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MATHEMATICS

Dynamics of Particles with Anisotropic Mass Depending on Time and Position

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Abstract—Representations of solutions of equations describing the diffusion and quantum dynamics of particles in a Riemannian manifold are discussed under the assumption that the mass of particles is anisotropic and depends on both time and position. These equations are evolution differential equations with second-order elliptic operators, in which the coefficients depend on time and position. The Riemannian manifold is assumed to be isometrically embedded into Euclidean space, and the solutions are represented by Feynman formulas; the representation of a solution depends on the embedding.

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We consider the following problem. Suppose that $E = \mathbb{R}^n$, S is a compact Riemannian submanifold in E and A is a Borel function on E taking values in the space $\mathcal{L}(E)$ (given a Banach space X, $\mathcal{L}(X)$ denotes the Banach space of all continuous linear operators on X). By Δ_A we denote the closed linear operator on the (real or complex) space $\mathfrak{L}_p(E)$, $1 \le p < \infty$, defined by $(\Delta_A\varphi)(x) := \mathrm{tr}A(x)\varphi''(x)$ for functions φ (assumed to be infinitely differentiable) which form a dense subspace of $\mathfrak{L}_p(E)$. In what follows, we also define operators Δ_A^S on $\mathfrak{L}_p(S)$ associated with Δ_A.

Let G be a function on $[0, a)$, $a > 0$, taking values in the space of $\mathcal{L}(E)$-valued Borel functions on E. We shall obtain Feynman formulas for the evolution families of operators (a definition is given below) on $\mathfrak{L}_p(S)$ generated by the families $\Delta_{G(t)}^S$ and $i\Delta_{G(t)}^S$ of operators associated with the families $\Delta_{G(t)}$ and $i\Delta_{G(t)}$.

The method which we use can be described as follows. If $A(x) = \mathcal{I}$ for each x, where $\mathcal{I}$ is the identity self-mapping of E, then Δ_A is the usual Laplacian acting on functions defined on E, and the associated operator Δ_A^S, which acts on functions on S, is the Laplace–Beltrami operator. In the situation under consideration, it can be obtained as follows. For each $\tau > 0$, let $F(\tau)$ be the integral operator on $\mathfrak{L}_p(S)$ defined by $(F(\tau)\varphi)(x) := c(\tau, x)\int_S e^{-\frac{(z-x)^2}{2\tau}}\varphi(z)dz$, where $(c(\tau, x))^{-1} = \int_S e^{-\frac{(z-x)^2}{2\tau}}dz$ and $F(0) = \mathcal{I}$. Then the function F is continuous in the topology of pointwise convergence on $\mathcal{L}(\mathfrak{L}_p(S))$, and $F'(0)$ is the Laplace–Beltrami operator $\Delta_{\mathcal{I}}^S$ on $\mathfrak{L}_p(S)$. It follows that if a function $F(\cdot)$ satisfies the estimate of Chernoff's theorem, then this function can be used to obtain Feynman approximations for the semigroup $e^{t\Delta_{\mathcal{I}}^S}$. Our method is based on the fact that a similar operator-valued function $F(\cdot)$ can be defined for any function A: $E \to \mathcal{L}(E)$ in such a way that $F'(0)\varphi = \Delta_A^S\varphi$ for $\varphi \in \mathrm{dom}\Delta_A^S$.

The paper is organized as follows. In the first section, we give definitions used in what follows and state a theorem similar to Chernoff's theorem for evolution families of operators (Chernoff's classical theorem deals with one-parameter semigroups of operators). In the next section, we derive Feynman formulas for evolution families of operators on $\mathfrak{L}_p(S)$ generated (a definition is given below) by the families of operators $\Delta_{G(t)}^S$ and $i\Delta_{G(t)}^S$. The last, third, section contains two conjectures on a relationship between stochastic processes in E ($=\mathbb{R}^n$) and in S that are generated by $\Delta_{G(t)}$ and $\Delta_{G(t)}^S$.

Main attention is focused on the algebraic structure of the problems, and assumptions of analytical character are not considered.

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1. DEFINITIONS AND PRELIMINARY RESULTS

Let $a > 0$, and let $\mathcal{A}(t)$ be a linear (unbounded in the general case) operator on a Banach space X for each $t \in [0, a)$; we assume that the domains of the operators $\mathcal{A}(t)$ coincide and denote the common domain by dom$\mathcal{A}$.

