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OPTIMAL MANAGEMENT

On the Optimal Control Function Diagrams in the Problem of the Movement of a Platform with Oscillators

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Abstract—We consider the problem of the time-optimal movement of a rigid body moving translationally along a horizontal straight line and carrying n linear oscillators. The only control force is applied to the platform and is limited in magnitude, and there is no friction. The system is transferred from a state of rest to a specified distance with vibration damping. The evolution of optimal control functions depending on the distance of the movement is investigated. A general approach to construct a visual diagram reflecting this evolution is proposed. To do this, a geometric interpretation of the necessary optimality conditions is used as properties of an auxiliary control curve in n-dimensional space. Numerical examples of constructing diagrams of the optimal control functions for a platform with three oscillators are given.

Keywords: optimal control function diagrams, platform with oscillators

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INTRODUCTION

We consider a system consisting of a load-bearing solid body and n linear springs with material points at the ends attached to it (Fig. 1). The springs are parallel to the horizontal axis along which the platform moves translationally. A single control force is applied to it, limited in magnitude by a predetermined value. Such a model can approximately describe small movements of a platform with elastic links or a vessel partially filled with liquid. Statements of control problems for such objects were formulated in [1], where, using the Pontryagin maximum principle [2], the problem of the fastest movement of a platform with one oscillator was solved. The optimal control turned out to be piecewise constant with three switchings [1, 3]. In the general case (for a platform with n oscillators), the problem has not yet been solved. The controllability of such a system is guaranteed if the natural frequencies of the oscillators are different in pairs [1, 4]. Despite the linearity of the system, control with a lack of external influences presents a certain complexity. The problems of moving such objects with vibration damping in finite time were recently considered in [5–7]. Algorithms were constructed that achieve the goal under the action of unknown distur-

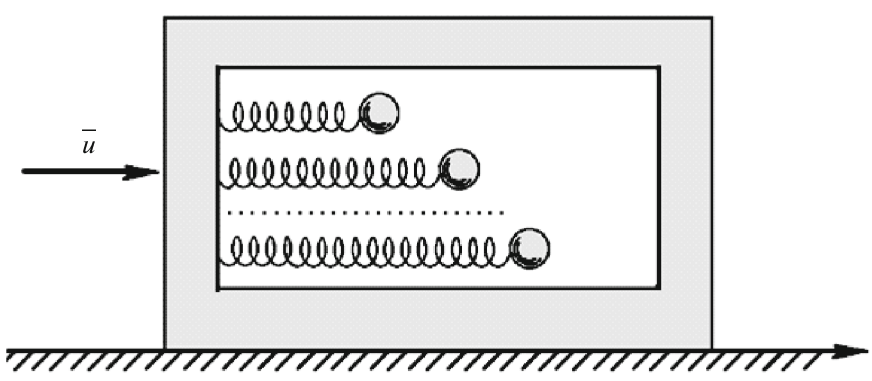

Fig. 1. Platform model with oscillators.

bances and unmeasured states of the oscillators [8, 9]. In [10], a control law is proposed that transfers the platform from n oscillators in finite time to the required state of rest with incomplete information about the state and disturbances, and for a certain type of initial states the asymptotic behavior of the time of movement depending on the number n.

In the problem of the optimal speed of moving a platform with n oscillators from one equilibrium state to another, when there is no friction and the phase state of the system at each moment is measurable, it is possible to use the symmetry of the boundary conditions. On this basis, the optimal control function was expressed through the values of the first n switching moments, preliminarily determined from a system of nonlinear equations [11]. For the case $n = 2$, the optimal modes were illustrated on the phase plane of one of the oscillators [11] and by constructing a visual image called a diagram of the optimal control functions [12]. The algorithm for constructing the diagram was based on the properties of the auxiliary control curve, which geometrically reflected the solutions of the conjugate system from the Pontryagin maximum principle. In this study, this approach is generalized to the case of n oscillators, taking into account the fact that for small movements of the platform, the optimal control on a half-trajectory has no more than n switchings [13].

1. STATEMENT OF THE PROBLEM AND ITS SYMMETRY PROPERTIES

Repeating the transformations of coordinates and time used in [11], we reduce the equations of motion of the system (Fig. 1) to a form with dimensionless variables and time:

$$\ddot{x}_0 = u, \quad \ddot{x}_i + \omega_i^2 x_i = u, \quad i = \overline{1,n}, \quad |u| \le 1, \tag{1.1}$$

where u is the control force and the frequency values are numbered in ascending order ($1 = \omega_1 < \omega_2 < \ldots < \omega_n$). The variables under the derivative signs are linear combinations of the coordinates of the platform and oscillators; in this case, the simultaneous vanishing of the values x_i, $i = \overline{1,n}$, corresponds to the unstressed states of the springs and the position of the supporting body is characterized by the variable x_0. If its value changes by $2b$ during the required platform movement, then in the middle of this segment we assign the origin of the coordinate x_0. We will introduce a similar double notation for the desired total time $2T$ of the movement of the system, which will give the boundary conditions a symmetrical appearance.

The problem of the optimal speed of the movement of a platform with n oscillators is formulated as follows: it is required to determine the control $u(t)$, $t \in [0, 2T]$, which transfers system (1.1) from the state

$$x_0(0) = -b, \quad x_i(0) = \dot{x}_i(0) = 0, \quad i = \overline{1,n}, \tag{1.2}$$

in the least amount of time $2T$ (previously unknown) into the state

$$x_0(2T) = b, \quad x_i(2T) = \dot{x}_i(2T) = 0, \quad i = \overline{1,n}. \tag{1.3}$$

We will consider the posed performance problem for the mutual variational problem for the maximum range $2b$ at the given time $2T$ (due to their monotonic interdependence). Generalizing the results of [12], we also study the evolution of the optimal control functions $u(t)$, $t \in [0, 2T]$, with value T changing.

Based on Pontryagin's maximum principle, it was shown [11] that in the optimal performance problem (1.1)–(1.3) in displaced time $\tilde{t} = t - T$, $\tilde{t} \in [-T, T]$, the optimal control $u(\tilde{t})$ and all solutions $x_i(\tilde{t})$, $i = \overline{0,n}$, will be odd functions, and $\dot{x}_i(\tilde{t})$, $i = \overline{0,n}$, will be even functions. Therefore, at the end of the optimal half-trajectory, the following relations are fulfilled:

$$u(T) = x_i(T) = 0, \quad i = \overline{0,n}. \tag{1.4}$$

Special controls are impossible here due to the linear independence of the known solutions of the conjugate system from the maximum principle. As a result, the optimal control will be piecewise constant with an odd number of switchings.

