ISSN 1064-5624, Doklady Mathematics, 2016, Vol. 93, No. 1, pp. 16–19. © Pleiades Publishing, Ltd., 2016.
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MATHEMATICS

Trace and Integrable Operators Affiliated with a Semifinite von Neumann Algebra

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Presented by Academician I.A. Ibragimov July 1, 2015

Received July 24, 2015

Abstract—New properties of the space of integrable (with respect to the faithful normal semifinite trace) operators affiliated with a semifinite von Neumann algebra are found. A trace inequality for a pair of projections in the von Neumann algebra is obtained, which characterizes trace in the class of all positive normal functionals on this algebra. A new property of a measurable idempotent are determined. A useful factorization of such an operator is obtained; it is used to prove the nonnegativity of the trace of an integrable idempotent. It is shown that if the difference of two measurable idempotents is a positive operator, then this difference is a projection. It is proved that a semihyponormal measurable idempotent is a projection. It is also shown that a hyponormal measurable tripotent is the difference of two orthogonal projections.

DOI: 10.1134/S1064562416010075

This paper continues the author's study initiated in papers [1–3]; we use the notation and terminology of these papers. In Section 2, we establish new properties of the space $L_1(\mathcal{M}, \tau)$ of integrable (with respect to the trace τ) operators affiliated with the semifinite von Neumann algebra $\mathcal{M}$. We show that if A and B are a hyponormal and a cohyponormal τ-measurable operator, respectively, and $AB \in L_1(\mathcal{M}, \tau)$, then $BA \in L_1(\mathcal{M}, \tau)$ and $\|BA\|_1 \le \|AB\|_1$; moreover, $\tau(AB) = \tau(BA)$, and for self-adjoint A and B, we have $\tau(AB) = \tau(BA) \in \mathbb{R}$. We prove that if $A \in L_1(\mathcal{M}, \tau)$, then $\tau(A^*) = \overline{\tau(A)}$. We obtain a trace inequality for a pair of projections in $\mathcal{M}$, which characterizes trace in the class of all positive normal functionals on $\mathcal{M}$.

In Section 3, we establish new properties of a τ-measurable idempotent ($A = A^2$). We obtain a useful factorization of such an operator; using it, we prove that $\tau(A) \in \mathbb{R}^+$ for an idempotent $A \in L_1(\mathcal{M}, \tau)$. Therefore, if $A, A^2 \in L_1(\mathcal{M}, \tau)$ and $A = A^3$, then $\tau(A) \in \mathbb{R}$. We show that if the difference of two τ-measurable idempotents is a positive operator, then this difference is a projection. We prove that a semihyponormal τ-measurable idempotent is a projection. We also show that a hyponormal τ-measurable tripotent ($A = A^3$) is the difference of two orthogonal projections.

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1. NOTATION AND DEFINITIONS

Suppose that $\mathcal{M}$ is a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$, $\mathcal{M}^{\text{pr}}$ is the lattice of projections on $\mathcal{M}$, I is the identity of $\mathcal{M}$, $P^{\perp} = I - P$ for $P \in \mathcal{M}^{\text{pr}}$, and $\mathcal{M}^+$ is the cone of positive elements in $\mathcal{M}$. If $P, Q \in \mathcal{M}^{\text{pr}}$, then the projection $P \wedge Q$ is defined by $(P \wedge Q)\mathcal{H} = P\mathcal{H} \cap Q\mathcal{H}$ and $P \vee Q = (P^{\perp} \wedge Q^{\perp})^{\perp}$ is the projection onto $\overline{\text{Lin}(P\mathcal{H} \cup Q\mathcal{H})}$.

A mapping φ: $\mathcal{M}^+ \to [0, +\infty]$ is called a trace if $\varphi(X + Y) = \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{M}^+$, $\lambda \ge 0$ (it is assumed that $0 \cdot (+\infty) \equiv 0$), and $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$. A trace φ is said to be faithful if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \ne 0$; it is semifinite if $\varphi(X) = \sup\{\varphi(Y)\colon Y \in \mathcal{M}^+, Y \le X, \varphi(Y) < +\infty\}$ for each $X \in \mathcal{M}^+$; and it is normal if $X_i \nearrow X$ ($X_i, X \in \mathcal{M}^+$) $\Rightarrow \varphi(X) = \sup \varphi(X_i)$. For a trace φ, we set $\mathfrak{M}_{\varphi}^+ = \{X \in \mathcal{M}^+\colon \varphi(X) < +\infty\}$ and $\mathfrak{M}_{\varphi} = \text{lin}_{\mathbb{C}}\mathfrak{M}_{\varphi}^+$. The restriction $\varphi \mid \mathfrak{M}_{\varphi}^+$ admits a well-defined extension by linearity to a functional on $\mathfrak{M}_{\varphi}$, which we denote by the same letter φ.