Definition 1. The evolution family on $[0, a)$ of continuous linear operators on a Banach space X generated by an operator-valued function $\mathcal{A}$ is a mapping $U^{\mathcal{A}}$ of the set V_a defined by $V_a = \{(\tau, t) \in \mathbb{R}^2 \mid 0 \le \tau < t < a\}$ to $\mathcal{L}(X)$ with the following properties:

(i) $U^{\mathcal{A}}(\tau, \tau) = \mathcal{I}$ for $\tau \in [0, a)$ (here and in what follows, $\mathcal{I}$ denotes the identity map of the corresponding space);

(ii) for each $\tau \in [0, a)$, the function $[\tau, a) \ni t \to f_\tau(t) := U^{\mathcal{A}}(\tau, t)$ is continuous with respect to the topology of pointwise convergence (the strong operator topology) on $\mathcal{L}(X)$, and $\frac{1}{\delta}(f_\tau(\tau + \delta)\varphi - \varphi) \to \mathcal{A}(\tau)\varphi$ in X as $\delta \to 0$ for all $\varphi \in$ dom$\mathcal{A}$;

(iii) for any $0 \le t_1 < t_2 < t_3 < a$, the relation $U^{\mathcal{A}}(t_2, t_3)U^{\mathcal{A}}(t_1, t_2) = U^{\mathcal{A}}(t_1, t_3)$ holds.

Remark 1. If $U^{\mathcal{A}}$ is an evolution family in the sense of the above definition, then, for any $\varphi \in X$ and any $\tau \in [0, a)$, the function $[\tau, a) \ni t \to f_\tau(t)\varphi \in X$ is a solution of the Cauchy problem for the equation $\psi'(t) = \mathcal{A}(t)\psi(t)$ with initial data (τ, φ).

Theorem 1 (generalization of Chernoff's theorem [2, 4, 5]). *Let X be a Banach space with norm $\|\cdot\|$, and let $\mathcal{L}_s(X)$ be the space $\mathcal{L}(X)$ with the topology of pointwise convergence; suppose that, for $a > 0$, F: $V_a \to \mathcal{L}_s(X)$ is a continuous mapping with the following properties:*

(i) $F(\tau, \tau) = \mathcal{I}$;

(ii) *there exists an $\alpha > 0$ such that if $\tau \in [0, a)$, $\delta > 0$, and $(\tau, \tau + \delta) \subset V_a$, then $\|F(\tau, \tau + \delta)\| \le \exp(\alpha\delta)$;*

(iii) *there exists a vector subspace D of X such that*

(a) *for any $x \in D$ and any $\tau \in [0, a)$, the limit*

$$F_2'(\tau, \tau) := \lim_{\delta \to 0} \frac{F(\tau, \tau + \delta)x - x}{\delta}$$

exists in X; this means that D is contained in the domain of the derived function $[\tau, a) \ni t \mapsto F(\tau, \tau + t) \in \mathcal{L}^s(X)$ at zero;

(b) *the operator $F_2'(\tau, \tau)$ with domain D defined by $D \ni x \mapsto F_2'(\tau, \tau)x \in X$ has a closure, which coincides with $\mathcal{A}(\tau)$.*

Then, for any $\tau \in [0, a)$ and $t > 0$ such that $(\tau, t) \in V_a$, the sequence of operators $\prod_{k=n}^{k=1} F\left(\tau + \frac{k-1}{b}t, \tau + \frac{k}{n}t\right) \in \mathcal{L}_s(X)$ converges to $U^{\mathcal{A}}(\tau, \tau + t)$ uniformly with respect to admissible $t \in [0, T]$ for any $T > 0$.

The proof of this theorem is similar to that of Chernoff's classical theorem [2, 4, 5].

2. FEYNMAN FORMULAS FOR EVOLUTION FAMILIES OF OPERATORS IN FUNCTION SPACES ON RIEMANNIAN MANIFOLDS

If X is a function space on a measure space and the values of the mapping F specified in Theorem 1 are integral operators on X, then the formula

$$\prod_{k=n}^{k=1} F\left(\tau + \frac{(k-1)t}{n}, \tau + \frac{kn}{n}\right) \to U^{\mathcal{A}}(\tau, \tau + t)$$

$$\text{as} \quad n \to \infty$$

is called the Feynman formula in X for the evolution family $U^{\mathcal{A}}$ on $[0, a)$ and the mapping F. Moreover, if the assumptions of Theorem 1 hold, then the Feynman formula in X for the evolution family $U^{\mathcal{A}}$ on $[0, a)$ and the mapping F is valid.

Let S be a compact Riemannian submanifold of the space $E = \mathbb{R}^n$; for $x \in S$, by T_xS we denote the tangent space to S at x and by $\mathrm{pr}_{T_xS}: \mathbb{R}^n \to T_xS$, the orthogonal projection. We use the symbol ∇ to denote the covariant derivative generated by the Riemannian (Levi-Civita) connection. Note that, given any function $\psi \in C^\infty(S)$, its second covariant derivative $\nabla^2\psi$: $S \to \mathcal{L}(TS)$ defines a linear map on each fiber T_xS of TS (we denote it by the same symbol).