2. NECESSARY CONDITIONS FOR OPTIMALITY AND THEIR GEOMETRIC INTERPRETATION

If the moment in time T precedes j control switching times $\tau_1, \tau_2, \ldots, \tau_j$, then we can express the values $x_i(T)$, $i = \overline{1,n}$, through them (by integrating the differential equations (1.1)) at the end of the half-trajectory [11]. They are equal to zero according to relations (1.4); therefore, in the notation

$$\gamma_k = T - \tau_k, \quad k = \overline{1,j}, \tag{2.1}$$

the system is obtained from n nonlinear equations:

$$(-1)^j - \cos\omega_i T + 2\sum_{k=1}^{j} (-1)^{k+1} \cos\omega_i\gamma_k = 0, \quad i = \overline{1,n}. \tag{2.2}$$

Further considerations will be based on the evolution of roots $\gamma_k, k = \overline{1,j}$, of system (2.2) when values T change. Each duration T of the half-trajectory corresponds to its number j moments of control switching, which is unknown in advance, and its own set of roots satisfying the inequality

$$T > \gamma_1 > \gamma_2 > \ldots > \gamma_j > 0. \tag{2.3}$$

If in system (1.1) all numbers ω_i, $i = \overline{1,n}$, are rational, then in the optimal performance problem (1.1)–(1.3) the following properties hold [11].

Remark 1. There is optimal movement with one control switching at $T = T_*$, where $T_* = 2n_*\pi$ and n_* is the common denominator of all irreducible fractions ω_i, $i = \overline{1,n}$.

Remark 2. If the control with switching moments $\tau_1, \tau_2, \ldots, \tau_j$, T satisfies the boundary conditions (2.2), then they will also satisfy the following controls:

(1) control with switching moments $\tau_1 + T_*$, $\tau_2 + T_*$, ..., $\tau_j + T_*$, $T + T_*$;

(2) control with switching moments $\tau_1 + \Delta$, $\tau_2 + \Delta$, ..., $\tau_{j-1} + \Delta$, $T + \Delta$ (given that $\Delta = T_* - 2T > 0$).

Remarks 1 and 2 make it possible to describe all scenarios of the optimal behavior of the system, periodically repeated as parameter T increases. If at least one of the values ω_i, $i = \overline{1,n}$, is irrational, then in system (2.2) there will not be a general period T_* of all functions; therefore, with duration T increasing, the number of different types of optimal movement will be infinite, not amenable to complete description.

Based on Pontryagin's maximum principle, the statement [11] was proved, which in terms $\gamma_k, k = \overline{1,j}$, can be formulated as follows.

Statement 1. If in problem (1.1)–(1.3) the half-trajectory is optimal in terms of speed and is implemented by piecewise constant control with n switchings between $t \in [0, T)$ at moments in time $\tau_1, \tau_2, \ldots, \tau_n$, then the control function has the form

$$u = \operatorname{sign}(\xi \det Q_{n+1}(t)), \quad Q_{n+1} = \begin{Vmatrix} T - t & \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ \sin\omega_1(T-t) & \sin\omega_1\gamma_1 & \sin\omega_2\gamma_2 & \cdots & \sin\omega_1\gamma_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sin\omega_n(T-t) & \sin\omega_n\gamma_1 & \sin\omega_n\gamma_2 & \cdots & \sin\omega_n\gamma_n \end{Vmatrix}, \tag{2.4}$$

and the sign of constant ξ is determined by a known (by condition) control value $u(t)$ at $t \in [0, \tau_1)$.

For each given duration of the half-trajectory T, first we need to find from system (2.2) the expected set of switching moment values. Then we need to check it for compliance with the necessary optimality conditions by applying Statement 1: if, during the action of control (2.3), other switching moments arise, in addition to the expected ones $\tau_1, \tau_2, \ldots, \tau_n$, then this schedule is not optimal. In the case when the number of switchings found from system (2.2) exceeds n, i.e., $j > n$, then for them the condition remains the same [11]: all these j values should automatically appear when control (2.4) is applied. If the number of roots in system (2.2) is $j < n$, then in the formula for the control function (2.4) instead of the determinant Q_{n+1} we take only its upper left minor in dimensions $(j+1)\times(j+1)$ [11].

The geometric meaning of such necessary optimality conditions can be illustrated using a line in n-dimensional space, called the control curve in [12]. To do this, note that the equality to zero of the determinant Q_{n+1} is equivalent to the condition

$$\begin{Vmatrix} 1 & 1 & \dots & 1 \\ \psi(\omega_1(T-t)) & \psi(\omega_1\gamma_1) & \dots & \psi(\omega_1\gamma_n) \\ \dots & \dots & \dots & \dots \\ \psi(\omega_n(T-t)) & \psi(\omega_n\gamma_1) & \dots & \psi(\omega_n\gamma_n) \end{Vmatrix} = 0, \tag{2.5}$$

where the auxiliary function

$$\psi(\rho) = \frac{\sin\rho}{\rho}, \tag{2.6}$$

predefined by value $\psi(0) = 1$, is introduced. According to Statement 1, condition (2.4) must be satisfied only when $T - t = \gamma_k$, $k = \overline{1,n}$, i.e., only when the first column of the determinant numerically coincides with one of n subsequent ones. Let us represent these n columns as vectors $\overline{OB_k}$, deferred from the origin O in an affine space with coordinates $(y_0, y_1, \dots, y_n)^{\mathrm{T}}$ and generating a hyperplane. Its intersection with the hyperplane $y_0 = 1$ will give an $(n-1)$-dimensional plane containing all points B_k (due to the identicality of their first coordinate). Therefore, the set of vectors $\overline{OB_k}$, $k = \overline{1,n}$, can be uniquely projected onto a subspace with coordinates $(y_1, \dots, y_n)^{\mathrm{T}}$, receiving the vectors contained in it $\overline{OA_k} = (\psi(\omega_1\gamma_k), \dots, \psi(\omega_n\gamma_k))^{\mathrm{T}}$, $k = \overline{1,n}$. Then it turns out that all the points A_k belong to the general curve defined in the n-dimensional space vector function

$$r = (\psi(\omega_1\rho), \dots, \psi(\omega_n\rho))^{\mathrm{T}}, \quad \rho \in [0, T]. \tag{2.7}$$

