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MATHEMATICS

On Admissible Changes of Variables for Sobolev Functions on (sub)Riemannian Manifolds[1]

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Abstract—We give a description of metric properties of measurable mappings of domains on Riemannian manifolds inducing isomorphisms of Sobolev spaces by the composition rule. We prove that any such mapping can be redefined on a set of measure zero to be quasi-isometric, when the exponent of summability is different from the dimension of a Riemannian manifold or to coincide with a quasi-conformal mapping otherwise.

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The present work can be seen as a natural continuation of studies started in [1–5] and dedicated to solving the following problem: what metric and analytic properties has a measurable mapping $\varphi\colon D \to D'$ inducing an isomorphism $\varphi^*\colon L^1_p(D') \to L^1_p(D)$ of Sobolev spaces by the composition rule: $\varphi^*(f) = f \circ \varphi$ (here D, D' are open connected sets (further domains) either in the Euclidean space $\mathbb{R}^n$, $n \geq 2$ (see [1–4]) or on a Carnot group (see [5])). In the mentioned papers there were obtained various proofs of the fact that an isomorphism of the operator φ^* implies, according to a relation between the indices of smoothness, summability and the dimension of the space, the property of mapping to be quasi-conformal or quasi-isometric with respect to a metric of domain adequate to the geometry of a function space.

Here we give a scetch for solving a similar problems for measurable mappings of domains on a Riemannian manifold which induce isomorphisms of Sobolev spaces with the generalized first derivatives. The used methods are based up on methods of papers [3, 4] which were successfully applied in [5] in solving the problem on the composition of Sobolev functions on Carnot groups. In papers [3–5] it was provided also a detailed history of the problem under consideration and an itemized bibliography.

1. Next we fix a complete Riemannian manifold $\mathbb{M} = (\mathbf{M}, g)$, $n = \dim_{\text{top}} \mathbf{M} \geq 2$, namely a smooth manifold $\mathbf{M}$, in every tangent space $T_x\mathbf{M}$ of which it is chosen an Euclidean metric g_x changing smoothly from a point to a point. The length of an absolutely continuous curve $\gamma\colon [a, b] \to \mathbb{M}$ is expressed by the integral $l(\gamma) = \int_a^b |\dot{\gamma}(t)|\, dt$. Here, $|\dot{\gamma}(t)| = \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}$ is the length of the tangent vector $\dot{\gamma}(t)$ in the Euclidean space $T_{\gamma(t)}\mathbb{M}$ with the scalar product $g_{\gamma(t)}$. The metric $d(x, y)$ on the Riemannian manifold $\mathbb{M}$ is defined as the infimum of the lengths of absolutely continuous curves with endpoints x and y.

2. Sobolev classes on $\mathbb{M}$. Let D be an open connected domain in the Riemannian manifold $\mathbb{M}$. Sobolev space $L^1_p(D)$ consists of locally integrable functions $f\colon D \to \mathbb{R}$ having the distribution gradient $\nabla f \in L_p(D)$. The semi-norm in $L^1_p(D)$ is defined as the value

$$\|f \mid L^1_p(D)\| = \|\nabla f \mid L_p(D)\| = \left(\int_D |\nabla f(x)|^p d\omega\right)^{\frac{1}{p}},$$

where $d\omega$ is an element of the n-dimensional volume, $\nabla f(x)$ is the distribution gradient of the function f at a point $x \in D$, and $|\nabla f(x)|$ is the length of the distribution gradient $\nabla f(x)$ in the Euclidean space $T_x\mathbb{M}$ with the scalar product g_x.

We will say (see [6]) that the mapping $\varphi\colon D \to \mathbb{M}$ belongs to the class $W^1_{p,\text{loc}}(D; \mathbb{M})$ if the following conditions hold.

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(A) For every $z \in \mathbb{M}$, the function $[\varphi]_z$: $D \ni x \mapsto d(\varphi(x), z)$ belongs to $W^1_{p,\,\mathrm{loc}}(D)$.

(B) The family of generalized gradients $(\nabla[\varphi]_z)_{z \in \mathbb{M}}$ has a majorant belonging to $L_{p,\,\mathrm{loc}}(D)$, i.e., there is a function $g \in L_{p,\,\mathrm{loc}}(D)$, which is independent of z, such that $|\nabla[\varphi]_z(x)| \le g(x)$ for almost all $x \in D$.