An operator on $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be affiliated with a von Neumann algebra $\mathcal{M}$ if it commutes with any unitary operator in the commutator subalgebra $\mathcal{M}'$ of $\mathcal{M}$. A self-adjoint operator is affiliated with $\mathcal{M}$ if and only if all projections in its spectral decomposition of unity belong to $\mathcal{M}$.

In what follows, τ is a faithful normal semifinite trace on $\mathcal{M}$. A closed operator X affiliated with $\mathcal{M}$ whose domain $\mathfrak{D}(X)$ is dense in $\mathcal{H}$ is said to be τ-measurable if, for any $\varepsilon > 0$, there exists a $P \in \mathcal{M}^{\text{pr}}$ such that

$P\mathcal{H} \subset \mathfrak{D}(X)$ and $\tau(P^{\perp}) < \varepsilon$. The set $\widetilde{\mathcal{M}}$ of all τ-measurable operators is a *-algebra under the passage to the dual operator, multiplication by a scalar, and the strong addition and multiplication operations obtained as the closures of the usual operations [4, 5]. Given a family $\mathcal{L} \subset \widetilde{\mathcal{M}}$, we denote its positive, Hermitian, and idempotent ($X = X^2$) parts by $\mathcal{L}^+$, $\mathcal{L}^{\mathrm{sa}}$, and $\mathcal{L}^{\mathrm{id}}$, respectively. We denote the partial order on $\widetilde{\mathcal{M}}^{\mathrm{sa}}$ generated by the proper cone $\widetilde{\mathcal{M}}^+$ by $\le$.

If X is a closed densely defined linear operator affiliated with $\mathcal{M}$ and $|X| = \sqrt{X^*X}$, then the spectral decomposition $P^{|X|}(\cdot)$ is contained in $\mathcal{M}$ and $X \in \widetilde{\mathcal{M}}$ if and only if there exists a $\lambda \in \mathbb{R}$ such that $\tau(P^{|X|}((\lambda, +\infty))) < +\infty$. If $X \in \widetilde{\mathcal{M}}$ and $X = U|X|$ is the polar decomposition of X, then $U \in \mathcal{M}$ and $|X| \in \widetilde{\mathcal{M}}^+$. Moreover, if $|X| = \int_0^{\infty} \lambda\, P^{|X|}(d\lambda)$ is the spectral decomposition, then $\tau(P^{|X|}((\lambda, +\infty))) \to 0$ as $\lambda \to +\infty$.

By $\mu_t(X)$ we denote the rearrangement of an operator $X \in \widetilde{\mathcal{M}}$, i.e., the nonincreasing right continuous function $\mu(X)$: $(0, \infty) \to [0, \infty)$ defined by

$$\mu_t(X) = \inf\{\|XP\| : P \in \mathcal{M}^{\mathrm{pr}}, \tau(P^{\perp}) \le t\}, \quad t > 0.$$

The set of τ-compact operators $\widetilde{\mathcal{M}}_0 = \{X \in \widetilde{\mathcal{M}} : \mu_\infty(X) \equiv \lim_{t \to \infty} \mu_t(X) = 0\}$ is an ideal in $\widetilde{\mathcal{M}}$ [6].

Let m be the linear Lebesgue measure on $\mathbb{R}$. The noncommutative Lebesgue L_p-space ($0 < p < \infty$) associated with $(\mathcal{M}, \tau)$ can be defined as the space $L_p(\mathcal{M}, \tau) = \{X \in \widetilde{\mathcal{M}} : \mu(X) \in L_p(\mathbb{R}^+, m)\}$ with F-norm (norm if $1 \le p < \infty$)) $\|X\|_p = \|\mu(X)\|_p$, $X \in L_p(\mathcal{M}, \tau)$. The restriction $\tau|\mathfrak{M}_\tau^+$ can be extended to a linear bounded functional on $L_1(\mathcal{M}, \tau)$, which we denote by the same letter τ. We have $\mathfrak{M}_\tau = \mathcal{M} \cap L_1(\mathcal{M}, \tau)$ and $L_p(\mathcal{M}, \tau) \subset \widetilde{\mathcal{M}}_0$ for all $0 < p < \infty$.