We call line (2.7) the *reference curve*. Its form depends on the combination of values of ω_i, $i = \overline{1,n}$, but the line always starts at the point $K_0(1,1,\dots,1)$ at $\rho = 0$ and passes through the point $O(0,0,\dots,0)$ at $\rho = T_*/2$ (if $T_*/2 < T$). Its actual section ends at point W with parameter $\rho = T$ and contains points A_k, $k = \overline{1,n}$, located sequentially according to the values of their parameters (2.3). In the displaced time $\tilde{t} \in [-T, T]$, we can follow the movement along the control curve of the representing point, which will pass the arc WK_0 twice (from the end W to the beginning K_0 and back). In this case, the determinant Q_{n+1} will change sign each time when passing through points A_k, i.e., at the points of intersection of the control curve with the hyperplane $A_1A_2 \dots A_n$. The geometric meaning of the necessary optimality condition is that the control curve should not have a hyperplane $A_1A_2 \dots A_n$ of other intersection points, except those whose parameters $\gamma_k, k = \overline{1,j}$, correspond to the planned switching moments $\tau_1, \tau_2, \dots, \tau_j$.

For example, when $n = 2$ (the case of a platform with two oscillators), the control curve lies in a two-dimensional plane. Figure 2 shows its image for values $\omega_1 = 1$ and $\omega_2 = 3$; and Fig. 3, for values $\omega_1 = 1$ and $\omega_2 = 7$.

For a platform with three oscillators ($n = 3$), the control curve lies in three-dimensional space (y_1, y_2, y_3). Figure 4 shows the cases $\omega_1 = 1$, $\omega_2 = 3$, and $\omega_3 = 7$ (the dotted line marks the fragments of the curve lying below the horizontal plane (y_1, y_2)). Projections of this line on the plane (y_1, y_2) and (y_1, y_3) look like those in Figs. 2 and 3.

Note that in Fig. 2, the initial section of the curve (from K_0 to the first point of inflection) is concave; i.e., it can only contain two collinear points A_1 and A_2, and cannot contain a third one. Therefore, here the straight line A_1A_2 is a hyperplane that has no other intersections with the current section of the control curve, except A_1 and A_2. It follows that for sufficiently small values of T (not exceeding the inflection point parameter), the optimal control has no more than two switching moments on the half-trajectory. Similarly, it can be shown that when $n = 3$ the initial section of the control curve cannot contain more than three coplanar points, i.e., on a short interval T, the optimal control of a platform with three oscillators will have no more than three switches. This pattern has been proved in [13] in the form of the following statement.

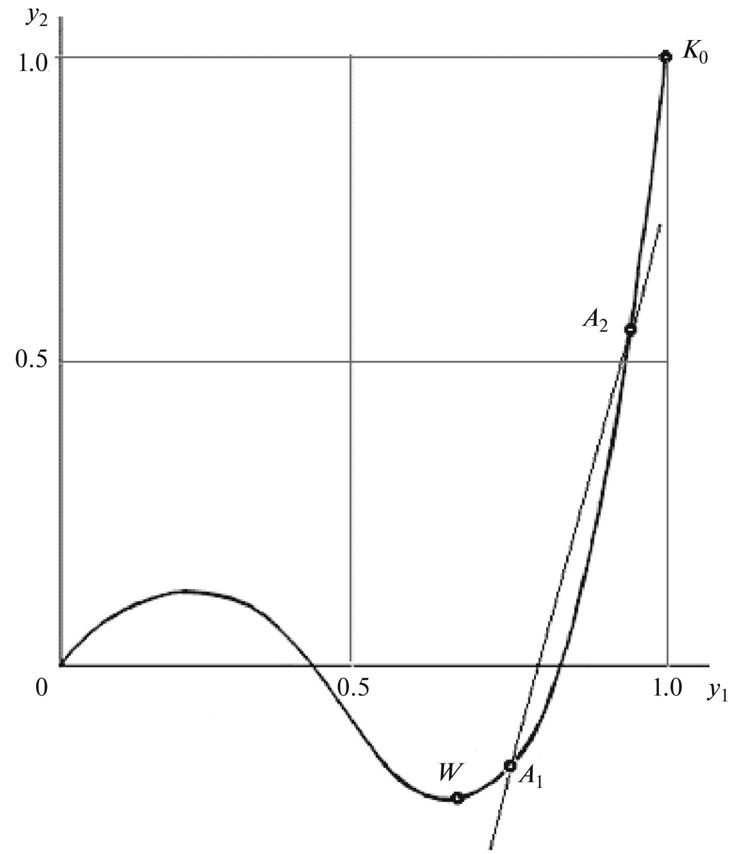

Fig. 2. Control curve for the case $n = 2$ at $\omega_2/\omega_1 = 3$.

Statement 2. In problem (1.1)–(1.3), there is a value T_0 that, satisfying the necessary optimality conditions for all $T \in (0, T_0]$ functions $u(t)$, $t \in [0, T)$, will not have more than n switching times.

The proof is reduced to substantiating the fact that, in a sufficiently small area WK_0 of the control curve (2.7), none of its $(n+1)$ points A_k, $k = \overline{1, n+1}$, can belong to a common hyperplane; i.e., vector $\overline{A_1A_{n+1}}$ cannot belong to the subspace generated by the vectors $\overline{A_1A_i}$, $i = \overline{2, n}$. The determinant constructed on these vectors will be positive [13], at least under the condition

$$T_0 = \sigma\pi/\omega_n, \quad \sigma = 0.597670. \tag{2.8}$$

This estimate is very rough and is determined by the style of the proof (by expanding the elements of the determinant into Maclaurin series). In Fig. 2 (where $n = 2$, $\omega_n = 3$), point W has the parameter $T = \pi/2$. In other words, for a platform with two oscillators at $\omega_1 = 1$ and $\omega_2 = 3$, in the range $T < \pi/2$, the optimal half-trajectories correspond to exactly two control switching moments.