One can verify: the mapping φ: $D \to \mathbb{M}$ belongs to the Sobolev class $W^1_{p,\,\mathrm{loc}}(D)$ if and only if it can be changed on a set of measure zero so that

(1) for every $z \in \mathbb{M}$, the function $[\varphi]_z$: $D \ni x \mapsto d(\varphi(x), z)$ belongs to the class $L_{p,\,\mathrm{loc}}(D)$;

(2) for any countable set of domains $U_i \Subset D$, $i \in \mathbb{N}$, constituting a topological base of neighborhoods of the domain D, with a local basis X_j, $j = 1, 2, \ldots, n$, of the tangent bundle over U_i, the mapping φ: $D \to \mathbb{M}$ is absolutely continuous on almost all integral lines of the vector fields X_j, $j = 1, 2, \ldots, n$ ($\varphi \in ACL(D)$);

(3) the derivative $X_j\varphi(x)$ is defined almost everywhere in U_i, $i \in \mathbb{N}$, and $|X_j\varphi| \in L_p(U_i)$ for all $j = 1, 2, \ldots, n$.

We denote by $D\varphi$: $T_x\mathbb{M} \to T_{\varphi(x)}\mathbb{M}$ the approximative differential of the mapping φ [7], which is defined almost everywhere in D, and denote by $J(x, \varphi)$ its Jacobian determinant $\det D\varphi$.

Definition 1. A homeomorphism Φ: $D \to D'$ of two domains D, $D' \subset M$ of the class $W^1_{n,\,\mathrm{loc}}(D)$ is called quasiconformal if there is a constant K such that $|D\Phi(x)|^n \le K|J(x, \Phi)|$ almost everywhere in D.

Definition 2. A homeomorphism Φ: $D \to D'$ of two domains D, $D' \subset \mathbb{M}$ of the class $W^1_{1,\,\mathrm{loc}}(D)$ is called quasi-isomeric if $|D\Phi(x)| \le M$ and $0 < \alpha \le |J(x, \Phi)|$ for almost all $x \in D$ where the constants M and α are independent of x.

Definition 3. Two domains D, $D' \subset \mathbb{M}$ are called $(1, p)$-equivalent if the restriction operators r_i: $L^1_p(D_1 \cup D_2) \to L^1_p(D_i)$, $r_i(f) = f|_{D_i}$, where $f \in L^1_p(D_1 \cup D_2)$, are isomorphisms.

The properties of $(1, p)$-equivalent domains in Euclidean space were studied in [8].

3. Composition operators and mappings of the class IL^1_p. Similarly to the paper [5] we introduce the main object of our study: the class IL^1_p of mappings on a Riemannian manifold.

Definition 4. Let D, D' be domains in the Rimannian manifold $\mathbb{M}$. A measurable mapping φ: $D \to D'$ belongs to the class IL^1_p, $p \in [1, \infty)$, if φ induces a composition operator of Sobolev spaces

$$\begin{gathered}\varphi^*\colon L^1_p(D') \cap \mathrm{Lip}_{\mathrm{loc}}(D') \to L^1_p(D),\\ \varphi^*(f) = f \circ \varphi, \quad f \in L^1_p(D') \cap \mathrm{Lip}_{\mathrm{loc}}(D'),\end{gathered} \tag{1}$$

such that

(1) the inequalities $K^{-1}\|f|L^1_p(D')\| \le \|\varphi^*(f) \mid L^1_p(D)\| \le K\|f|L^1_p(D')\|$ hold for any function $f \in L^1_p(D') \cap \mathrm{Lip}_{\mathrm{loc}}(D')$ where the constant K is independent of a function f (here $\mathrm{Lip}_{\mathrm{loc}}(D')$ is the totality of Lipschitz functions on a domain D');

(2) the image $\varphi^*(L^1_p(D') \cap \mathrm{Lip}_{\mathrm{loc}}(D'))$ is dense in $L^1_p(D)$.

In the paper [6] it was shown that the second condition of this definition is independent of the first one.

4. Case $p \neq n$. A complete description of measurable mappings of domains on the Riemannian manifold $\mathbb{M}$, inducing isomorphisms of Sobolev spaces L^1_p in the sense of Definition 5, when $p \neq n$, is given below.

