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MATHEMATICS

On the Solvability of Certain Discrete Equations and Related Estimates of Discrete Operators

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Abstract—The solvability of discrete equations with Calderón–Zygmund kernels is studied, the limit passage from the discrete to the continuous case is substantiated, and estimates of the errors of discrete solutions are obtained.

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1. A function $K(x, y)$ defined on $\mathbf{R}^m \times (\mathbf{R}^m \backslash \{0\})$ is called a Calderón–Zygmund kernel [10, 11] if it satisfies the following conditions:

(i) $K(x, tx) = t^{-m}K(x, y)$ for any $x \in \mathbf{R}^m$ and any $t > 0$;

(ii) $\int\limits_{S^{m-1}} K(x, \omega)d\omega = 0$ for any $x \in \mathbf{R}^m$;

(iii) $|K(x, y)| \le C$ and $K(x, \omega)$ is differentiable on S^{m-1} for any $x \in \mathbf{R}^m$, where S^{m-1} is the unit sphere of m-space and C is a constant.

Let $\mathbf{Z}^m$ be the integer lattice in m-space $\mathbf{R}^m$. We set $K(0) = 0$ and denote the restriction of a kernel $K(x)$ to $h\mathbf{Z}^m$, $h > 0$, by K_d. Let u_d be a function of a discrete argument defined on the lattice $h\mathbf{Z}^m_h$.

For the multidimensional singular integral operator (v.p. means "valeur principale," i.e., principal value)

$$(Ku)(x) = \text{v.p.}\int\limits_D K(x-y)u(y)dy, \quad x \in D \subset \mathbf{R}^m,$$

we consider the discrete analogue

$$(K_d u_d)(x) = \sum_{\tilde{y} \in h\mathbf{Z}^m} K_d(\tilde{x}-\tilde{y})u_d(\tilde{y})h^m, \quad \tilde{x} \in h\mathbf{Z}^m,$$

the sum of the series is understood as the limit

$$\lim_{N \to \infty} \sum_{\tilde{y} \in h\mathbf{Z}^m \cap Q_N} K_d(\tilde{x}-\tilde{y})u_d(\tilde{y})h^m,$$

of partial sums, where

$$Q_N = \{x \in \mathbf{R}^m: \max_{1 \le k \le m} |x_k| \le N\}.$$

We use the symbol ℓ_h^2 to denote the Hilbert space $L_2(h\mathbf{Z}^m)$ of functions of a discrete argument with inner product

$$(u_d, \upsilon_d) = \sum_{\tilde{x} \in h\mathbf{Z}^m} u_d(\tilde{x})\overline{\upsilon_d(\tilde{x})}h^m$$

and norm

$$\|u_d\|_{\ell_h^2} = \left(\sum_{\tilde{x} \in h\mathbf{Z}^m} |u_d(\tilde{x})|^2 h^m\right)^{1/2}.$$

Theorem 1. *The estimate*

$$\|K_d u_d\|_{\ell_h^2} \le c\|u_d\|_{\ell_h^2},$$

holds, where the constant c does not depend on h.

The symbol of the operator K is defined as the Fourier transform of the kernel $K(x)$ in the sense of principal value [10, 11], that is, as

$$\sigma(\xi) = \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int\limits_{\varepsilon < |x| < N} K(x)e^{i\xi \cdot x}dx.$$

With the discrete operator K_d we associate a symbol $\sigma_d(\xi)$, too, where $\xi \in [-\pi h^{-1}, \pi h^{-1}]^m$; this symbol is determined by the multidimensional Fourier series

$$\sigma_d(\xi) = \sum_{\tilde{x} \in h\mathbf{Z}^m} K(\tilde{x})e^{i\tilde{x} \cdot \xi}h^m,$$

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where the partial sums are over the discrete cubes $Q_N \cap h\mathbf{Z}^m$, and is a periodic function on $\mathbf{R}^m$ with main cube of periods $[-\pi h^{-1}, \pi/h^{-1}]^m$.