Below, to describe the evolution of the optimal control functions $u(t)$, $t \in [0, 2T]$, with change in parameter T, we will use the visual representation proposed in [12] and called the diagram of the optimal control functions.

For example, Fig. 5 shows a diagram for the case $n = 2$, $\omega_1 = 1$, and $\omega_2 = 3$. The axis is plotted upward along the axis of symmetry T. Each horizontal section of the diagram, read from left to right, symbolizes one piecewise constant optimal control function $u(\tilde{t})$, $\tilde{t} \in [-T, T]$. The gray horizontal segments correspond to time intervals where $u = 1$, and the white segments, to time intervals where $u = -1$. For example, the section EJ gives the optimal control function with five switchings at $T = \pi/2$. The section EF encodes the graph $u(t)$ for the half-trajectory shown in Fig. 6 in unshifted time $t \in [0, T]$. The switching moments $\tau_1 = \pi/10$ and $\tau_2 = 3\pi/10$ correspond to the roots $\gamma_1 = 2\pi/5$ and $\gamma_2 = \pi/5$ of system (2.2) at $j = 2$,

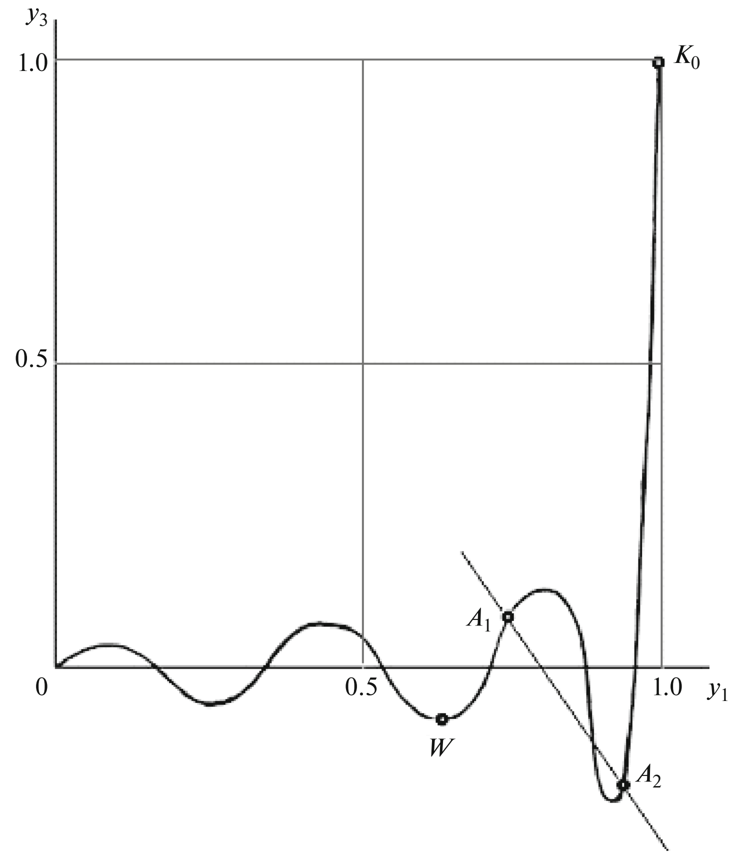

Fig. 3. Control curve for the case $n = 2$ at $\omega_2/\omega_1 = 7$.

$T = \pi/2$. On the control curve (Fig. 2), they correspond to points W ($\rho = \pi/2$), A_1 ($\rho = 2\pi/5$), and A_2 ($\rho = \pi/5$). Here the straight line A_1A_2 has no other intersections with the arc WK_0; thus, the function $u(t)$ satisfies the necessary optimality conditions in the performance problem (1.1)–(1.3).

For the considered integer values $\omega_1 = 1$ and $\omega_2 = 3$, according to Remark 1, the quantity $T_* = 2\pi$ is the period for repeating the shape of the diagram. Its continuation, according to Remark 2, is constructed by mirroring the triangle OBC (with its shaded areas) relative to the horizontal BC, which would give a symmetrical triangle O_1BC. When continuing the vertical axis T inside the angle BOC, squares OBO_1C with their colored areas will be repeated periodically.

Next, we will consider the patterns that allow us to construct diagrams of the control functions in the problem of the optimal performance of moving a platform with n oscillators.

3. PROPERTIES OF THE DIAGRAM OF OPTIMAL CONTROL FUNCTIONS FOR SMALL VALUES OF PARAMETER T

Let us consider the structure of the diagram at its very bottom, i.e., in the vicinity of vertex O. Since for $T \in (0, T_0]$, according to Statement 2, the number of control switching moments $u(t)$, $t \in [0, T)$, does not exceed n, no more than n curves will emerge from vertex O. In the notation $\gamma_k = T - \tau_k$, $k = \overline{1, n}$, where inequalities (2.3) are satisfied, we can consider these curves as graphs of functions $\gamma_i(T)$, $i = \overline{1, m}$, $m \le n$. With parameter T increasing, such an evolution corresponds to a change in the roots of the system of n equations (2.2), written at $j = n$. With fewer control switches $m < n$ in this system, we can formally consider $\gamma_j \equiv 0$, $j = \overline{m + 1, n}$.