Theorem 1. *Let $p \in [1, \infty)\setminus\{n\}$, and D, D' be domains on the Riemannian manifold $\mathbb{M}$ (here n is a topological dimension of $\mathbb{M}$). A measurable mapping φ: $D \to D'$ belongs to the class IL^1_p if and only if φ coincides with some quasi-isometry Φ: $D \to \Phi(D)$ almost everywhere, for which the domains $\Phi(D)$ and D' are $(1, p)$-equivalent.*

5. Case $p = n$. The main result, when $p = n$, is

Theorem 2. *Let $p = n$, and D, D' be domains on the Riemannian manifold $\mathbb{M}$ of topological dimension n. A measurable mapping φ: $D \to D'$ belongs to the class IL^1_n if and only if φ coincides with some quasiconformal mapping Φ: $D\setminus\Sigma \to \mathbb{M}$ almost everywhere (here $\Sigma \subset D$ is some closed set in D of zero capacity in the space W^1_n), for which the domains $\Phi(D\setminus\Sigma)$ and D' are $(1, n)$-equivalent.*

6. For mappings belonging to IL^1_p, the following properties are valid.

Proposition 1. (1) *The domain of a mapping $\varphi \in IL^1_p$, $p \in [1, \infty)$, can be reduced to the set $T = \bigcup_k T_k$, $|D\setminus T| = 0$, where $\{T_k\}$ is an increasing, by inclusion, sequence of bounded sets of positive measure, consisting of points of positive density.*

(2) *The mapping φ is continuous on every T_k.*

(3) *On the set T, Luzin property $\mathcal{N}$ and $\mathcal{N}^{-1}$-Luzin property hold for the mapping φ.*

(4) *The mapping φ: $T \to D'$ is injective.*

(5) *The image $\varphi(T)$ is dense in D', and $|D' \setminus \varphi(T)| = 0$.*

Operator (1) is extended to $L^1_p(D)$ with preserving properties of composition operator.

Lemma 1. *Let a measurable mapping φ: $D \to D'$ belong to IL^1_p. Then the operator φ^*: $L^1_p(D') \cap \mathrm{Lip}_{\mathrm{loc}}(D') \to L^1_p(D)$ is extended by continuity to the* *operator* $\widetilde{\varphi}^*: L_p^1(D') \to L_p^1(D)$ *with the following properties:*

(1) *the value of the operator* $\widetilde{\varphi}^*: L_p^1(D') \to L_p^1(D)$ *on classes* $[f] \in L_p^1(D')$ *can be found by formula:*

$$\widetilde{\varphi}^*([f]) = \begin{cases} g \circ \varphi \ \text{ for } p \le n, \text{ where} \\ f \ \text{ is an arbitrary representative of the class } [f], \\ \tilde{f} \circ \varphi \ \text{ for } p > n, \text{ where} \\ \tilde{f} \ \text{ is a continuous representative of the class } [f]; \end{cases}$$

(2) $K^{-1}\|f|L_p^1(D') \le \|\widetilde{\varphi}^*(f)|L_p^1(D)\| \le K\|f|L_p^1(D')\|$;

(3) $\widetilde{\varphi}^*: L_p^1(D') \to L_p^1(D)$ *is an isomorphism.*

7. Case $p > n$. Recall that a mapping $\varphi: U \to \mathbb{M}$, $U \subset \mathbb{M}$, is called locally bilipschitz, if, for every point $x \in U$, there are a neighborhood $V \subset U$ and a constant L_V for which the relations $L_V^{-1} d(y, z) \le d(\varphi(y), \varphi(z)) \le L_V d(y, z)$ hold for all $y, z \in V$. Kâˆ¥f | L1p(Dâ ²)âˆ¥;

Lemma 2. *Let* D, $D' \subset \mathbb{M}$ *be two domains and* $\varphi: D \to D'$ *be a measurable mapping such that, for any bounded function* $f \in L_p^1(D')$, $p > n$, *the following conditions hold:*

(1) $\tilde{f} \circ \varphi \in L_p^1(D)$,

(2) $K^{-1}\|f|L_p^1(D')\| \le \|\tilde{f} \circ \varphi|L_p^1(D)\| \le K\|f|L_p^1(D')\|$,

where $\tilde{f}$ *is a continuous representative of f, and K is a positive constant. Then the mapping* φ *coincides almost everywhere with some locally bilipschitz mapping.*

8. Case $p \le n$. Consideration of this case begins with multi-path arguments, the ultimate purpose of which is a proof of approximative differentiability of the mappings φ. Notice the following lemma, which is a base for proving Proposition 1 for $p \le n$.

