Omega}(\varphi)$ of the set

$$\hat{\Omega}(\varphi) = \{x \in \mathbf{R}^n\colon \varphi_{\inf} - \tau < \varphi(x) < \varphi_{\sup} + \tau\}, \quad (3)$$

where $\tau > 0$, is always semipermeable for trajectories of the system. From this point of view, set (3) is positively invariant only in the case depending on the signs of $\dot{\varphi}(x)$ on the connected components of $\partial\hat{\Omega}(\varphi)$, while, in the other cases, the system has no compact invariant set attracting all its trajectories.

Theorem 6. *If an equilibrium is globally asymptotically stable, then, for any localizing function φ, the localizing set $\Omega(\varphi)$ is positively invariant.*

Applying Theorem 5 sequentially with various localizing functions, we can find new localizing sets.

Theorem 7 [9, 10]. *Given a set $Q \subset \mathbf{R}^n$ and a sequence of localizing functions $h_i \in C^1(Q)$, $i = 1, 2, \ldots$, all compact invariant sets of the system contained in Q are contained in the sets*

$$K_0 = Q, \quad K_i = \Omega(h_i, K_{i-1}), \quad i > 0. \quad (4)$$

The localizing sets K_i are such that $Q \supseteq K_1 \supseteq K_2 \supseteq \ldots$.

The construction of localizing sets K_i in Theorem 7 is called an iterative procedure.

By an extended iterative procedure, we mean the construction of localizing sets $\hat{K}_i$ (open in $\mathbf{R}^n$) according to the following theorem.

Theorem 8. *Given a set $Q \subset \mathbf{R}^n$, a neighborhood $\hat{Q}$ of Q, a sequence of localizing functions $h_i \in C^1(Q)$ ($i = 1, 2, \ldots$), and a sequence of nonnegative numbers τ_i ($i = 1, 2, \ldots$), all compact invariant sets of the system contained in Q are contained in the sets $\hat{K}_0 = \hat{Q}$ and $\hat{K}_i$, $i > 0$,*

$$\hat{K}_i = \hat{\Omega}(h_i, \hat{K}_{i-1}) = \{x \in \hat{K}_{i-1}\colon h_{i,\inf}(\hat{K}_{i-1}) - \tau_i < h_i(x) < h_{i,\sup}(\hat{K}_{i-1}) + \tau_i\},$$

$$h_{i,\inf}(\hat{K}_{i-1}) = \inf_{S(h_i, \hat{K}_{i-1})} h_i(x),$$

$$h_{i,\sup}(\hat{K}_{i-1}) = \sup_{S(h_i, \hat{K}_{i-1})} h_i(x),$$

$S(h_i, \hat{K}_{i-1}) = \{x \in \hat{K}_{i-1}\colon \dot{h}_i(x) = 0\}$. *All localizing sets $\hat{K}_i$, $i = 0, 1, \ldots$, are open in $\mathbf{R}^n$ and $\hat{Q} \supseteq \hat{K}_1 \supseteq \hat{K}_2 \supseteq \ldots$.*

The proof of Theorem 8 follows from the fact that $K_i \subset \hat{K}_i$ for $i \ge 0$, so the sets $\hat{K}_i$ also contain all compact invariant sets contained in Q.

The boundary $\partial\hat{K}_i$ of the set $\hat{K}_i$ consists of points of the sets $\partial\hat{Q}$,

$$\{x \in \hat{K}_{j-1}\colon\ h_j(x) = h_{j,\inf}(\hat{K}_{j-1}) - \tau_j\}$$

and

$$\{x \in \hat{K}_{j-1}\colon\ h_j(x) = h_{j,\sup}(\hat{K}_{j-1}) + \tau_j\},$$

$j = 1, 2, \ldots, i$. If $\partial\hat{K}_i \cap \partial\hat{Q} = \varnothing$, then the positive invariance of $\hat{K}_i$ and the absence of a compact invariant set attracting all trajectories of the system depend on the signs of the derivatives $\dot{h}_j$ on the corresponding connected components of the boundary of $\hat{K}_i$.

The above results allow us to check whether or not the conditions of Theorems 1 and 2 hold. For example, an equilibrium P is a unique compact invariant set of the system contained in Q if there is a sequence of localizing sets (4) contracting to the point P, i.e., $\bigcap_{i=0}^{i=\infty} K_i = P$. The second condition of Theorem 1 and the third condition of Theorem 2 can be verified with the help of the sequence of localizing sets from Theorem 8. In some cases, it may happen that it suffices to consider only a few terms of these sequences.

