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MATHEMATICS

The Structure of the Positive Discrete Spectrum of the Evolution Operator Arising in Branching Random Walks

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Abstract—A branching random walk (BRW) with continuous time and a finite number of branching sources located at points of a multidimensional lattice is considered. The definition of weakly supercritical BRWs, whose discrete spectrum contains a unique positive eigenvalue, is introduced. Conditions for a supercritical BRW to be weakly supercritical are determined.

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Consider a branching random walk (BRW) with continuous time on a lattice $\mathbb{Z}^d$, $d \geq 1$, assuming that there is a single particle in the system at the initial moment of time, which is located at some point x, and branching, i.e., birth and death of particles, occur at N lattice points $x_1, x_2, \ldots, x_N$, called branching sources.

Apparently, a BRW model with one source was first considered in [1] for a simple random walk with pure birth, i.e., without death of particles at the source. The more general case of a symmetric or symmetrizable BRW with finite variance of jumps and a single source has subsequently been studied by many authors (see, e.g., [2, 3]). A general model of a BRW with finitely many branching sources at which the walk symmetry can be both preserved and violated was introduced and studied in [4].

The presence of a positive eigenvalue in the spectrum of the BRW evolution operator ensures the exponential growth of the number of particles at each lattice point and on the entire lattice (i.e., the BRW is supercritical) [5]. For this reason, the authors of the works mentioned above usually restricted themselves to determining only the highest eigenvalue. At the same time, in a number of situations, the information on whether a positive eigenvalue is unique or nonunique and, in the latter case, on the location of the other eigenvalues of the evolution operator may be important for analyzing the behavior of the corresponding BRW.

For example, the uniqueness of a positive eigenvalue substantially facilitates the study of the propagation of particle fronts [6]. However, in the presence of two and more sources on $\mathbb{Z}^d$, the behavior of solutions of differential equations for the moments of numbers of BRW particles is determined not only by the value of the leading positive eigenvalue but also by the mutual arrangement of the positive eigenvalues of the evolution operator [4].

In this connection, here we study conditions for the emergence of a simple isolated positive eigenvalue in the spectrum of the evolution operator with increasing the intensity of the branching sources. We also study the process of the appearance of positive eigenvalues with further increasing the intensity of sources. We show that the appearance of eigenvalues and their multiplicity are determined not only by the intensities of sources but also by their spatial configuration. This study makes it possible to reveal the difference in the behaviors of processes on lattice and continuous (see, e.g., [7]) structures.

PROBLEM STATEMENT

Let $A = (a(x, y))_{x, y \in \mathbb{Z}^d}$ be the matrix of transition intensities in the random walk, where $a(x, y) \geq 0$ for $x \neq y$, $a(x, x) < 0$,

$$a(x, y) = a(y, x) = a(0, y - x) = a(y - x)$$

and $\sum_z a(z) = 0$. Suppose that the matrix A is irreducible, i.e., for each $z \in \mathbb{Z}^d$, there exists a set of vectors z_1, $z_2, \ldots, z_k \in \mathbb{Z}^d$ such that $z = \sum_{n=0}^{\infty} z_i$ and $a(z_i) \neq 0$ for $i = 1$, $2, \ldots, k$. The mechanism of branching at the sources does not depend on the walk and is determined by an

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infinitesimal generating function $f(u) := \sum_{n=0}^{\infty} b_n u^n$, where $b_n \geq 0$ for $n \neq 1$, $b_1 < 0$, and $\sum_n b_n = 0$. As usual, it is assumed that each particle evolves independently of the other particles. We also assume that there exist all derivatives $\beta_r := f^{(r)}(1)$, $r \in \mathbb{N}$, i.e., all moments of the number of direct offsprings of each particle are finite; for brevity, we use the notation $\beta := \beta_1$. The finiteness of all moments is used in the proof of limit theorems on the behavior of the number of particles by the method of moments [8]. In what follows, it suffices to assume only the existence of β.

In BRW models, there arise multipoint perturbations of the generator $\mathcal{A}$ of a symmetric random walk [4]; in the case where the intensities of the sources are identical, the corresponding evolution operators have the form

$$\mathcal{H}_\beta = \mathcal{A} + \beta \sum_{i=1}^{N} \Delta_{x_i},$$

where $x_i \in \mathbb{Z}^d$, $\mathcal{A}$: $l^p(\mathbb{Z}^d) \to l^p(\mathbb{Z}^d)$, $p \in [1, \infty]$, is the symmetric operator generated by the matrix A and acting by the rule $(\mathcal{A}u)(z) := \sum_{z' \in \mathbb{Z}^d} a(z - z')u(z')$, $\Delta_x = \delta_x \delta_x^T$, and $\delta_x = \delta_x(\cdot)$ denotes the column vector on the lattice taking the value 1 at x and vanishing at the other points. The perturbation $\beta \sum_i \Delta_x$ of the linear operator $\mathcal{A}$ may lead to the appearance of positive eigenvalues of $\mathcal{H}_\beta$; the multiplicity of every such eigenvalue does not exceed the number N of summands in the last sum [4].