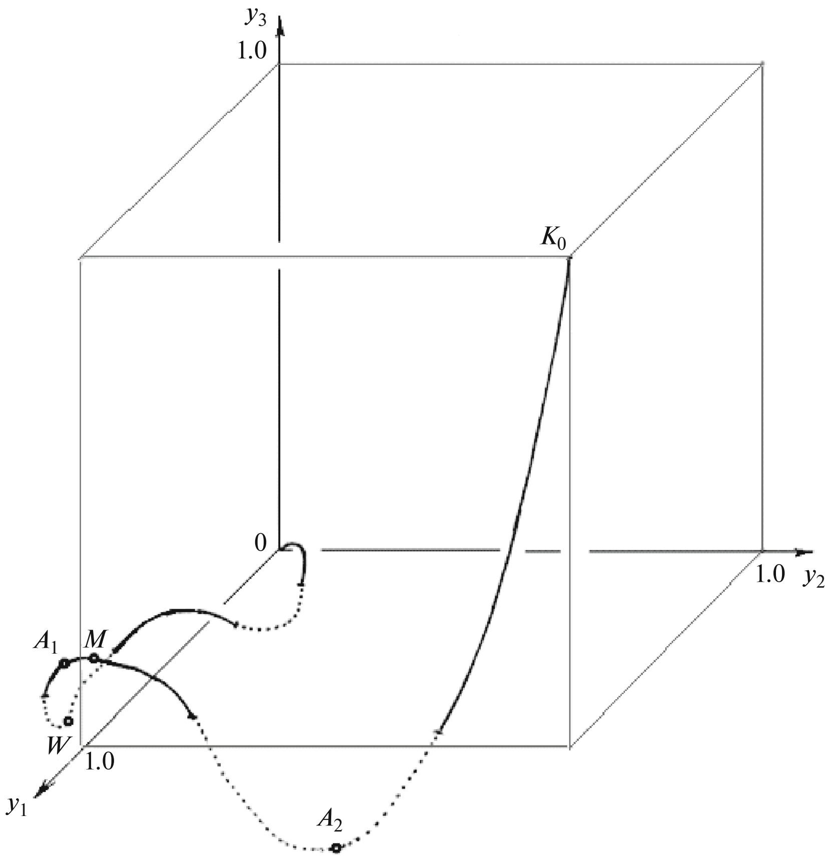

Fig. 4. Control curve for the case $n = 3$ at $\omega_1 = 1$, $\omega_2 = 3$, $\omega_3 = 7$.

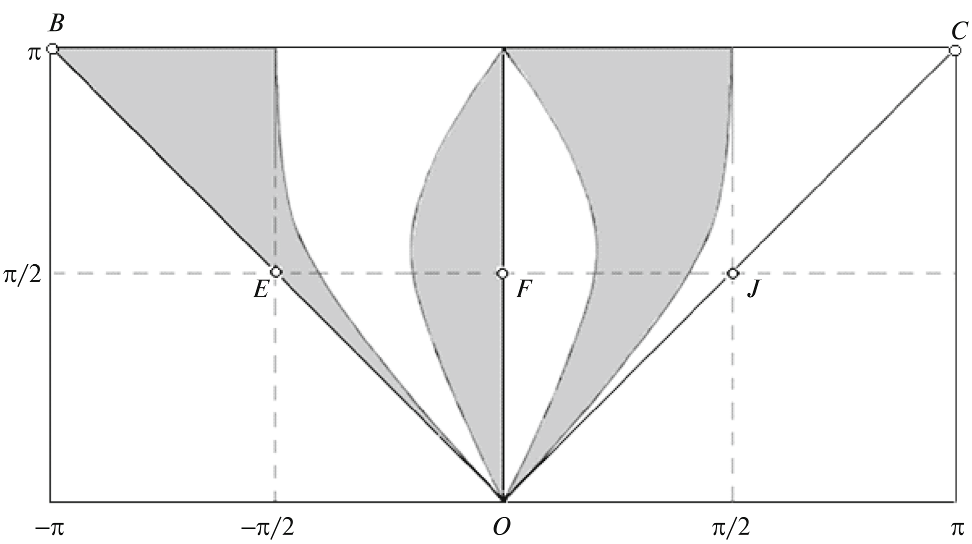

Fig. 5. Diagram of optimal control functions for the case $n = 2$ at $\omega_2/\omega_1 = 3$.

Further, under the assumption that the T values are small (and that means all $\gamma_k, k = \overline{1,n}$), we use expansions in the Maclaurin series:

$$\cos(\omega_i T) \approx 1 + \sum_{j=1}^{n} (-1)^j \frac{(\nu_i t)^j}{(2j)!}, \quad \cos(\omega_i \gamma_k) \approx 1 + \sum_{j=1}^{n} (-1)^j \frac{(\nu_i \theta_k)^j}{(2j)!},$$

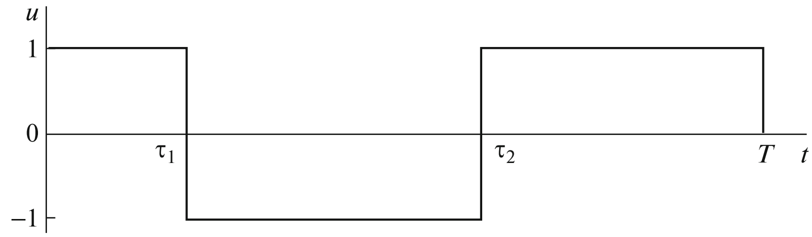

Fig. 6. Graph of a function $u(t)$ when $n = 2$ at $\omega_2/\omega_1 = 3$, $T = \pi/2$.

$$t = T^2, \quad \nu_i = \omega_i^2, \quad i = \overline{1,n}, \quad \theta_k = \gamma_k^2, \quad k = \overline{1,n}. \tag{3.1}$$

Substituting relations (3.1) into (2.2), we rearrange the terms, highlighting the expressions in them:

$$p_j = \frac{(-1)^j}{(2j)!}\left[\frac{t^j}{2} + \sum_{k=1}^{n}(-1)^k\theta_k^j\right], \quad j = \overline{1,n}. \tag{3.2}$$

Then, introducing the vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)^{\mathrm{T}}$, we transform system (2.2) to the form

$$\Omega\mathbf{p} = \mathbf{0}, \quad \Omega = \begin{bmatrix} 1 & \nu_1 & \ldots & \nu_1^{n-1} \\ 1 & \nu_2 & \ldots & \nu_2^{n-1} \\ \ldots & \ldots & \ldots & \\ 1 & \nu_n & \ldots & \nu_n^{n-1} \end{bmatrix}. \tag{3.3}$$

Since the Vandermonde determinant

$$\det\Omega = \prod_{1\le i<j\le n}(\nu_i - \nu_j) \neq 0, \quad \text{at} \quad \omega_j \neq \omega_i \ (j \neq i),$$

from matrix relation (3.3) it follows that $p_i = 0$, $i = \overline{1,n}$, which leads to the system of n equations in relation to n variables $\alpha_i = \theta_i/t$, $i = \overline{1,n}$:

$$\sum_{k=1}^{n}(-1)^{k+1}\alpha_k^i = \frac{1}{2}, \quad i = \overline{1,n}. \tag{3.4}$$

It can be shown that such a system (3.4) always has n roots $\alpha_1 > \alpha_2 > \ldots > \alpha_n > 0$, and the values α_1 and α_n, α_2 and α_{n-1}, etc., complement each other to unity. We present the details of the calculations separately for the two cases.