An operator $X \in \widetilde{\mathcal{M}}$ is said to be semihyponormal if $|X| \ge |X^*|$, hyponormal if $X^*X \ge XX^*$, and cohyponormal if X^* is hyponormal.

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ is the *-algebra of all bounded linear operators on $\mathcal{H}$ and $\tau = \mathrm{tr}$ is the canonical trace, then $\widetilde{\mathcal{M}}$ coincides with $\mathcal{B}(\mathcal{H})$ and $\widetilde{\mathcal{M}}_0$ is the ideal of compact operators on $\mathcal{H}$. We have

$$\mu_t(X) = \sum_{n=1}^{\infty} s_n(X) \chi_{[n-1, n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^{\infty}$ is the sequence of s-numbers of X and χ_A is the indicator function of $A \subset \mathbb{R}$. In this case, the space $L_p(\mathcal{M}, \tau)$ is the Schatten–von Neumann ideal $\mathfrak{S}_p$, $0 < p < \infty$.

2. ON INTEGRABLE OPERATORS

Theorem 1. *If* $X, Y \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$ *and* $XY \in L_1(\mathcal{M}, \tau)$, *then* $YX \in L_1(\mathcal{M}, \tau)$ *and* $\tau(XY) = \tau(YX) \in \mathbb{R}$.

Corollary 1. *If* $X, Y \in \mathcal{B}(\mathcal{H})^{\mathrm{sa}}$ *and* $XY \in \mathfrak{S}_1$, *then* $YX \in \mathfrak{S}_1$ *and* $\mathrm{tr}(XY) = \mathrm{tr}(YX) \in \mathbb{R}$.

Theorem 2. *If* $X \in L_1(\mathcal{M}, \tau)$, *then* $\tau(X^*) = \overline{\tau(X)}$.

Theorem 3. *Let* $0 < p < \infty$, *and let* $A, B \in \widetilde{\mathcal{M}}$ *be operators such that A is hyponormal and B is cohyponormal. If* $AB \in L_p(\mathcal{M}, \tau)$, *then* $BA \in L_p(\mathcal{M}, \tau)$ *and* $\|BA\|_p \le \|AB\|_p$; *for* $p = 1$, $\tau(AB) = \tau(BA)$.

Theorem 4. *Let* $A \in \widetilde{\mathcal{M}}_0$, *and let* $V \in \mathcal{M}$ *wiith* $\|V\| \le 1$. *If* $V^*AV = A$, *then* $VA = AV$.

Theorem 5. *Let* φ *be a trace on the von Neumann algebra* $\mathcal{M}$. *Then* $\varphi(P + Q + |P - Q|) \le 2\varphi(P \vee Q)$ *for all* $P, Q \in \mathcal{M}^{\mathrm{pr}}$.

Corollary 2. *For all* $P, Q \in \mathcal{M}^{\mathrm{pr}}$, $\varphi(P + Q + |P - Q| + 2(P^{\perp} \wedge Q^{\perp})) \le 2\varphi(I)$.

Corollary 3. *If* $\varphi(I) < \infty$, *then* $\varphi(|P - Q| + 2(P \wedge Q)) \le \varphi(P + Q)$ *for all* $P, Q \in \mathcal{M}^{\mathrm{pr}}$.

The inequalities of Theorem 5 and Corollaries 2 and 3 become equalities in the cases (i) $Q = P^{\perp}$, (ii) $P \le Q$, and (iii) $Q \le P$. Under the conditions of Corollary 3, we have $\varphi(P \wedge Q) \le \frac{1}{2}\varphi(P + Q - |P - Q|)$ for all $P, Q \in \mathcal{M}^{\mathrm{pr}}$.

Theorem 6. *For a positive normal functional* φ *on the von Neumann algebra* $\mathcal{M}$, *the following conditions are equivalent:*

(i) φ *is a trace;*

(ii) $\varphi(P + Q + |P - Q|) \le 2\varphi(P \vee Q)$ *for all* $P, Q \in \mathcal{M}^{\mathrm{pr}}$;

(iii) $\varphi(|P - Q| + 2(P \wedge Q)) \le \varphi(P + Q)$ *for all* $P, Q \in \mathcal{M}^{\mathrm{pr}}$.

