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CONTROL THEORY

Stabilization of Nonlinear Discrete-Time Dynamic Control Systems with a Parameter and State-Dependent Coefficients

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Abstract—A numerical-analytical algorithm for designing nonlinear stabilizing regulators for the class of nonlinear discrete-time control systems is proposed that significantly reduces computational costs. The resulting regulator is suboptimal with respect to the constructed quadratic functional with state-dependent coefficients. The conditions for the stability of the closed-loop system are established, and a stability result is stated. Numerical results are presented showing that the nonlinear regulator designed is superior to the linear one with respect to both nonlinear and standard time-invariant cost functionals. An example demonstrates that the closed-loop system is uniformly asymptotically stable.

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At present, the design of stabilizing control in nonlinear systems is an important problem in control theory (see [1, 2]). The goal of this paper is to study the class of nonlinear discrete-time control systems in which a formal small parameter can be identified so that the systems can be transformed into a quasilinear form. A suitable quadratic functional with state-dependent coefficients is constructed, and a stabilizing regulator is designed using the discrete-time state dependent Riccati equation (D-SDRE) technique [3–5].

Consider the nonlinear discrete-time control system

$$x(t+1) = f(x(t)) + g(x(t))u,\qquad x(0) = x^0,\quad t = 0, 1, \dots . \tag{1}$$

Assume that (1) can be transformed into a linear control system with coefficients that are nonlinear with respect to the state and depend on a positive parameter μ:

$$\begin{gathered} x(t+1) = A(x(t), \mu)x(t) + B(x(t), \mu)u(t) \\ = (A_0 + \mu A_1(x(t)))x(t) + (B_0 + \mu B_1(x(t)))u(t), \\ x(0) = x^0, \\ x(t) \in X \in R^n,\quad u \in R^r,\quad t = 0, 1, \dots, \\ 0 < \mu \le \mu_0, \end{gathered} \tag{2}$$

where μ_0 is a given number that is not necessarily small; A_0 and B_0 are constant matrices; A_0, $A_1(x) \in R^{n\times n}$, B_0, $B_1(x) \in R^{n\times r}$, and $X \subset R^n$ is a given bounded subset of the state space.

The task is to find a sufficiently smooth control $u(x, \mu)$ with $0 < \mu \le \mu_0$ such that the equilibrium in the closed-loop system corresponding to (2) is uniformly asymptotically stable in the sense of Lyapunov, i.e., $x(t) \to 0$ for $t \to \infty$ as uniformly in $t \ge t_0$, $\mu \in (0, \mu_0]$.

A control function is sought in the form of the nonlinear state feedback

$$u(x, \mu) = K(x, \mu)x = (K_0 + \mu K_1(x))x, \tag{3}$$

where K_0 and $K_1(x)$ are a constant and a varying square matrix, respectively.

The matrix $K_1(x)$ is used to take into account the disturbances $A_1(x)$ and $B_1(x)$ in the coefficients of system (2). The closed-loop system for (2) along the nonlinear control $u(x, \mu)$ (see (3)) has the form

$$\begin{gathered} x(t+1) = (A_0 + \mu A_1(x(t)))x(t) \\ + (B_0 + \mu B_1(x(t)))(K_0 + \mu K_1(x(t)))x(t) \\ = [(A_0 + \mu A_1(x(t))) + (B_0 + \mu B_1(x(t))) \\ \times (K_0 + \mu K_1(x(t)))]x(t) = A_{cl}(x(t), \mu)x(t), \\ x(0) = x^0. \end{gathered} \tag{4}$$

The matrices K_0 and $K_1(x)$ are chosen using the criterion

$$I(u) = \frac{1}{2}\sum_{t=0}^{\infty}(x^{\mathrm{T}}(Q_0 + \mu Q_1(x))x + u^{\mathrm{T}}R_0u) \to \min, \tag{5}$$

where Q_0 and R_0 are constant matrices such that $Q_0 \ge 0$ and $R_0 > 0$ and $Q_1(x) \ge 0$, so that the resulting regulator (3) is stabilizing in (2).

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An algorithm for designing nonlinear stabilizing regulators for (2), (5) is based on the formal solution of the discrete-time matrix algebraic Riccati equation

$$A^{\mathrm{T}}(x,\mu)PA(x,\mu) - P - A^{\mathrm{T}}(x,\mu)PB(x,\mu) \times (R_0 + B^{\mathrm{T}}(x,\mu)PB(x,\mu))^{-1} \times B^{\mathrm{T}}(x,\mu)PA(x,\mu) + Q(x,\mu) = 0, \quad (6)$$

and relies on the scheme proposed for the continuous time case in [6].