Case 1. If n is even, i.e., $n = 2k$, $k \in N$, then we can introduce new variables $\beta_1, \beta_2, \ldots, \beta_k$, representing

$$\alpha_i = \frac{1}{2} + \beta_i, \quad i = \overline{1,k}, \quad \alpha_j = \frac{1}{2} - \beta_{n-j+1}, \quad j = \overline{k+1,n}, \tag{3.5}$$

where

$$1/2 > \beta_1 > \beta_2 > \ldots > \beta_k > 0. \tag{3.6}$$

When substituting expressions (3.5) into system (3.4), every second equation will repeat the previous one, so that only k equations remain, generating in new variables $\delta_i = (-1)^{i+1}\beta_i$, $i = \overline{1,k}$, the symmetric system

$$\sum_{i=1}^{k}\delta_i^{2j-1} = \left(\frac{1}{4}\right)^j, \quad j = \overline{1,k}.$$

It is solved by introducing symmetric polynomials, whose numerical values can be found using the standard procedures, and then (according to Vieta's theorem) become coefficients of the generating algebraic equation of the kth degree with roots $\delta_1, \delta_2, \ldots, \delta_k$. At $k > 4$, these equations can be solved only numerically, but for small k the roots are in explicit form:

(1) at $k = 1$ (i.e., $n = 2$), we have $\beta_1 = 1/4$ (i.e., $\alpha_1 = 3/4$, $\alpha_2 = 1/4$),

(2) at $k = 2$ (i.e., $n = 4$), we have $\beta_1 = (\sqrt{5}+1)/8$, $\beta_2 = (\sqrt{5}-1)/8$ (i.e., $\alpha_1 = (5+\sqrt{5})/8$, $\alpha_2 = (3+\sqrt{5})/8$, $\alpha_3 = (5-\sqrt{5})/8$, $\alpha_4 = (3-\sqrt{5})/8$),

(3) at $k = 3$ (i.e., $n = 6$), a generating equation

$$\delta^3 - \frac{1}{4}\delta^2 - \frac{1}{8}\delta + \frac{1}{64} = 0$$

arises, whose roots are found using the Cardano formulas

$$\delta_i = \frac{1}{12} + \frac{\sqrt{7}}{6}\cos\left[\varphi + \frac{2\pi}{3}(i-1)\right]_i, \quad i = \overline{1,3}, \quad \varphi = \frac{1}{3}\arccos\left(-\frac{1}{\sqrt{28}}\right).$$

They correspond to the numerical values $\beta_1 \approx 0.45048441$, $\beta_2 \approx 0.31174485$, and $\beta_3 \approx 0.11126044$, giving according to formulas (3.5) all the roots α_i, $i = \overline{1,6}$, of system (3.4).

Case 2. If n is odd, i.e., $n = 2k+1$, $k \in N$, the roots of system (3.4) will be expressed through new variables in the form

$$\alpha_i = \frac{1}{2} + \beta_i, \quad i = \overline{1,k}, \quad \alpha_{k+1} = \frac{1}{2}, \quad \alpha_j = \frac{1}{2} - \beta_{n-j+1}, \quad j = \overline{k+2,n}, \tag{3.7}$$

where inequality (3.6) is again satisfied.

When substituting expressions (3.7) into system (3.4), its odd equations degenerate into identities, and its even equations can only be expressed in terms of β_i^2, $i = \overline{1,k}$; thus, when replacing $\tilde{\alpha}_i = 4\beta_i^2$, $i = \overline{1,k}$, the system will exactly repeat the form of system (3.4) for $n = k$. In other words, we get the values $\beta_i = \sqrt{\tilde{\alpha}_i/4}$, where $\tilde{\alpha}_i$, $i = \overline{1,k}$, are the roots of system (3.4) for $n = k$. Thus, we get the following equations:

(1) at $n = 3$, roots $\alpha_1 = (2+\sqrt{2})/4$, $\alpha_2 = 1/2$, $\alpha_3 = (2-\sqrt{2})/4$,

(2) at $n = 5$, roots $\alpha_1 = (2+\sqrt{3})/4$, $\alpha_2 = 3/4$, $\alpha_3 = 1/2$, $\alpha_4 = 1/4$, $\alpha_5 = (2-\sqrt{3})/4$,

(3) at $n = 7$, roots $\alpha_1 = (2+\sqrt{2+\sqrt{2}})/4$, $\alpha_2 = (2+\sqrt{2})/4$, $\alpha_3 = (2+\sqrt{2-\sqrt{2}})/4$, $\alpha_4 = 1/2$, $\alpha_5 = (2-\sqrt{2-\sqrt{2}})/4$, $\alpha_6 = (2-\sqrt{2})/4$, $\alpha_7 = (2-\sqrt{2+\sqrt{2}})/4$, etc.

Remark 3. Substituting roots α_i, $i = \overline{1,n}$, of system (3.4) into the expression

$$S = \sum_{j=1}^{n}(-1)^{j+1}\alpha_j^{n+1} - 1/2, \tag{3.8}$$

at any n we get the inequality $S < 0$.

It can be shown (we omit the details) that when $n = 2k$, $k \in N$, expression (3.8) takes the value $S = -(2k+1)/2^{4k+1}$, and when $n = 2k+1$, $k \in N$, it takes the value $S = -(k+1)/2^{4k+2}$.

Since among the roots of system (3.4), which have the original meaning $\alpha_i = \gamma_i^2/T^2$, $i = \overline{1,n}$, there are no zero values, and the system itself is generated by representation (3.1), under the assumption of the smallness of T, the validity of the following property is shown.

Statement 3. In the performance problem (1.1)–(1.3), for a sufficiently small value of T, the optimal control function $u(t)$, $t \in [0, T)$, has exactly n switchings. On the corresponding diagram from vertex O, n curves $\gamma_i(T)$, $i = \overline{1,n}$, emerge having the following angular coefficients at this point:

$$k_i = \left.\frac{d\gamma_i}{dT}\right|_{T=0} = \sqrt{\alpha_i}, \quad i = \overline{1,n}, \tag{3.9}$$

where α_i, $i = \overline{1,n}$, are the roots of system (3.4).