Accordingly, by the symbol of the discrete singular equation

$$(aI + K_d)u_d = \upsilon_d \quad (1)$$

we understand the function $a + \sigma_d(\xi)$, $\xi \in [-\pi h^{-1}, \pi h^{-1}]^m$.

Let $D = \mathbf{R}^m$. We define a discrete space $H^\alpha_\gamma(h\mathbf{Z}^m)$ as the space of functions $u_d(\tilde{x})$ of a discrete argument with weight $\omega(\tilde{x}) = (1 + |\tilde{x}|)^\alpha$ and norm

$$\|u_d\|_{\alpha,\gamma} = \|\omega \cdot u_d\|_\gamma, \quad 0 < \gamma < 1, \quad 0 < \alpha + \gamma < m,$$

where

$$\|u_d\|_\gamma = \max_{\tilde{x} \in h\mathbf{Z}^m} |u_d(\tilde{x})| + \max_{\tilde{x}, \tilde{y} \in h\mathbf{Z}^m} \left|\frac{u_d(\tilde{x}) - u_d(\tilde{y})}{\tilde{x} - \tilde{y}}\right|.$$

Theorem 2. *The operator* K_d *boundedly acts on the space* $H^\alpha_\gamma(h\mathbf{Z}^m)$, *and its norm does not depend on h.*

Now, we set $D = \mathbf{R}^m_+$, $\mathbf{R}^m_+ = \{x \in \mathbf{R}^m: (x_1, \ldots, x_m), x_m > 0\}$, introduce the notation $x' = (x_1, \ldots, x_{m-1})$, and consider a weight function of the form

$$\omega(\tilde{x}) = (1 + |\tilde{x}|)^\alpha \left(\frac{\tilde{x}_m}{1 + \tilde{x}_m}\right)^\beta,$$

by $\mathbf{Z}^m_+$ we denote the discrete half-space $\{\tilde{x} \in \mathbf{Z}^m: \tilde{x}_m > 0\}$.

We define a discrete space $H^{\alpha,\beta}_\gamma(h\mathbf{Z}^m_+)$ as the space of functions $u_d(\tilde{x})$ of a discrete argument with norm

$$\|u_d\|_{\alpha,\beta,\gamma} = \|\omega \cdot u_d\|_\gamma, \quad 0 < \gamma < 1,$$
$$0 < \alpha + \gamma < m, \quad \gamma < \beta < \gamma + 1.$$

Theorem 3. *The operator* K_d *boundedly acts on the space* $H^{\alpha,\beta}_\gamma(h\mathbf{Z}^m_+)$, *and its norm does not depend on h.*

Let P_h denote the operator of restriction to the lattice $h\mathbf{Z}^m$; to each function defined on $\mathbf{R}^m$ this operator assigns the set of its discrete values at the points of the lattice $h\mathbf{Z}^m$.

The measure of approximation [4] of the operators K and K_d in a linear normed space X of functions on $\mathbf{R}^m$ is defined as the operator norm

$$\|P_h K - K_d P_h\|_{X_d},$$

where X_d is the normed space of functions on the lattice $h\mathbf{Z}^m$ with norm induced by that on the space X.

As the space X_d we shall use, in addition to ℓ^2_h, the space C_h of functions u_d of a discrete argument $\tilde{x} \in h\mathbf{Z}^m$ with norm

$$\|u_d\|_{C_h} = \max_{\tilde{x} \in h\mathbf{Z}^m} |u_d(\tilde{x})|.$$

In other words, C_h is the space of the restrictions of functions $u \in C(\mathbf{R}^m)$ to the lattice $h\mathbf{Z}^m$. It is worth mentioning here that the operator K is not bounded on the space $C(\mathbf{R}^m)$ but is bounded on $L_2(\mathbf{R}^m)$; it is well known that, if the right-hand side υ of the equation

$$(aI + K)u = \upsilon \quad (2)$$

has certain smoothness properties (e.g., satisfies the Höder condition), then a solution of Eq. (2) (if it exists in $L_2(\mathbf{R}^m)$) has the same smoothness properties [10]. Equations of type (2) often arise in applied problems, and it is required to substantiate a numerical solution method for these equations (see, e.g., [5, 9, 10]).