Lemma 3. *Let* D, $D' \subset \mathbb{M}$ *and a mapping* $\varphi: D \to D'$ *belong to* IL_p^1, $p \in [1, \nu]$. *Then there exists a set of measure zero to be removed from the domain of* φ *such that the following property holds on the reduced domain* $\mathrm{Dom}_1\varphi$: *for any two balls* $B_1, B_2 \subset D$ *with* $\overline{B}_1 \cap \overline{B}_2 = \emptyset$, *the intersection of images has measure zero, i.e.,* $|\varphi(B_1 \cap \mathrm{Dom}_1\varphi) \cap \varphi(B_2 \cap \mathrm{Dom}_1\varphi)| = 0$.

Fix a ball $Q \subset \mathbb{M}$. Define the following set function: $\Psi_Q: B \mapsto |\varphi(B \cap \mathrm{Dom}_1\varphi) \cap Q|$, i.e., to every ball $B \subset D$ the function Ψ_Q compares the measure of the intersection of the image of the ball with the ball Q. By Lemma 3, the function Ψ_Q has the following property of additivity: $\Psi_Q(B_1 \cup B_2) = \Psi_Q(B_1) + \Psi_Q(B_2)$ for any balls B_1, $B_2 \subset D$ such that $\overline{B}_1 \cap \overline{B}_2 = \emptyset$. Using this property and applying the arguments of the proof in [9, Theorem 3], we can show that a finite derivative $\Psi'_Q(x) = \lim\limits_{r \to 0} \dfrac{\Psi_Q(B(x, r))}{|B(x, r)|}$ is defined for almost all $x \in D$, and the inequality $\int_U \Psi'_Q(x)dx \le \Psi_Q(U)$ holds where U is a finite union of balls, the closures of which are desjoined. We denote by Σ_Ψ the set of measure zero on which the derivative Ψ'_Q either is not defined or equals to ∞. Then a finite derivative Ψ'_Q is defined in all points of the complement $D \backslash \Sigma_\Psi$. Let $J_{\varphi, Q}(x) = \Psi'_Q(x)$.

Lemma 4. *Let* D, $D' \subset M$ *and a mapping* $\varphi: D \to D'$ *belong to* IL_p^1, $p \in [1, \infty)$. *If* $u \in L_p^1(D') \cap \mathrm{Lip}_{\mathrm{loc}}(D')$ *and* $\|u\,|L_p^1(D')\| \le 1$ *then*

$$|\nabla(u \circ \varphi)|(x) \le K \cdot J_{\varphi, Q}^{\frac{1}{p}}(x) \qquad (2)$$

almost everywhere in $D \cap \varphi^{-1}(Q)$, *where K is some constant independent of Q.*

From (2) we derive the following property.

Lemma 5. *Let* D, $D' \subset \mathbb{M}$ *and a mapping* $\varphi: D \to D'$ *belong to* IL_p^1, $p \in [1, \nu]$. *Then* φ *is approximatively differentiable almost everywhere along integral lines of basis vector fields* X_j, $j = 1, 2, \ldots, n$.

The approximative differentiability of the mapping φ almost everywhere along the integral lines of the basis vector fields implies the total approximative differentiability of φ almost everywhere in D [7]. Hence [7], the set D can be represented as a countable union $D = S \cup \bigcup_i E_i$ so that $\varphi \in \mathrm{Lip}(E_i)$, and the measure of S equals zero.

Every set E_i is contained in a countable union of sets $F_{ki} = \left\{z\colon\ d(\varphi(x), \varphi(z)) \ge \frac{1}{k} d(x, z),\ x \in E_i \cap B\left(z, \frac{1}{k}\right)\right\}$, i.e., $E_i \subset \bigcup_k F_{ki}$, (see [7]). Then the set D can be represented as a union $S \cup \bigcup_j D_j$ so that the mapping φ is bilipschitz on every D_j. We may assume that the domain of the mapping φ is the set $\mathrm{Dom}_2\varphi = \bigcup_j D_j \cap \mathrm{Dom}_1\varphi$, and φ is bilipschitz on $D_j \cap \mathrm{Dom}_1\varphi$, $j \in \mathbb{N}$.