Due to the smoothness of the functions involved in (2.2) and taking the above into account, there is a range $T \in [0, T_0)$ in which the functions $\gamma_i(T)$, $i = \overline{1,n}$, are continuous and differentiable; thus, if we apply the expansion

$$\gamma_i(T) \approx \gamma_i(0) + \dot{\gamma}_i(0)T + \ddot{\gamma}_i(0)T^2/2, \quad i = \overline{1,n}, \tag{3.10}$$

then $\gamma_i(0) = 0$, $\dot{\gamma}_i(0) = \sqrt{\alpha_i}$, $i = \overline{1,n}$.

Note that system (3.3) is generated by approximation (3.1) taking into account n degrees of variable $t = T^2$. If expansion (3.1) is supplemented with degree t^{n+1}, implying an increase in the discussed range T, after substitution into system (2.2), a new component would appear in it, along with expressions (3.2):

$$h = \frac{(-1)^{n+1}}{(2(n+1))!}\left[\frac{t^{n+1}}{2} + \sum_{k=1}^{n}(-1)^k \theta_k^{n+1}\right]. \tag{3.11}$$

Then instead of the homogeneous matrix equation (3.3), there will be the inhomogeneous one

$$\Omega p = -h\mathbf{q}, \quad \mathbf{q} = (\nu_1^n, \nu_2^n, \dots, \nu_n^n)^{\mathrm{T}}. \tag{3.12}$$

From it, using Cramer's method, the following value will follow: $p_1 = -h \det\Omega_1/\det\Omega$, where matrix Ω_1 obtained from Ω replacing the first column with column $\mathbf{q}$. This is why

$$\det\Omega_1 = (-1)^{n+1}\prod_{k=1}^{n}(\nu_k)\det\Omega,$$

where

$$p_1 = -h(-1)^{n+1}\prod_{k=1}^{n}(\nu_k) \tag{3.13}$$

Comparing relations (3.8) and (3.11), we obtain

$$-h(-1)^{n+1} = \frac{t^{n+1}}{(2(n+1))!}S;$$

i.e., the sign of the new expression (3.13) for p_1 will coincide with the sign of the previous expression (3.8) for S. Thus, for a sufficiently small value of T, according to Statement 2 and Remark 3, we previously had estimates for $p_1 = 0$ and $S < 0$. Now with a slight increase in parameter T, the sign of expression S will be preserved, and the sign of the expression

$$p_1 = \frac{t}{2!}(\alpha_1 - \alpha_2 + \dots + (-1)^{n+1}\alpha_n - 1/2) \tag{3.14}$$

will become negative, which leads to the following conclusion.

Remark 4. For sufficiently small values of T, the expression $w = (\dot{\gamma}_1^2 - \dot{\gamma}_2^2 + \dots + (-1)^{n+1}\dot{\gamma}_n^2)$ decreases, i.e.,

$$\frac{dw}{dT} < 0. \tag{3.15}$$

This allows for sufficiently small values of T to clarify the sign of the optimal control $u(t)$, $t \in [0, T)$, at $t = 0$. For this purpose, we introduce into consideration the time function t, defined on the interval $t \in [0, T]$ and depending on parameter τ:

$$v(t,\tau) = \begin{cases} 0 & \text{at} \quad 0 \le t \le \tau, \\ 1 & \text{at} \quad \tau < t \le T. \end{cases}$$

Then the piecewise constant control function $u(t)$, $t \in [0, T]$, characterized by switching moments τ_1, τ_2, ..., τ_n, T (where $0 < \tau_1 < \dots < \tau_n < T$) and taking the value $u(0) = 1$, can be represented in the form

$$u(t) = v(t,0) + 2\sum_{i=1}^{n}(-1)^i v(t,\tau_i). \tag{3.16}$$

If at the initial moment there were $u(0) = -1$, in formula (3.16) the right-hand side would be multiplied by -1. For the differential equation $\ddot{x}_0 = u$, $x_0(0) = \dot{x}_0(0) = 0$, characterizing the movement of the platform from the zero position under the influence of control (3.16), the solution will be

$$x_0(t) = \frac{1}{2}(t-0)^2 + \sum_{i=1}^{n}(-1)^i(t-\tau_i)^2.$$

Therefore, the movement of the platform in time T in notation (2.1) takes the form

$$x_0(T) = \frac{1}{2}T^2 + \sum_{i=1}^{n}(-1)^i\gamma_i^2 \tag{3.17}$$

(with the right side multiplied by -1 in the case $u(0) = -1$).

If for small values of T in expressions (3.10) we limit ourselves to linear components, when they are substituted into (3.17), the right-hand side will degenerate to zero, since the right-hand side in (3.14) is zero. Therefore, to find the sign of the quantity $x_0(T)$, we will need expression (3.10), taking into account the quadratic component. Substituting it into (3.17), canceling similar terms, and neglecting quantities of the fourth order of smallness, we obtain

$$x_0(T) \approx -T^3[\dot{\gamma}_1(0)\ddot{\gamma}_1(0) - \dot{\gamma}_2(0)\ddot{\gamma}_2(0) + \ldots + (-1)^{n+1}\dot{\gamma}_n(0)\ddot{\gamma}_n(0)] \tag{3.18}$$

(with multiplication by -1 in the case $u(0) = -1$). Since the expression in square brackets (3.18) is negative due to inequality (3.15), the range $x_0(T)$ will be positive only if $u(0) = 1$. Thus, the following property is true.

Statement 4. In the performance problem (1.1)–(1.3) for a sufficiently small value of T, the optimal control function $u(t)$, $t \in [0, T)$, at the initial stage (with $t \in [0, \tau_1)$) is positive.

4. GENERAL APPROACH TO CONSTRUCT A DIAGRAM OF THE OPTIMAL CONTROL FUNCTIONS

All of the above allows us to begin constructing a diagram of the optimal control functions, by deriving from the vertex O the curves $\gamma_i(T)$, $i = \overline{1,n}$, with angular coefficients (3.9), and painting the gray and white areas between them in accordance with Statement 4.

