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MATHEMATICS

Arithmetic Sums of Polynomial Values

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Abstract—New estimates are obtained for the mean values of Bernoulli polynomials in polynomials with real or rational coefficients.

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In this paper, we study sums involving the first-degree Bernoulli polynomial $\rho(x) = \frac{1}{2} - \{x\}$, where $\{x\}$ is the fractional part of the number x. First, we consider complete rational arithmetic sums of the form

$$S = S\left(\frac{f(x)}{q}\right) = \sum_{x=1}^{q} \rho\left(\frac{f(x)}{q}\right),$$

where $q > 1$ is an integer and $f(x) = a_n x^n + \ldots + a_1 x + a_0$ is a polynomial with integer coefficients such that $(a_n, \ldots, a_1, q) = 1$. A fairly useful tool for estimating the modulus of such sums is the following multiplication formula for Bernoulli polynomials. Given a positive integer m such that $(a, m) = 1$, for any real x, we have

$$\rho(mx) = \rho(x) + \rho\left(x + \frac{a}{m}\right) + \ldots + \rho\left(x + \frac{a(m-1)}{m}\right).$$

Below is an analogue of the estimate for the complete rational exponential sum found by L.-K. Hua.

Theorem 1. *It is true that*

$$|S| \ll q^{1-1/n}.$$

In 1924 Franel [7] (see also [8–10]) found that, for any positive integers m and n,

$$\int_0^1 \rho(m\alpha)\rho(n\alpha)d\alpha = \frac{(m, n)^2}{12mn},$$

where (m, n) denotes the greatest common divisor of m and n.

Now let $f(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_n x^n$ be a polynomial of degree n with real coefficients. Consider the integral

$$I = \int_0^1 \rho(f(x))dx.$$

Let $\alpha = \max\{|\alpha_n|, \ldots, |\alpha_1|\}$. Then we have an analogue of Vinogradov's estimate for trigonometric integrals:

$$|I| \ll \min\{1, \alpha^{-1/n}\}.$$

In 2014 my student M.Sh. Shakhsadilov obtained the following result.

Theorem 2. *Let* $n \geq 1$, $\alpha_0, \alpha_1, \ldots, \alpha_n$ *be real numbers; and let*

$$f(x) = \alpha_n x^n + \ldots + \alpha_1 x + \alpha_0,$$

$$\beta_r(x) = \frac{f^{(r)}(x)}{r!}, \quad r = 1, 2, \ldots, n,$$

$$H = H(\bar{\alpha}) = H(\alpha_n, \ldots, \alpha_1, \alpha_0) = \min_{a \leq x \leq b} \max_{1 \leq r \leq n} |\beta_r(x)|^{1/r},$$

$$J = \int_a^b \rho(f(x))dx.$$

Then the integral J satisfies the estimate

$$|J| \leq \min(b - a; 4en^2 H^{-1}).$$

Finally, consider a sum of the form

$$S = S(\bar{\alpha}) = S(\alpha_0, \alpha_1, \ldots, \alpha_n) = \sum_{x \leq P} \rho(\alpha_0 + \alpha_1 x + \ldots + \alpha_n x^n),$$

where $P \geq 1$.

Note that $S(\alpha_0, \alpha_1, \ldots, \alpha_n)$ is a 1-periodic function of each variable α_s, $0 \leq s \leq n$, so it is sufficient to consider S inside the $(n+1)$-dimensional unit cube Π_n.

An integral $J = J(P; n; k)$ of the form

$$J = \int_0^1 \ldots \int_0^1 \left|\sum_{x \leq P} \rho(\alpha_0 + \alpha_1 x + \ldots + \alpha_n x^n)\right|^{2k} d\alpha_0 d\alpha_1 \ldots d\alpha_n$$

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is called the mean value of the sum $S = S(\bar{\alpha})$. It is an analogue of Vinogradov's well-known integral in his mean value theorem for Weyl exponential sums [1–6].

Below is the mean value theorem.

Theorem 3. *Let* $\tau \geq 0$ *be an integer,* $k \geq n\tau$, *and* $P \geq 1$. *Then*

$$J = J(P; n; k) \leq DP^{2k - \frac{n(n+1)}{2} + \delta(\tau)},$$

where

$$\delta(\tau) = \frac{n(n+1)}{2}\left(1 - \frac{1}{n}\right)^{\tau}$$

and

$$D = D(\tau) = (n\tau)^{6n\tau}(2n)^{4n(n+1)\tau}.$$

Note that $J = J(P; n, k)$ satisfies the upper bound

$$J \geq cP^{2k - \frac{n(n+1)}{2}},$$

where $c = 4^{-2k}(4(n+1))^{-n-1}$.

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