Proof. We have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii); let us prove the implication (iii) $\Rightarrow$ (i). By virtue of the monotonicity of the functional φ, condition (iii) implies the weaker inequality

$$\varphi(|P - Q|) \le \varphi(P + Q) \text{ for all } P, Q \in \mathcal{M}^{\mathrm{pr}},$$

and φ is a trace by virtue of Theorem 3.4(v) in [7].

For other characterizations of trace, see [7–9] and the references therein.

3. ON IDEMPOTENT OPERATORS

Example 1. Suppose that $0 < p, q < \infty$ and $a_n = 2^{n+1} n^{-q}$, $n \in \mathbb{N}$. Let us endow the von Neumann algebra $\mathcal{M} = \bigoplus_{n=1}^{\infty} \mathbb{M}_2(\mathbb{C})$ with the faithful normal finite trace $\tau = \bigoplus_{n=1}^{\infty} 2^{-n}\,\mathrm{tr}_2$ and set $A = \bigoplus_{n=1}^{\infty}\begin{pmatrix}1 & a_n\\ 0 & 0\end{pmatrix}$. We have $A = A^2$, $A \in L_p(\mathcal{M}, \tau)$ if $pq > 1$, and $A \notin L_p(\mathcal{M}, \tau)$ if $pq \le 1$.

Proposition 1. *Suppose that* $0 < p, q, r \le \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, *and* $A \in \tilde{\mathcal{M}}^{\mathrm{id}}$. *If* $A \in L_p(\mathcal{M}, \tau) \cap L_q(\mathcal{M}, \tau)$, *then* $A \in L_r(\mathcal{M}, \tau)$ *and* $\|A\|_r \le \|A\|_p\|A\|_q$.

If $A \in \mathcal{M}^{\mathrm{id}}$, then $\mu_t(A) \in \{0\} \cup [1, \|A\|]$ for all $t > 0$ (see Lemma 3.8, (1) in [10]).

Theorem 7. *Suppose that* $0 < p < \infty$, $P \in \tilde{\mathcal{M}}^{\mathrm{id}}$, *and* $Q \in \mathcal{M}^{\mathrm{id}}$. *If* $A \equiv P - Q \in L_p(\mathcal{M}, \tau)$, *then* $A^2 \in L_p(\mathcal{M}, \tau)$.

Corollary 4. *If* $A \in L_1(\mathcal{M}, \tau)$, $A = A^3$, *and* $A - A^2 \in \mathcal{M}$, *then* $\tau(A) \in \mathbb{R}$.

Theorem 8. *For an operator* $A \in \tilde{\mathcal{M}}$, *the following conditions are equivalent:*

(i) $A = A^2$;

(ii) $A = |A^*||A|$.

Corollary 5. *If* $A \in \tilde{\mathcal{M}}^{\mathrm{id}}$, *then there exists a unitary operator* $S \in \mathcal{M}^{\mathrm{sa}}$ *such that* $4|\Re A| \le |A|^2 + |A^*|^2 + S(|A|^2 + |A^*|^2)S$. *Therefore*, $2\tau(|\Re A|) \le \tau(|A|^2 + |A^*|^2)$. *If, in addition,* $A \in L_1(\mathcal{M}, \tau)$, *then* $\tau(A) \in \mathbb{R}^+$.

Proof. If $X, Y \in \tilde{\mathcal{M}}^{\mathrm{sa}}$, then $(X \pm Y)^2 \ge 0$; therefore, $-X^2 - Y^2 \le XY + YX \le X^2 + Y^2$ and, according to [11, Theorem 1 and Section 2], there exists a unitary operator $S \in \mathcal{M}^{\mathrm{sa}}$ such that $2|XY + YX| \le X^2 + Y^2 + S(X^2 + Y^2)S$. By Theorem 8, we can represent the operator $2\Re A$ in the form $2\Re A = A + A^* = |A^*||A| + |A||A^*|$; we set $X = |A^*|$ and $Y = |A|$.

If $A \in L_1(\mathcal{M}, \tau)$, then we have $\sqrt{|A|}\,|A^*|\sqrt{|A|} \in L_1(\mathcal{M}, \tau)^+$ and $\tau(A) = \tau(|A^*||A|) = \tau(\sqrt{|A|}\,|A^*|\sqrt{|A|}) \ge 0$ by virtue of Theorem 3 in [1].

Theorem 9. *If* $P, Q \in \tilde{\mathcal{M}}^{\mathrm{id}}$ *and* $A \equiv P - Q \in \tilde{\mathcal{M}}^{+}$, *then* $A \in \mathcal{M}^{\mathrm{pr}}$ *and* $QA = AQ = 0$.