In a similar manner, we can prove that a positively invariant set is contained in the basin of attraction of an asymptotically stable equilibrium [13].

4. EXAMPLES

As an example, consider the two-dimensional system

$$\begin{gathered} \dot{x} = a(x,y), \quad \dot{y} = b(x,y), \\ a(x,y), \quad b(x,y) \in C^1(\mathbf{R}^2) \end{gathered} \tag{5}$$

with equilibrium $O(0, 0)$. Assume that, in the phase plane of system (5), there is a line $L = \{y = lx, l > 0\}$ such that the sets $A = \{a(x, y) = 0\}$ and $B = \{b(x, y) = 0\}$ intersect L only at the point $O(0, 0)$; the set $A \backslash O(0, 0)$ is contained in the set

$$X = \{x \geq 0, y > lx\} \cup \{x \leq 0, y < lx\},$$

which is bounded by L and the Oy axis; and the set $B \backslash O(0, 0)$ is contained in the similar set

$$Y = \{x > 0, 0 \leq y < lx\} \cup \{x < 0, 0 \geq y > lx\}.$$

Let us show that the only compact invariant set of system (5) is the point O. Indeed, consider the set

$$Q(p) = \{|x| < p, |y| < lp\}. \tag{6}$$

If there exists another compact invariant set, then, at some $p = p_0$, it is contained in set (6). Starting from an arbitrary set $Q = Q(p)$, we construct iteration sequence (4) corresponding to the localizing functions

$$h_{2m-1} = x, \quad h_{2m} = y, \quad m = 1, 2, \ldots.$$

As a result, we obtain a sequence of localizing sets K_n, $n = 0, 1, \ldots$. It is easy to prove that, for any positive number p_1, the set $Q(p_1)$ contains some localizing set K_n. Therefore, the sequence of sets K_n contracts to the point O, which, hence, is the only compact invariant set in Q.

The asymptotic stability of an equilibrium of system (5) depends on the signs of $a(x, y)$ and $b(x, y)$ in Y and X, respectively. If $xa(x, y) < 0$ in Y and $yb(x, y) < 0$ in X, then the zero position of the system is asymptotically stable and globally asymptotically stable. The last conclusion follows from Theorems 1 and 2, since, in this case, the bounded neighborhood (6) of the zero equilibria is positively invariant.

Remark. The results that the constructed iteration sequence contracts to the zero equilibrium and neighborhood (6) is positively invariant follow only from the inclusions $A \backslash O(0, 0) \subset X$ and $B \backslash O(0, 0) \subset Y$ and from the fact that the signs of the right-hand sides of the system satisfy the above-indicated condition, whereas the structure of A and B is of no importance. It is easy to see that these properties are preserved in the following more general case of sets X and Y. Let $\gamma(u)$, $u \in [0, +\infty)$, $\gamma(0) = (0, 0)$ be a continuous curve in the first quarter that issues from the origin, and let the coordinates $x(u)$ and $y(u)$ of a point on this curve be nondecreasing functions. Extend γ to a plane curve Γ that is symmetric about the coordinate axes. Then, as a set X (Y), we can use connected components of $\mathbf{R}^2 \backslash \Gamma$ that contain the points of the vertical axis (horizontal axis).

5. PROOFS OF THEOREMS 1 AND 2

Assume without loss of generality that an equilibrium of system (1) is the point $x = 0$.

If the equilibrium is asymptotically stable or globally asymptotically stable, then, according to the inversion theorem [8], in its basin of attraction R_A there exists a positive definite function $V(x) \in C^1(R_A)$ that grows to infinity as $x \to \partial R_A$ on R_A and is such that its derivative along the solutions of the system is a negative definite function in R_A. Therefore, for any $c > 0$, the set $\{x\colon V(x) < c\}$ is bounded, open, and positively invariant and any trajectory of the system hits this set. Therefore, the second condition of Theorem 1 and the third condition of Theorem 2 are satisfied.