Equation (6) is related to the optimality conditions in the time-invariant linear quadratic optimal control problem, but we use (6) as a baseline heuristic construction. Obviously, the solution of Riccati equation (6) now depends on the state x. P is sought in the form of a linear function of μ, i.e.,

$$P(x,\mu) = P_0 + \mu P_1(x). \quad (7)$$

Substituting (7) into (6) and formally assuming that μ is small, we expand the left-hand side of the resulting equation in a series for every x. Equating coefficients of like powers of μ in the resulting expansion yields a system of matrix equations for determining the terms in (7).

Define the matrix $\tilde{R}_0 = R_0 + B_0^{\mathrm{T}} P_0 B_0$. Now, combining the coefficients of μ^0 and μ^1 separately, we obtain corresponding algebraic matrix equations for P_0 and $P_1(x)$:

$$A_0^{\mathrm{T}} P_0 A_0 - P_0 - A_0^{\mathrm{T}} P_0 B_0 (R_0 + B_0^{\mathrm{T}} P_0 B_0)^{-1} \times B_0^{\mathrm{T}} P_0 A_0 + Q_0 = 0, \quad (8)$$

$$A_{cl,0}^{\mathrm{T}} P_1(x) A_{cl,0} - P_1(x) = -C(x), \quad (9)$$

where $A_{cl,0} = A_0 - B_0 \tilde{R}_0^{-1} B_0^{\mathrm{T}} P_0 A_0$ and

$$C(x) = A_0^{\mathrm{T}} P_0 A_1(x) + A_1^{\mathrm{T}}(x) P_0 A_0 - A_0^{\mathrm{T}} P_0 B_1(x) \tilde{R}_0^{-1} B_0^{\mathrm{T}} P_0 A_0 - A_0^{\mathrm{T}} P_0 B_0 \tilde{R}_0^{-1} B_1^{\mathrm{T}}(x) P_0 A_0 - A_1^{\mathrm{T}}(x) P_0 B_0 \tilde{R}_0^{-1} B_0^{\mathrm{T}} P_0 A_0 - A_0^{\mathrm{T}} P_0 B_0 \tilde{R}_0^{-1} B_0^{\mathrm{T}} P_0 A_1(x) + A_0^{\mathrm{T}} P_0 B_0 \tilde{R}_0^{-1} B_0^{\mathrm{T}} P_0 B_1(x) \tilde{R}_0^{-1} B_0^{\mathrm{T}} P_0 A_0 + A_0^{\mathrm{T}} P_0 B_0 \tilde{R}_0^{-1} B_1^{\mathrm{T}}(x) P_0 B_0 \tilde{R}_0^{-1} B_0^{\mathrm{T}} P_0 A_0 + Q_1(x).$$

By taking into account the form of the optimal regulator in the time-invariant linear quadratic optimal control problem formally obtained from (2), (5) by setting $\mu = 0$, the desired regulator for problem (2), despite the state-dependent coefficients, is sought in the same form, taking into account (7):

$$u(x,\mu) = -[R_0 + (B_0 + \mu B_1(x))^{\mathrm{T}}(P_0 + \mu P_1(x)) \times (B_0 + \mu B_1(x))]^{-1}(B_0 + \mu B_1(x))^{\mathrm{T}} \times (P_0 + \mu P_1(x))(A_0 + \mu A_1(x))x. \quad (10)$$

Now, formally assuming that μ is small, we represent (10) in the form

$$u(x,\mu) = u_0(x) + \mu u_1(x) = K_0 x + \mu K_1(x)x, \quad (11)$$

where

$$K_0 = -\tilde{R}_0^{-1} B_0^{\mathrm{T}} P_0 A_0, \quad (12)$$

$$K_1(x) = \tilde{R}_0^{-1}\{[B_0^{\mathrm{T}} P_1(x) B_0 - B_0^{\mathrm{T}} P_0 B_1(x) - B_1^{\mathrm{T}}(x) P_0 B_0]\tilde{R}_0^{-1} B_0^{\mathrm{T}} P_0 A_0 - [B_0^{\mathrm{T}} P_0 A_1(x) + B_1^{\mathrm{T}}(x) P_0 A_0 + B_0^{\mathrm{T}} P_1(x) A_0]\}. \quad (13)$$

Combining (11) with (12) and (13) yields

$$u(x,\mu) = -\tilde{R}_0^{-1} B_0^{\mathrm{T}} P_0 A_0 x + \mu \tilde{R}_0^{-1}\{[B_0^{\mathrm{T}} P_1(x) B_0 - B_0^{\mathrm{T}} P_0 B_1(x) - B_1^{\mathrm{T}}(x) P_0 B_0]\tilde{R}_0^{-1} B_0^{\mathrm{T}} P_0 A_0 - [B_0^{\mathrm{T}} P_0 A_1(x) + B_1^{\mathrm{T}}(x) P_0 A_0 + B_0^{\mathrm{T}} P_1(x) A_0]\}x. \quad (14)$$

To analyze the stability of the closed-loop system (4), in view of (7)–(9), we introduce the Lyapunov function

$$V(x) = x^{\mathrm{T}} P(x,\mu)x = x^{\mathrm{T}}(P_0 + \mu P_1(x(t)))x.$$

Define the matrix $D(x(t),\mu) = P(x(t),\mu) - A_{cl}(x,\mu)^{\mathrm{T}} P(x(t+1),\mu) A_{cl}(x,\mu)$.