Let $p(t, x, y)$ denote the transition probability of the random walk; naturally, the function $p(t, x, y)$ is determined by the transition intensities $a(x, y)$ (see, e.g., [9, 10]). The Green function of the operator $\mathcal{A}$ can be represented in the form of the Laplace transform of the transition probability $p(t, x, y)$:

$$G_\lambda(x, y) := \int_0^\infty e^{-\lambda t} p(t, x, y)\,dt, \quad \lambda \geq 0.$$

The BRW analysis substantially depends on whether the quantity $G_0 = G_0(0, 0)$ is finite or infinite. If the variance of jumps (here and in what follows, $|z|$ denotes the Euclidean norm of z) is finite,

$$\sum_z |z|^2 a(z) < \infty \tag{1}$$

then $G_0 = \infty$ at $d = 1, 2$ and $G_0 < \infty$ at $d \geq 3$ (see, e.g., [2]). If, for all $z \in \mathbb{Z}^d$ with sufficiently large norm, the asymptotic relation

$$a(z) \sim \frac{H\left(\frac{z}{|z|}\right)}{|z|^{d+\alpha}}, \quad \alpha \in (0, 2), \tag{2}$$

holds, where $H(\cdot)$ is a continuous positive function symmetric on the sphere $\mathbb{S}^{d-1} = \{z \in \mathbb{R}^d: |z| = 1\}$, then $G_0 = \infty$ for $d = 1$ and $\alpha \in [1, 2)$ and G_0 is finite if $d = 1$ and $\alpha \in (0, 1)$ or $d \geq 2$ and $\alpha \in (0, 2)$ [11]. Condition (2), unlike (1), leads to the divergence of the series $\sum_z |z|^2 a(z)$ and, thereby, to the infinity of the variance of jumps.

THE DISCRETE SPECTRUM OF THE EVOLUTION OPERATOR

Let β_c denote the least intensity of the source with the property that, for $\beta > \beta_c$, the spectrum of $\mathcal{H}_\beta$ contains positive eigenvalues.

Theorem 1. *Suppose that a* BRW *is based on a symmetric spatially homogeneous irreducible random walk and one of conditions* (1) *or* (2) *holds. If* $G_0 = \infty$, *then* $\beta_c = 0$ *for* $N \geq 1$. *If* $G_0 < \infty$, *then* $\beta_c = G_0^{-1}$ *for* $N = 1$ *and* $0 < \beta_c < G_0^{-1}$ *for* $N \geq 2$.

In the case where $G_0 < \infty$, $N = 2$, and condition (1) holds, the quantity β_c was evaluated in [4]:

$$\beta_c = (G_0 + \tilde{G}_0)^{-1}, \tag{3}$$

where $\tilde{G}_0 = G_0(x_1, x_2)$. Although, condition (1) is inessential for the argument of [4], and relation (3) remains valid under condition (2).

Additional information about the structure of the discrete spectrum of $\mathcal{H}_\beta$ is provided by the following theorem.

Theorem 2. *Let* $N \geq 2$. *For* $\beta > \beta_c$, *the operator* $\mathcal{H}_\beta$ *can have at most* N *positive eigenvalues*

$$\lambda_0(\beta) > \lambda_1(\beta) \geq \ldots \geq \lambda_{N-1}(\beta) > 0 \tag{4}$$

counting their multiplicity; here, the eigenvalue $\lambda_0(\beta)$ *is simple. Moreover, there exists a value* $\beta_{c_1} > \beta_c$ *such that, for* $\beta \in (\beta_c, \beta_{c_1})$, *the operator has a unique eigenvalue* $\lambda_0(\beta)$.

In the general case, finding eigenvalues (4) is a difficult problem; it can be solved by using the following assertion proved in [4] in the more general case of different intensities of sources.

Theorem 3. *An eigenvalue* λ *belongs to the discrete spectrum of* $\mathcal{H}_\beta$ *if and only if the system of linear equations*

$$V_i - \beta \sum_{j=1}^{N} G_\lambda(x_i, x_j) V_j = 0, \quad i = 1, 2, \ldots, N, \tag{5}$$

has a nontrivial solution with respect to the variables $\{V_i\}_{i=1}^N$.