The first condition of Theorem 1 and the second condition of Theorem 2 are also satisfied, since, if $\varphi = V(x)$ is used as a localizing function, then the set $S(V, R_A) = \{x \in R_A : \dot{V}(x) = 0\}$ coincides with the equilibrium $x = 0$ due to the fact that $\dot{V}(x)$ is negative definite. Therefore,

$$V_{\inf}(R_A) = \inf_{S(V,R_A)} V(x) = V(0) = 0,$$

$$V_{\sup}(R_A) = \sup_{S(V,R_A)} V(x) = V(0) = 0$$

and, by Theorem 5, all compact invariant sets of the system contained in R_A are contained in the localizing set $\Omega(V, R_A) = \{x \in R_A: V(x) = 0\}$, which coincides with the zero equilibrium, since $V(x)$ is positive definite.

The necessity of the conditions of Theorems 1 and 2 is proved.

Let us prove the sufficiency of the conditions of Theorem 1.

Suppose that, in a neighborhood U, the point $x = 0$ is a unique compact invariant set. For any neighborhood O of $x = 0$, consider a neighborhood $\tilde{O}$ of $x = 0$ that, together with its closure, is contained in the set $O \cap U$. According to the second condition of Theorem 1, $\tilde{O}$ contains a bounded positively invariant neighborhood W of $x = 0$. Any trajectory of the system beginning in W remains within W and, hence, remains within O, which means that the equilibrium $x = 0$ is stable. All such trajectories are bounded and tend to their ω-limit sets, which coincide with $x = 0$. This follows from the fact that the ω-limit sets of these bounded trajectories are compact invariant sets contained in U, which coincide with $x = 0$ according to the first condition of the theorem. Therefore, the equilibrium $x = 0$ is asymptotically stable and Theorem 1 is proved.

Let us prove the sufficiency of the conditions of Theorem 2.

Let $x_0 \in \mathbf{R}^n$, and let $x(t)$, $t \geq 0$, with $x(t)|_{t=0} = x_0$ be a trajectory of the system. Suppose that, as $t \to +\infty$, this trajectory hits a bounded positively invariant set at some time. Then this trajectory is bounded for $t \to +\infty$ and its ω-limit set is compact and, by the second condition of the theorem, coincides with $x = 0$. Therefore, the trajectory $x(t)$ tends to $x = 0$ as $t \to +\infty$. Since x_0 is an arbitrary point of $\mathbf{R}^n$, the basin of attraction of $x = 0$ coincides with $\mathbf{R}^n$. This means that the equilibrium $x = 0$ is globally asymptotically stable, since it is asymptotically stable according to the first condition of the theorem. Theorem 2 is proved.

ACKNOWLEDGMENTS

This work was supported by the Ministry of Education and Science of the Russian Federation (state work no. 736 "Organization of Scientific Research" and project no. 1.644.2014/K) and by the Russian Foundation for Basic Research (project nos. 14-01-00424 and 15-07-06484).

REFERENCES

1. A. M. Lyapunov, *The General Problem of the Stability of Motion* (Gostekhizdat, Moscow, 1950; Taylor and Francis, London, 1992).
2. N. N. Krasovskii, Mat. Sb. **40** (1), 57–64 (1956).
3. L. Yu. Anapol'skii, V. D. Irtegov, and V. M. Matrosov, "Methods for constructing Lyapunov functions," in *Advances in Science and Technology: General Mechanics* (VINITI, Moscow, 1975), Vol. 2, p. 53–112 [in Russian].
4. V. V. Rumyantsev, Differ. Uravn. **19** (5), 739–776 (1983).
5. J. Björnsson, P. Giesl, S. F. Hafstein, and C. M. Kellett, Discrete Continuous Dyn. Syst. **35** (9), 4019–4039 (2015).
6. A. I. Barkin, Autom. Remote Control **76** (7), 1156–1168 (2015).
7. J. P. LaSalle, IRE Trans. Circuit Theory **7**, 520–527 (1960).
8. H. K. Khalil, *Nonlinear Systems* (Prentice Hall, New Jersey, 2002).
9. A. P. Krishchenko, Differ. Equations **41** (12), 1669–1676 (2005).
10. A. P. Krishchenko, Differ. Equations **31** (11), 1826–1833 (1995).
11. A. P. Krishchenko, Comput. Math. Appl. **34** (2–4), 325–332 (1997).
12. A. P. Krishchenko, Differ. Equations **51** (11), 1432–1439 (2015).
13. A. P. Krishchenko and K. E. Starkov, Commun. Nonlin. Sci. Numer. Simul. **33** (4), 174–183 (2016).

Translated by I. Ruzanova