Corollary 6. *If* $P, Q \in \tilde{\mathcal{M}}^{\mathrm{id}}$ *and* $A \equiv P - Q \le 0$, *then* $-A \in \mathcal{M}^{\mathrm{pr}}$ *and* $PA = AP = 0$.

Corollary 7. *If* $P, R \in \tilde{\mathcal{M}}^{\mathrm{id}}$, $B \equiv P + R \in \tilde{\mathcal{M}}^{+}$, *and* $B \ge I$, *then* $B - I \in \mathcal{M}^{\mathrm{pr}}$.

Corollary 8. *If* $S, T \in \tilde{\mathcal{M}}$ *with* $S^2 = T^2 = I$, *and* $A \equiv S - T \in \tilde{\mathcal{M}}^{+}$, *then* $\frac{1}{2}A \in \mathcal{M}^{\mathrm{pr}}$.

Lemma 1. *Suppose that* $A \in \tilde{\mathcal{M}}$ *and* $A = A^n$ *for some* $n \in \mathbb{N}$, $n \ge 2$. *If* $A \notin \tilde{\mathcal{M}}_0$, *then* $\mu_t(A) \ge 1$ *for all* $t > 0$.

Proof. Suppose that, on the contrary, $a = \mu_\infty(A) \in (0, 1)$. Choose a number $\varepsilon > 0$ for which $(a + \varepsilon)^n < a$ and let $t > 0$ be such that $\mu_{t/n}(A) \in [a, a + \varepsilon]$. Recall that $\mu_{s+t}(XY) \le \mu_s(X)\mu_t(Y)$ for all $X, Y \in \tilde{\mathcal{M}}$ and $s, t > 0$ (see [6, 12]). Therefore, $\mu_t(A) = \mu_t(A^n) \le (\mu_{t/n}(A))^n \le (a + \varepsilon)^n < a$; we have arrived at a contradiction.

Theorem 10. *If* $A \in \tilde{\mathcal{M}}^{\mathrm{id}}$ *and* A (or A^*) *is semihyponormal, then A is normal and, thereby,* $A \in \mathcal{M}^{\mathrm{pr}}$.

Corollary 9. *If* $A \in \tilde{\mathcal{M}}^{\mathrm{id}}$ *and A is hyponormal or cohyponormal, then A is normal and, thereby,* $A \in \mathcal{M}^{\mathrm{pr}}$.

Proof. If $X, Y \in \tilde{\mathcal{M}}^{+}$, then it follows from $X \le Y$ that $\sqrt{X} \le \sqrt{Y}$. Therefore, each τ-measurable hyponormal operator is semihyponormal.

Theorem 11. *If* $A \in \tilde{\mathcal{M}}$, $A = A^3$, *and A is hyponormal or cohyponormal, then A is normal; thereby,* $A \in \mathcal{M}^{\mathrm{sa}}$ *and* $A = P - Q$ *for some* $P, Q \in \mathcal{M}^{\mathrm{sa}}$ *with* $PQ = 0$.

Corollary 10. *Let* $A \in \tilde{\mathcal{M}}$. *If, for some* $\lambda \in \mathbb{C}$, *the operator* $A_\lambda = \lambda I + A$ *is hyponormal* (*or cohyponormal*) *and* $A_\lambda = A_\lambda^3$, *then* $A \in \mathcal{M}$.

Corollary 11. *If* $A, A^2 \in L_1(\mathcal{M}, \tau)$ *and* $A = A^3$, *then* $\tau(A) \in \mathbb{R}$.

Proof. In the decomposition $A = P - Q$ with $P = \frac{A + A^2}{2}$ and $Q = \frac{A^2 - A}{2}$ [13, Proposition 1], we have $P = P^2$, $Q = Q^2$, and $P, Q \in L_1(\mathcal{M}, \tau)$. Since $\tau(P)$, $\tau(Q) \in \mathbb{R}^+$ by virtue of Corollary 5, it follows that $\tau(A) = \tau(P) - \tau(Q) \in \mathbb{R}$.

Remark 1. For the von Neumann algebra $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and the trace $\tau = \mathrm{tr}$, the assertions of Theorems 10 and 11 were proved by the author in [14] (Lemma 3 and Theorem 2, respectively). The assertion of Theorem 4 in the special case of an operator $A \in \mathfrak{S}_2$ and an isometry V was proved in [15, Lemma 3.1].

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research, project no. 15-41-02433.

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Translated by O. Sipacheva