We define a discrete space $C_h(\alpha, \beta)$ as the space of functions of a discrete argument $\tilde{x} \in h\mathbf{Z}^m$ which have finite norm

$$\|u_d\|_{C_h(\alpha,\beta)} = \|u_d\|_{C_h} + \sup_{\tilde{x}, \tilde{y} \in h\mathbf{Z}^m} \frac{|\tilde{x} - \tilde{y}|^\alpha}{(\max\{1 + |\tilde{x}|, 1 + |\tilde{y}|\})^\beta},$$

and satisfy the conditions

$$|u_d(\tilde{x})| \le \frac{c}{(1 + |\tilde{x}|)^{\beta - \alpha}},$$

$$|u_d(\tilde{x}) - u_d(\tilde{y})| \le c \frac{|\tilde{x} - \tilde{y}|^\alpha}{(\max\{1 + |\tilde{x}|, 1 + |\tilde{y}|\})^\beta},$$

$$\forall \tilde{x}, \tilde{y} \in h\mathbf{Z}^m, \quad \alpha, \beta > 0, \quad 0 < \alpha < 1.$$

A continuous analogue of these spaces is the space $H^\alpha_\beta(\mathbf{R}^m)$ of functions which are continuous on $\mathbf{R}^m$ and satisfy the Hölder condition with exponent $0 < \alpha < 1$ and weight $(1 + |x|)^\beta$ (see [1]). Results of [1] imply, in particular, that the operator K is a linear bounded operator K: $H^\alpha_\beta(\mathbf{R}^m) \to H^\alpha_\beta(\mathbf{R}^m)$, where $m < \beta < \alpha + m$.

For the spaces $C_h(\alpha, \beta)$, the following theorem is valid.

Theorem 4. *The following estimate holds:*

$$\|K_d u_d\|_{C_h(\alpha,\beta)} \le c\|u_d\|_{C_h(\alpha,\beta)},$$

where $m < \beta < \alpha + m$ *and the constant c does not depend on h.*

Below we give an estimate of the measure of approximation of the operators K and K_d in the space $C_h(\alpha, \beta)$. This will allow us to estimate the error of the solution under the change of the continual operator K by its discrete analogue K_d.

Theorem 5. *The measure of approximation of the operators K and* K_d *satisfies the inequality*

$$\|P_h K - K_d P_h\|_{C_h(\alpha,\beta)} \le c h^{\tilde{\alpha}},$$

where the constant c does not depend on h, $\tilde{\alpha} < \alpha$, *and* $\tilde{\beta} > \beta$.

By a discrete solution we mean the solution of equation (1) with right-hand side $P_h\upsilon$.

Theorem 6. *For the discrete solution, the following estimate is valid:*

$$|u(\tilde{x}) - u_d(\tilde{x})| \le c h^{\tilde{\alpha}}.$$

2. To study equations in a half-space, we need a special periodic analogue of Riemann's classical boundary value problem, which naturally arises in studying discrete convolutions by using the discrete Fourier transform.

Let Π_+ and Π_- be the upper and the lower half-strip in the complex plane **C**:

$$\Pi_{\pm} = \{z \in \mathbf{C}: z = t + is, t \in [-\pi; \pi], \pm s > 0\}.$$

By the periodic Riemann problem we understand the following problem: Find a pair $\Phi^{\pm}(z)$ of analytic functions on $\Pi_{\pm}$ whose boundary values satisfy the following linear relation on the interval $[-\pi; \pi]$ as $s \to 0\pm$:

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in [-\pi; \pi], \tag{3}$$

where $G(t)$ and $g(t)$ are functions on $[-\pi; \pi]$ such that $G(-\pi) = G(\pi)$, and $g(-\pi) = g(\pi)$.