We denote by $D\varphi$ the approximative differential of φ. Notice, that $J(x, \varphi) = \det D\varphi = \lim\limits_{r \to 0} \dfrac{|\varphi(B(x, r))|}{|B(x, r)|}$ almost everywhere in D.

Lemma 6. *Let the mapping* φ *belong to* IL_p^1, $p \in [1, \infty)$, *and* $\psi = \varphi^{-1}$: $\varphi(\mathrm{Dom}_2\varphi) \to \mathrm{Dom}_2\varphi$ (*see Proposition* 1). *Then*

(1) *the following estimates hold when* $p \in [1, \infty)\setminus\{n\}$:

$$|D\varphi|(x) \le L, \quad |J(x, \varphi)| \ge \alpha_1 \ \text{ and } \quad |D\psi|(y) \le L', \quad |J(y, \psi)| \ge \alpha; \tag{3}$$

(2) *the following estimates hold when* $p = n$:

$$|D\varphi|^n(x) \le L|J(x, \varphi)| \ \text{ and } |D\psi|^n(y) \le L'|J(x, \psi)| \tag{4}$$

for almost all $x \in D$ *and almost all* $y \in \varphi(\mathrm{Dom}_2\varphi)$ *where* $J(x, \varphi) = \det D\varphi(x)$ *and* $J(y, \psi) = \det D\psi(y)$.

In addition, $\varphi(\mathrm{Dom}_2\varphi) \subset D'$ *is a set of full measure and the mapping* ψ: $\varphi(\mathrm{Dom}_2\varphi) \to \mathrm{Dom}_2\varphi$, *inverse to* φ, *induces the operator* ψ^*: $L_p^1(D) \to L_p^1(D')$ *by the composition rule, which is inverse to* φ^*.

Lemma 2 and (3) imply the assertion of Theorem 1.

9. The proof of Theorem 2 is based on the following lemma, which is a modification of a statement in [3].

We fix an arbitrary closed set $F \subset D$ of a positive measure without isolated points, on which φ is continuous, $D'_F = D'\setminus\varphi(F)$. The capacity $\mathrm{Cap}(K; L^1_{n,F}(D))$ of the compact $K \subset D_F$ in the space $L^1_{n,F}(D)$ equals the value $\inf\|g| L^1_{n,F}(D)\|^n$ where the infimum is taken over all continuous functions $g \in L^1_{n,F}(D)$ such that $g \ge 1$ on K (here $L^1_{n,F}(D) = \{u \in L^1_n(D)$: $u(x) = 0$ for almost all $x \in F\}$). In a standard way, this capacity can be extended to arbitrary sets in D_F [10].

Definition 5. A function f, defined quasi-everywhere on D_F, will be called a quasi-continuous if for every $\varepsilon > 0$ there exists an open set $U_\varepsilon \subset D_F$ such that $\overline{\mathrm{Cap}}(U_\varepsilon;\ L^1_{n,F}(D)) < \varepsilon$ and the restriction of f to the complement $D_F\setminus U_\varepsilon$ is continuous.

Lemma 7. *Let* D, D' *be domains on the Riemannian manifold* $\mathbb{M}$ *of the topological dimension* n. *There are exist some set* $S_\varphi \subset D_F$ *of zero capacity and quasi-continuous mapping* φ_0: $D_F\setminus S_\varphi \to \overline{D'_F}$ *such that* φ_0 *coincides with* φ *almost everywhere on* D_F. *For the mapping* φ_0, *the estimate*

$$\overline{\mathrm{Cap}}(\varphi_0(B_j) \cap D'_F;\ L^1_{n,\varphi(F)}(D')) \le K^{-n}\overline{\mathrm{Cap}}(B_j;\ L^1_{n,F}(D)) \tag{5}$$

is valid where $B_j \Subset D_F$ *is an arbitrary ball of some countable system constituting a base of open sets in* $U \subset D_F$.

By modifying further the circuit of arguments in [11], we obtain a homeomorphic mapping φ_0 on $D_F\setminus S_\varphi$ belonging to $W^1_{n,\mathrm{loc}}(D_F)$. Together with the inequalities (4) we get a quasi-conformal mappings φ_0 on $D_F\setminus S_\varphi$. The choice of F is arbitrary, so φ_0 is quasiconformal on $D\setminus S_\varphi$.

Thus, the result of [12, Theorem 2.2] is proved without additional restrictions. Theorems 1 and 2 can be extended to sub-Riemannian manifolds, at least for the compactly embedded domains.

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