In the presence of a certain "symmetry" in the arrangement of sources, some of the eigenvalues (4) of the operator $\mathcal{H}_\beta$ may coincide (i.e., they may be nonsimple).

Example 1. Let $\mathcal{A} = \kappa\Delta$, $\kappa > 0$, be a lattice Laplacian; suppose that $N \geq 2$ and $x_1, x_2, \ldots, x_N$ are points at which sources of the same intensity β are located. We assume that these points are vertices of a regular simplex, i.e., a simplex with equal edge lengths. Such simplices in $\mathbb{Z}^d$ are formed, e.g., by any combination of points being the vertices of some of the basis vectors. By virtue of Theorem 3, the existence of a nontrivial solution for the linear equation (5) for some β is equivalent to the vanishing of the determinant

$$\det\begin{bmatrix} G_\lambda(x_1, x_1) - \frac{1}{\beta} & \ldots & G_\lambda(x_1, x_N) \\ G_\lambda(x_2, x_1) & \ldots & G_\lambda(x_2, x_N) \\ \ldots & \ldots & \ldots \\ G_\lambda(x_N, x_1) & \ldots & G_\lambda(x_N, x_N) - \frac{1}{\beta} \end{bmatrix}. \quad (6)$$

It follows from the symmetry and homogeneity of the random walk that, for each $i = 1, 2, \ldots, N$, we have

$$G_\lambda(x_i, x_i) = G_\lambda(0, x_i - x_i) = G_\lambda(0, 0) = G_\lambda.$$

Note that the Green function has the alternative representation

$$G_\lambda(x, y) = \frac{1}{(2\pi)^d} \int\limits_{[-\pi, \pi]^d} \frac{e^{i(\theta, y - x)}}{\lambda - \phi(\theta)} d\theta, \quad (7)$$

where $\phi(\theta)$ is the Fourier transform of the transition intensities $a(z)$. Note also that, in the case where $G_\lambda(x, y)$ is the Green function of a lattice Laplacian, i.e., $\mathcal{A} = \kappa\Delta$, the values of the function $\phi(\theta)$ do not change under any permutation of the coordinates of the vector $\theta = \{\theta_1, \theta_2, \ldots, \theta_d\}$. This observation and (7) imply that the values $G_\lambda(x_i, x_j)$ do not depend on the choice of i and j, provided that $i \neq j$: all of them coincide with each other, so that the following quantity is well defined:

$$\tilde{G}_\lambda \equiv G_\lambda(x_i, x_j), \quad i \neq j.$$

Thus, the vanishing of determinant (6) can be written in the form

$$\det\begin{bmatrix} G_\lambda - \frac{1}{\beta} & \ldots & \tilde{G}_\lambda \\ \tilde{G}_\lambda & \ldots & \tilde{G}_\lambda \\ \ldots & \ldots & \ldots \\ \tilde{G}_\lambda & \ldots & G_\lambda - \frac{1}{\beta} \end{bmatrix} = 0,$$

which is equivalent to the equation

$$\left(G_\lambda + (N-1)\tilde{G}_\lambda - \frac{1}{\beta}\right)\left(G_\lambda - \tilde{G}_\lambda - \frac{1}{\beta}\right)^{N-1} = 0.$$

In this case, the eigenvalue $\lambda_0(\beta)$ is found from the equation

$$G_\lambda + (N-1)\tilde{G}_\lambda = \frac{1}{\beta},$$

and the coinciding eigenvalues $\lambda_1(\beta) = \ldots \ldots = \lambda_{N-1}(\beta)$, from the equation

$$G_\lambda - \tilde{G}_\lambda = \frac{1}{\beta}.$$

The critical values β_c and β_{c_1} are calculated explicitly as

$$\beta_c = (G_0 + (N-1)\tilde{G}_0)^{-1}, \quad \beta_{c_1} = (G_0 - \tilde{G}_0)^{-1}.$$

Remark 1. The value β_{c_1} depends on the distance ρ between the sources and does not depend on the number N of sources; i.e., $\beta_{c_1} = \beta_{c_1}(\rho) > 0$. At the same time, β_c depends not only on the distance between the sources but also on N, i.e., $\beta_c = \beta_c(\rho, N)$, and $\beta_c(\rho, N) \to 0$ as $N \to \infty$ for fixed ρ. Moreover, $\beta_c(\rho, N) \equiv 0$ under condition (1) for $d = 1, 2$ and under condition (2) for $d = 1$ and $\alpha \in [1, 2)$.