Definition 2. The operator Δ_A^S on $\mathfrak{L}_p(S)$, $1 \le l < \infty$, associated with Δ_A is defined by

$$(\Delta_A^S\psi)(x) := \mathrm{tr}(\mathrm{pr}_{T_xS}A(x)|_{T_xS}\nabla^2\psi(x)).$$

Remark 2. In the space $\mathcal{L}(T_xS)$ $\mathrm{pr}_{T_xS}A(x)|_{T_xS}\nabla^2\psi(x) = (\mathrm{pr}_{T_xS}A(x))(\mathrm{pr}_{T_xS}\nabla^2\psi(x))$.

Let $a > 0$, and let F^j, $j = 1, 2$, be mappings of V_a to $\mathcal{L}(\mathfrak{L}_p(S))$, $1 \le p < \infty$, defined by the following relations (throughout the paper, we assume that all operators $G(t)(x)$ are invertible):

$$(F^1(\tau, t)\varphi)(x) := c_1(\tau, t, x)\int_S e^{-\frac{((G(\tau)(x))^{-1}(x-z), x-z)}{2(t-\tau)}}\varphi(z)dz,$$

$$F^1(0) = \mathcal{I},$$

$$(F^2(\tau, t)\varphi)(x) := c_2(\tau, t, x)\int_S e^{-\frac{((G(\tau)(z))^{-1}(x-z), x-z)}{2(t-\tau)}}\varphi(z)dz,$$

$$F^2(0) = \mathcal{I},$$

where $\varphi \in \mathfrak{L}_p(S)$, and

$$\frac{1}{c_1(\tau, t, x)} := \int_S e^{-\frac{((G(\tau)(x))^{-1}(x-z), x-z)}{2(t-\tau)}}dz,$$

$$\frac{1}{c_2(\tau, t, x)} := \int_S e^{-\frac{((G(\tau)(z))^{-1}(x-z), x-z)}{2(t-\tau)}} dz.$$

Theorem 2. *If* $\varphi \in \mathrm{dom}\,\Delta^S_{G(\tau)}$, *then*

$$((F^j)'(\tau, \tau)_2(\varphi))(x) = \frac{1}{2}(\Delta^S_{G(\tau)}\varphi)(x), \quad j = 1, 2.$$

Let $X = \mathfrak{L}_p(S)$, $1 \le p < \infty$; consider the operators $\mathcal{A}(t)$, $t \in [0, a)$, on X defined by $\mathcal{A}(t) := \Delta^S_{G(t)}$.

Theorem 3. *The Feynman formula is valid in* $\mathfrak{L}_p(S)$, $1 \le p < \infty$, *for the evolution family* $U^{\mathcal{A}}$ *on* $[0, a)$ *and each of the functions* F^1 *and* F^2.

The proof of Theorem 3 uses Theorems 1 and 2.

Let $a > 0$, and let F^{Qj}, $j = 1, 2$, be mappings of V_a to $\mathcal{L}(\mathfrak{L}_2(S))$ defined by

$$(F^{Q1}(\tau, t)\varphi)(x) := c_1^Q(\tau, t, x)$$
$$\times \int_S e^{-\frac{(i(G(\tau)(x))^{-1}(x-z), x-z)}{2(t-\tau)}} \varphi(z)dz, \quad F^{Q1}(0) = \mathcal{I},$$

$$(F^{Q2}(\tau, t)\varphi)(x) := c_2^Q(\tau, t, x)$$
$$\times \int_S e^{-\frac{(i(G(\tau)(z))^{-1}(x-z), x-z)}{2(t-\tau)}} \varphi(z)dz, \quad F^{Q2}(0) = \mathcal{I},$$

where $\varphi \in \mathfrak{L}_2(S)$, and

$$\frac{1}{c_1^Q(\tau, t, x)} := \int_S e^{-\frac{(i(G(\tau)(x))^{-1}(x-z), x-z)}{2(t-\tau)}} dz,$$

$$\frac{1}{c_2^Q(\tau, t, x)} := \int_S e^{-\frac{(i(G(\tau)(z))^{-1}(x-z), x-z)}{2(t-\tau)}} dz.$$

Here, we use the natural regularizations of integrals of complex exponentials.

Let $X = \mathfrak{L}_2(S)$, and let $\mathcal{A}^Q(t)$, $t \in [0, a)$ be the operators on X defined by $\mathcal{A}^Q(t) := i\Delta^S_{G(t)}$.

Theorem 4. *The Feynman formula is valid in* $\mathfrak{L}_2(S)$ *for the evolution family* $U^{\mathcal{A}^Q}$ *on* $[0, a)$ *and each of the functions* F_1^Q *and* F_2^Q.