To solve the problem under consideration, we introduce the integral

$$\Phi(z) = \frac{1}{4\pi i}\int_{-\pi}^{\pi} \varphi(x)\cot\frac{x-z}{2}dx, \quad z \in \Pi_{\pm},$$

which is similar to a Cauchy-type integral. For this integral, we obtain corresponding analogues of Sokhotskii's formulas [2, 6].

Theorem 7. *If* $\varphi(t)$ *satisfies the Hölder condition on the interval* $[-\pi; \pi]$, $\varphi(-\pi) = \varphi(\pi)$, *then the boundary values* $\Phi^{\pm}(t)$ *of* $\Phi(z)$ $(z = t + is)$ *as* $s \to 0\pm$ *can be expressed as*

$$\Phi^+(\tau) = \frac{1}{4\pi i}\int_{-\pi}^{\pi} \varphi(t)\cot\frac{t-\tau}{2}dt + \frac{\varphi(\tau)}{2} + C,$$

$$\Phi^-(\tau) = \frac{1}{4\pi i}\int_{-\pi}^{\pi} \varphi(t)\cot\frac{t-\tau}{2}dt - \frac{\varphi(\tau)}{2} + C;$$

where the integral is understood in the sense of principal value.

The index κ of problem (3) is defined as the increment of the argument of the function G on the interval $[-\pi, \pi]$ divided by 2π. In what follows, we assume that the function $G(t)$ satisfies the Hölder condition. We introduce the following notation:

$$\Gamma(z) = \frac{1}{4\pi i}\int_{-\pi}^{\pi} \ln[e^{it\kappa}G(t)]\cot\frac{t-z}{2}dt + \frac{1}{4\pi i}\int_{-\pi}^{\pi} \ln[e^{-it\kappa}G(t)]dt,$$

$$X^+(z) = e^{\Gamma^+(z)}, \quad X^-(z) = e^{-it\kappa}e^{\Gamma^-(z)},$$

$$S_\kappa(z) = \sum_{k=0}^{\kappa} c_k e^{-izk} \quad (\kappa > 0).$$

Theorem 8. *If the index* κ *of the problem for the strip is nonnegative, then the homogeneous problem* has $\kappa + 1$ *linearly independent solutions:*

$$\Phi^+(z) = S_\kappa(z)e^{\Gamma^+(z)},$$

$$\Phi^-(z) = e^{-iz\kappa}S_\kappa(z)e^{\Gamma^-(z)} \quad (k = 0, 1, \ldots, \kappa).$$

The general solution contains $\kappa + 1$ *arbitrary constants. For a negative index, the problem is unsolvable.*

Theorem 9. *In the case of a nonnegative* κ, *the inhomogeneous problem for the strip is solvable for any right-hand side, and its general solution is given by*

$$\Phi(z) = \frac{X(z)}{4\pi i}\int_{-\pi}^{\pi} \frac{g(t)}{X^+(t)}\cot\frac{t-z}{2}dt + X(z)S_\kappa(z).$$

In the case of $\kappa < 0$, *the inhomogeneous problem is generally unsolvable. It is solvable if and only if the right-hand side satisfied* $-\kappa - 1$ *additional solvability conditions. Under these conditions, the solution is*

$$\Phi(z) = \frac{X(z)}{4\pi i}\int_{-\pi}^{\pi} \frac{g(t)}{X^+(t)}\cot\frac{t-z}{2}dt.$$

3. Here we consider discrete convolutions over a half-space with Calderón–Zygmund kernels from the point of view of their solvability, describe their relationship with Riemann's periodic problem with a parameter, and perform a comparison with the continuous case.