Remark 2. In Example 1, it is not necessary to take a lattice Laplacian for $\mathcal{A}$. It suffices to require that the values $\phi(\theta)$ of the Fourier transform of the intensity function $a(z)$ be invariant under any permutation of the coordinates of the vector $\theta = \{\theta_1, \theta_2, \ldots, \theta_d\}$. This condition is satisfied if the values of the function $a(z)$ do not change under any permutation of the coordinates of the vector $z = \{z_1, z_2, \ldots, z_d\}$.

WEAKLY SUPERCRITICAL BRANCHING RANDOM WALKS

Of special interest is the study of the asymptotic behavior of a BRW as $\beta \downarrow \beta_c$, i.e., as $\beta \to \beta_c$, $\beta > \beta_c$.

Definition 1. If there exists an $\varepsilon_0 > 0$ such that, for $\beta \in (\beta_c, \beta_c + \varepsilon_0)$, the operator $\mathcal{H}_\beta$ has one (counting multiplicity) positive eigenvalue $\lambda(\beta)$ satisfying the condition $\lambda(\beta) \to 0$ as $\beta \downarrow \beta_c$, then we say that the supercritical BRW is weakly supercritical at β close to β_c.

This definition gives rise to the question of whether any supercritical BRW is weakly supercritical. The following theorem, whose proof essentially uses the Perron–Frobenius theorem on the spectrum of positive operators, answers this question in the affirmative.

Theorem 4. *Each supercritical* BRW *is weakly supercritical as* $\beta \downarrow \beta_c$.

For a weakly supercritical BRW with finite variance of jumps, an asymptotic expansion of $\lambda(\beta)$ as $\beta \downarrow \beta_c$ was obtained in [12]. In the case where the variance of jumps is not finite, an analysis of the asymptotic behavior of $\lambda(\beta)$ is based on the following local limit theorem.

Theorem 5. *Suppose that the BRW under consideration is based on a symmetric spatially homogeneous irreducible random walk satisfying the condition*

$$\lim_{\|z\| \to \infty} a(z)\|z\|^{d+\alpha} = c, \tag{8}$$

where $c > 0$ *is a constant*, $\alpha \in (0, 2)$, *and* $\|z\|$ *denotes the max norm of* z. *Then*

$$\lim_{t \to \infty} t^{d/\alpha} p(t, x, y) = h_{\alpha, d},$$

where $h_{\alpha, d}$ *is a positive constant not depending on x and y for any fixed* α *and d.*

The proof of this theorem is based on an asymptotic representation of the Fourier transform of the transition intensities of the random walk [11] and on the multidimensional counterpart of Watson's well-known lemma [13]. Condition (8) is more restrictive than (2); the question of to what extent this constraint is essential requires further analysis.

Under the conditions of Theorem 5, the function $p(t, x, y)$ asymptotically (in t) behaves as the monotone function $h_{\alpha, d} t^{-d/\alpha}$. This observation and the Tauberian theorems [14] imply the assertion about the behavior of G_λ at small λ.

Theorem 6. *As* $\lambda \downarrow 0$, *the following asymptotic relations hold:*

$$G_\lambda \sim \begin{cases} \gamma_{1,\alpha} \lambda^{\frac{1-\alpha}{\alpha}}, & d = 1, \quad 1 < \alpha < 2, \\ -\gamma_{1,1} \ln \lambda, & d = 1, \quad \alpha = 1, \\ G_0 - \gamma_{1,\alpha} \lambda, & d = 1, \quad 0 < \alpha < 1, \\ G_0 - \gamma_{d,\alpha} \lambda, & d \geq 2, \quad 0 < \alpha < 2, \end{cases}$$

where $\gamma_{i,\alpha}$ *is a positive constant for any fixed* d, $i \in \mathbb{N}$, *and* α.

By virtue of this theorem, the asymptotic behavior of $\lambda(\beta)$ as $\beta \downarrow \beta_c$ has the form

$$\lambda_0(\beta) \sim \begin{cases} c_{1,\alpha} \beta^{\frac{\alpha}{\alpha-1}}, & d = 1, \quad 1 < \alpha < 2, \\ e^{-c_{1,1}/\beta}, & d = 1, \quad \alpha = 1, \\ c_{1,\alpha}(\beta - \beta_c), & d = 1, \quad 0 < \alpha < 1, \\ c_{d,\alpha}(\beta - \beta_c), & d \geq 2, \quad 0 < \alpha < 2, \end{cases}$$

where $c_{i,\alpha}$ is a positive constant for any $i \in \mathbb{N}$ and α.

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