Remark 3. Theorem 4 provides a solution of the Cauchy problem for a Schrödinger-type equation in which the Hamiltonian may be even asymmetric. Similar formulas can also be obtained for symmetric Hamiltonians.

Remark 4. The Feynman formulas obtained above are approximations of integrals of so-called chronological exponentials. In fact, according to the usual definition of such integrals, these integrals are equal to the limits of integrals of Feynman formulas. We also mention that, in the case under consideration, the corresponding chronological exponential can be regarded as the generalized density of a probability measure or a Feynman pseudomeasure on a space of functions taking values in S.

Remark 5. Feynman formulas can also be obtained for equations of the form

$$\dot{f}(t) = \frac{1}{2}\Delta^S_{G(t)}f(t) + (\nabla(f(t)), h(t)) + V(t)f(t),$$

where $h(t)$ is a vector field on S and $V(t)$ is a function on S for each t. The multiple integrals in these formulas approximate integrals over spaces of functions taking values in S; the integrands contain exponentials of stochastic integrals determined by the vector field h as multipliers.

3. CONJECTURES

In this section, we formulate two conjectures related to the results obtained above. The Green function of the Cauchy problem for the equation $\dot{f}(t) = \frac{1}{2}\Delta^S_{G(t)}f(t)$ (with respect to the unknown function f taking values in $\mathfrak{L}_p(S)$) and the Green function of the Cauchy problem for the equation $\dot{\varphi}(t) = \frac{1}{2}\Delta_{G(t)}\varphi(t)$ (with respect to the unknown function φ taking values in $\mathfrak{L}_p(E)$) generate probability measures μ_S and μ_E on the space of functions on $[0, a)$ taking values in S and E, respectively (or, which is the same thing, random processes in S and E). For each $\varepsilon > 0$, let S^ε be the ε-neighborhood of the manifold S in E, and let $C_b([0, a], S)$, $C_b([0, a], S^\varepsilon)$, and $C_b([0, a], E)$ be the spaces of continuous functions on $[0, a]$ taking values in S, S^ε, and E, respectively, and vanishing at $b \in S$. We assume that μ_S and μ_E are countably additive on $C_b([0, a], S)$ and on $C_b([0, a], E)$. Let $\bar{\nu}^\varepsilon$ be the restriction of the measure μ_E to $C_b([0, a], S^\varepsilon)$; we set $\nu^\varepsilon := \bar{\nu}^\varepsilon/\mu_E(C_b([0, a], S^\varepsilon)$ and define μ^ε to be the probability measure on $C_b([0, a], E)$ being the image of ν^ε under the canonical embedding of $C_b([0, a], S)$ into $C_b([0, a], E)$.

Conjecture 1. For suitable G, there exists a probability measure ν_S on $C_b([0, a], E)$ concentrated on $C_b([0, a], S)$ and such that $\mu^\varepsilon \to \nu_S$ in the weak topology on the space $\mathcal{M}(C_b([0, a], E))$ of measures on $C_b([0, a], E)$ determined by the duality between the space $\mathcal{M}(C_b([0, a], E))$ and an appropriate space consisting of bounded infinitely differentiable cylindrical functions on $C_b([0, a], E)$ all of whose derivatives are bounded. Moreover, the measure ν_S is equivalent to μ_S (it would be useful to find the corresponding Radon–Nikodym density). In the case where $G(t)(x)$ is the identity mapping for each $t \in [0, a]$ and each $x \in E$, this conjecture was proved and the Radon–Nykodim density was found in [7].

Let η^ε be the probability measure generated on $C_b([0, a], S^\varepsilon)$ by the Green function of the Cauchy–Neumann problem for the equation $\dot{\psi} = \frac{1}{2}\Delta_{G(t)}\psi(t)$ with respect to the unknown function ψ taking values in $\mathfrak{L}_p(S^\varepsilon)$, and let γ^ε be the measure on $C_b([0, a], E)$ being the image of η^ε under the canonical embedding of $C_b([0, a], S^\varepsilon)$ into $C_b([0, a], E)$.

Conjecture 2. In the space $\mathcal{M}(C_b([0, a], E))$ endowed with the topology defined above, $\gamma^\varepsilon \to \mu_S$. This was proved in [6] in the case where $G(t)(x) = \mathcal{I}$ for all $t \in [0, a]$ and all $x \in E$.

Remark 6. The formulas given above are usually referred to as Lagrangian formulas, because the functional integrals approximated by these formulas contain action in Lagrangian form. However, the corresponding Hamiltonian formulas can be obtained as well [3]. Such formulas are useful if the classical Hamiltonian function is not the sum of the kinetic and the potential energy.

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