Consider the equation

$$(P_+M_1 + P_-M_2)u = \upsilon, \tag{4}$$

where M_1 and M_2 are Calderón–Zygmund operators (similar to K in Eq. (2)) with kernels $M_1(x)$ and $M_2(x)$; by P_+ and P_- we understand the operators of restriction to the half-space $\mathbf{R}^m_{\pm} = \{x = (x_1, \ldots, x_m), \pm x_m > 0\}$.

The solvability of Eq. (4) can be studied by means of the theory of Riemann's classical boundary value problem [2, 6]. Denoting the Fourier transform by F, we obtain the relations

$$FP_+ = Q_{\xi'}F, \quad FP_- = P_{\xi'}F,$$

$$P = \frac{1}{2}(I + H_{\xi'}), \quad Q = \frac{1}{2}(I - H_{\xi'}).$$

Here, $H_{\xi'}$ denotes the Hilbert transform with respect to the variable ξ_m and $\xi' = (\xi_1, \ldots, \xi_{m-1})$:

$$(H_{\xi'}u)(\xi', \xi_m) \equiv \frac{1}{\pi i}\text{v.p.}\int_{-\infty}^{+\infty} \frac{u(\xi', \tau)}{\tau - \xi_m}d\tau.$$

Thus, Eq. (4) transforms into the following equation with parameter ξ':

$$\frac{\sigma_{M_1}(\xi', \xi_m) + \sigma_{M_2}(\xi', \xi_m)}{2}\tilde{U}(\xi) + \frac{\sigma_{M_1}(\xi', \xi_m) + \sigma_{M_2}(\xi', \xi_m)}{2\pi i}$$

$$\times \text{v.p.} \int_{-\infty}^{+\infty} \frac{\tilde{U}(\xi', \eta)}{\eta - \xi_m} d\eta = \tilde{F}(\xi). \quad (5)$$

This equation corresponds to Riemann's boundary value problem (with parameter ξ') with coefficient

$$G(\xi', \xi_m) = \sigma_{M_1}(\xi', \xi_m)\sigma_{M_2}^{-1}(\xi', \xi_m).$$

To ensure the unique solvability of Eq. (4), we must require that the index $G(\xi', \xi_m)$ with respect to the variable ξ_m vanish.

The symbol of the Calderón–Zygmund operator is very specific. This is a homogeneous function of degree 0; thus, essentially, it is defined on the unit sphere S^{m-1}. Let $m \geq 3$. Fix $\xi' \in S^{m-2}$ and suppose that $G(0, -1) = G(0, +1)$. As ξ_m varies between $-\infty$ and $+\infty$, the argument of $G(\xi)$ takes values on an arc of the great circle joining the points $(0, -1)$ and $(0, +1)$. At the same time, the symbol takes values on a closed curve in the complex plane. For different ξ', all these curves are homotopic, i.e., all of them have the same index κ. The condition $\kappa = 0$ ensures the existence and uniqueness of a solution to equation (4).

We proceed to the discrete equation

$$(P_+M_1^d + P_-M_2^d)u_d = \upsilon_d \quad (6)$$

in the discrete space $L_2(h\mathbf{Z}^m)$, assuming that the $P_\pm$ in (6) are the operators of restriction to $h\mathbf{Z}_\pm^m$ and M_1^d and M_2^d are the discrete Calderón–Zygmund operators generated by the kernels $M_1(x)$ and $M_2(x)$, which are bounded operators on the spaces $L_2(h\mathbf{Z}^m)$.

The discrete Fourier transform for functions of a discrete argument on the lattice $h\mathbf{Z}^m$ is given by

$$u(\tilde{x}) \mapsto \frac{1}{(2\pi)^m} \sum_{\tilde{x} \in h\mathbf{Z}^m} u(\tilde{x}) e^{-i\tilde{x}\cdot\xi} h^m \equiv \tilde{u}(\xi),$$

$$\xi \in [-h^{-1}\pi, h^{-1}\pi]^m.$$

This Fourier transform has the same properties as the classical transform [8].