Theorem. *Let the following conditions be satisfied:*

(I) *The coefficients of the matrices* $A_1(x)$, $B_1(x)$, $Q_1(x)$ *are continuous bounded functions on* X.

(II) *The triplet of matrices* $(A_0, B_0, Q_0^{1/2})$ *is stabilizable and detectable.*

(III) *There exists* $Q_1(x) > 0$ *such that* $C(x)$ *is a positive definite matrix* $\forall x \in X$.

(IV) *There are* $G \subseteq X$, $\mu_0 > 0$, *and a constant positive definite matrix* $D^- > 0$ *such that* $D(x,\mu) > D^-$ *holds uniformly in* $x \in G$, and $\mu \in (0, \mu_0]$.

Then there are $\{0\} \in G_1 \subseteq G$ and $\mu_0 > 0$ *such that the equilibrium* $x(t) \equiv 0$ *of system* (2), (14) *is uniformly asymptotically stable in the sense of Lyapunov for* $x^0 \in G_1$ *and* $\mu \in (0, \mu_0]$; *i.e., regulator* (14) *is stabilizing in system* (2) *for any* $x^0 \in G_1$ *and* $\mu \in (0, \mu_0]$.

Thus, the algorithm for designing a stabilizing regulator for system (1) can be described as follows:

1. Transform system (1) into (2), where A_0, B_0, and Q_0 satisfy condition (*I*).
2. Find P_0 by solving Eq. (8).
3. Choose $Q_1(x) > 0$ such that $C(x)$ is a positive definite matrix for any $x \in X$.
4. Determine. $P_1(x)$ as the solution of the discrete-time Lyapunov equation (9) with the help of the well-known formula

$$P_1(x) = \sum_{i=0}^{\infty} (A_{cl,0}^{\mathrm{T}})^i C(x) (A_{cl,0})^i \text{ [7]}.$$

5. Find the desired stabilizing regulator by using formula (14).

The class of problems for which the conditions of the theorem are satisfied is nonempty. Consider an illustrative example, namely, the problem of stabilizing a discrete-time inverted pendulum [4] governed by the dynamic equation

$$x(t+1) = \begin{bmatrix} 1 & T_s \\ \frac{T_s g \sin(x_1)}{L x_1} & 1 - \frac{T_s \gamma}{ML} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

$$x(0) = \begin{pmatrix} 2.75 \\ 0.5 \end{pmatrix}$$

(the first and second coordinates correspond to the deflection angle of the pendulum and the angular velocity, respectively, while $u(t)$ is the scalar control).

The model parameters are $T_s = 0.05$, $M = 0.1$, $L = 0.1$, $g = 9.8$, and $\gamma = 0.05$. The parameter T_s is used as μ. The matrices A_0, A_1, B_0, and B_1 are specified as

$$A_0 = \begin{bmatrix} 1 & T_s \\ \frac{T_s g}{L} & 1 - \frac{T_s \gamma}{ML} \end{bmatrix}, \quad A_1(x) = \begin{bmatrix} 0 & 0 \\ \frac{g \sin(x_1)}{L x_1} - \frac{g}{L} & 0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_1(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The matrices Q_0, $Q_1(x)$, and R_0 in the cost functional are defined as

$$Q_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_1(x) = \begin{bmatrix} 1500 + 0.1x_1^2 & 0 \\ 0 & 30 + 0.1x_2^2 \end{bmatrix},$$

$$R_0 = 1.$$

Then $P_0 = \begin{bmatrix} 195.234 & 13.633 \\ 13.633 & 2.049 \end{bmatrix}$, $C(x) = A_{cl}^{\mathrm{T}} P_0 A_1(x) + A_1(x)^{\mathrm{T}} P_0 A_{cl} + Q_1(x)$ is a positive definite matrixwith this choice of $Q_1(x)$. Moreover, D^- can be defined as $D^- = \begin{bmatrix} 40 & 2 \\ 2 & 2 \end{bmatrix}$, and $t_0 = 0$. With $u(x, \mu)$ and $u_0(x)$, for $N = 40$ $I(u(x, \mu)) = 1.748 \times 10^3$, and $I(u_0(x)) = 2.912 \times 10^3$; i.e., the regulator constructed is roughly 1.6 times more efficient than the linear one. A gain of the same order is observed if the regulators are compared in terms of the standard cost functional in the time-invariant control problem with matrices Q_0 and R_0. The best efficiency of the nonlinear regulator is achieved by tuning the parameters related to choosing the coefficients of the matrices A_0, $A_1(x)$, B_0, $B_1(x)$, Q_0, $Q_1(x)$, and R_0.

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