According to the above considerations, we define a periodic analogue of the Hilbert transform with respect to the variable ξ_m ($\xi \in [-\pi, \pi]^m$, ξ' is fixed) by

$$(H_{\xi'}^{\text{per}} u)(\xi_m) = \frac{1}{2\pi i} \int_{-\pi h^{-1}}^{\pi h^{-1}} u(t) \cot\frac{h(t - \xi_m)}{2} dt.$$

The periodic analogues of the projections are

$$P_{\xi'}^{\text{per}} = \frac{1}{2}(I + H_{\xi'}^{\text{per}}), \quad Q_{\xi'}^{\text{per}} = \frac{1}{2}(I - H_{\xi'}^{\text{per}}).$$

Finally, the periodic analogue of Eq. (5) is

$$\frac{\sigma_{1,h}(\xi', \xi_m) + \sigma_{2,h}(\xi', \xi_m)}{2} \tilde{U}(\xi)$$

$$+ \frac{\sigma_{1,h}(\xi', \xi_m) + \sigma_{2,h}(\xi', \xi_m)}{4\pi i}$$

$$\times \text{v.p.} \int_{-\pi h^{-1}}^{\pi h^{-1}} \tilde{U}(\xi', \eta) \cot\frac{h(\eta - \xi_m)}{2} d\eta = \tilde{F}(\xi), \quad (7)$$

where $\sigma_{1,h}$ and $\sigma_{2,h}$ are the symbols of the discrete operators M_1 and M_2.

Equation (7) is naturally related to a Riemann's periodic boundary value problem, and the condition for the unique solvability of this problem is given by Theorem 8. In the case under consideration, this is the condition

$$\text{Ind}\,\sigma_{1,h}(\cdot, \xi_m)\sigma_{2,h}^{-1}(\cdot, \xi_m) = 0.$$

Theorem 10. *Equations* (4) *and* (6) *are solvable or unsolvable simultaneously.*

4. It follows from results of the preceding section that, theoretically, we can expect the convergence of a discrete solution to a continuous one with decreasing the step size of the lattice. However, in practice, finding a solution of the discrete equation (1), which is an infinite system of linear algebraic equations, comes across the problem of choice of a finite approximation.

On our view, more pragmatic is the following scheme of finite approximation. Given a discrete kernel K_d and a right-hand side v_d, we construct their periodic approximations by restricting them to $Q_N \cap h\mathbf{Z}^m$ and periodically extending the restrictions to $h\mathbf{Z}^m$. We denote these approximations by $K_{d,N}$ and $\upsilon_{d,N}$, respectively. Instead of Eq. (1), we consider the equation

$$au_{d,N}(\tilde{x}) + \sum_{\tilde{y} \in \mathbf{Z}_h^m} K_{d,N}(\tilde{x} - \tilde{y})u_{d,N}(\tilde{x})h^m$$

$$= \upsilon_{d,N}(\tilde{x}), \quad \tilde{x} \in \mathbf{Z}_h^m, \quad (8)$$

in fact, this is a finite system of linear algebraic equations so-called cyclic convolution [7]. The apparatus of the discrete Fourier transform and properties of the symbol of a multidimensional singular integral make it possible to substantiate the solvability of Eq. (8) at large N, and applying the fast Fourier transform, we can avoid solving systems of linear algebraic equations and restrict ourselves to twice calculating Fourier transforms (direct and inverse). Moreover, a comparison of the numerical results for the simplest types of test equations (both regular and singular) obtained by using projection methods [3] and fast Fourier transform [7] demonstrated their close coincidence and a serious gain in time when the latter was used even in the one-dimensional case [12]. As the dimension increases, the difference becomes more significant.

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