The evolution of the values $\gamma_i(T)$, $i = \overline{1,n}$, can be described by differentiating relations (2.2) with respect to parameter T. Reducing each ith equation by $2\omega_i$, we get the system

$$\sum_{k=1}^{j}(-1)^{k+1}\dot{\gamma}_k \sin\omega_i\gamma_k = \frac{1}{2}\sin\omega_i T, \quad i = \overline{1,n}. \tag{4.1}$$

It is solved (due to $j = n$) with respect to the derivatives

$$\frac{d\gamma_i}{dT} = \frac{\Delta_i}{\Delta}, \quad i = \overline{1,n}, \tag{4.2}$$

$$\Delta = \eta D, \quad \Delta_i = \eta_i D_i, \quad \eta = \prod_{k=1}^{n}(\omega_k\gamma_k), \quad \eta_i = \eta T/(2\gamma_i),$$

where

$$D = \begin{Vmatrix} \psi(\omega_1\gamma_1) & \psi(\omega_1\gamma_2) & \ldots & \psi(\omega_1\gamma_n) \\ \psi(\omega_2\gamma_1) & \psi(\omega_2\gamma_2) & \ldots & \psi(\omega_2\gamma_n) \\ \ldots & \ldots & \ldots & \ldots \\ \psi(\omega_n\gamma_1) & \psi(\omega_n\gamma_2) & \ldots & \psi(\omega_n\gamma_n) \end{Vmatrix}. \tag{4.3}$$

Determinants D_i, $i = \overline{1,n}$, are obtained from D by replacing γ_i by T in the ith column while simultaneously moving it to a place to the left of the first column.

When numerically integrating equations (4.2), at each stage it is necessary to check the necessary optimality condition: a hyperplane drawn through the points A_k, $k = \overline{1,n}$ (with the coordinates in the form of columns of the determinant D), should not intersect the control curve at other points. For this purpose, for each next set T, γ_i, $i = \overline{1,n}$, we can tabulate the parameter $t \in [0, T)$ to make sure that relation (2.5) is satisfied only n times. Otherwise, the integration interval is considered complete, since equations (4.2) retain their meaning only for those values of T that correspond exactly to n roots of system (2.2).

The special case $n = 2$ was studied in [12], where the diagram could consist of several layers separated by horizontal lines $T = T_k$, $k \in N$. Each layer had its own continuous evolution of values $\gamma_i(T)$, $i = \overline{1,j}$, with its own specific quantity j; i.e., another system of differential equations was integrated.

For an arbitrary value of n, completion of the first layer of the diagram (with $T = T_1$), there can also be two variants. If $T_1 = T_*$ (trivial variant), then the diagram consists of one layer, inside which at all optimal half-trajectories the control has n switchings. For the nontrivial variant, which will be of interest later, at the end of the first layer with $T_1 < T_*$, system (2.2) corresponds to the number of roots $j < n$. Such a decrease in the number of roots can occur either due to the vanishing of the function $\gamma_n(T)$ or due to the coincidence of values $\gamma_i = \gamma_{i+1}$, $i \in \overline{1, n-1}$, etc. Moreover, if system (2.2) corresponds to a solution in which two neighboring values $\gamma_i = \gamma_{i+1}$ numerically coincide, then it will be satisfied by a solution in which these two values are replaced by zeros. Thus, the existence of a control with the number of switchings $j < n$ at some value T would imply for system (2.2) the presence of a solution $(\gamma_1, \gamma_2, \ldots, \gamma_j, 0, \ldots, 0)$, i.e., the validity of the conditions

$$\gamma_m = 0, \quad m = \overline{j+1, n}. \tag{4.4}$$

Remark 5. Condition (4.4) can only be satisfied for isolated points on the numerical axis T; i.e., it cannot continue on a continuous interval of this axis.

Indeed, if we write system (4.1) with $j < n$ “in differentials” (by multiplying the equations by dT), it will not be possible to express from it $d\gamma_i$, $i = \overline{1, j}$, through dT, since the resulting system (of n equations with j unknown) is inconsistent (due to the linear independence of the rows). Therefore, for the parameter $T + dT$, there is no continuation in the form $\gamma_i + d\gamma_i$, $i = \overline{1, j}$.

The corollary follows from Remark 5.

Remark 6. For the continuous evolution of the functions $\gamma_i(T)$, $i = \overline{1, j}$ their number should be $j \geq n$.

Let us present two more statements related to special cases of relations (4.4).

Remark 7. If at some value of T system (2.2) admits the set of roots for $\gamma_m = 0$, $m = \overline{2, n}$, the corresponding control (with three switchings at times $T - \gamma_1$, T, and $2T - \gamma_1$) at $u(0) = 1$ satisfies the necessary optimality conditions.

Indeed, if in problem (1.1)–(1.3) for some n, there is a regime with three switchings, then it also exists for the system obtained from (1.1) by formally eliminating the last $(n - 1)$ equations. However, for such a platform with one oscillator, the three-switching mode is optimal [1, 3]. The maximum range $x_0(2T)$ achievable at $n = 1$ will be no less than for system (1.1) at $n > 1$, i.e., for a system with restrictions in the form of additional $(n - 1)$ differential equations.

Remark 8. If at some value T system (2.2) admits the set of roots for $\gamma_m = 0$, $m = \overline{3, n}$, the corresponding control (with five switchings at times $T - \gamma_1$, $T - \gamma_2$, T, $2T - \gamma_2$, and $2T - \gamma_1$) at $u(0) = 1$ will satisfy the necessary optimality conditions in the case when at least one curve

$$y_k = \psi(\omega_k \rho), \quad y_s = \psi(\omega_s \rho), \quad \rho \in [0, T], \tag{4.5}$$

on the surface (y_k, y_s) has only two points in common with the line A_1A_2 drawn through the points taken on the curve A_1 and A_2 with parameters γ_1 and γ_2.

The proof is similar to Remark 6: in problem (1.1)–(1.3), if for some n there is a mode with five switchings, then it also exists for a system of lower order with respect to variables x_0, x_k, and x_s. The necessary optimality condition in geometric form for the control curve (4.5) is satisfied for it; therefore, the range achieved by the platform with two oscillators $x_0(2T)$ will be the maximum possible range. For the original system (1.1), constrained by additional differential equations, the range cannot be greater.

In Remark 7, there were no additional geometric requirements for modes with three switchings because the set was generated by only one point (with parameter γ_1); i.e., there is a zero-dimensional plane, and it obviously cannot have other intersections with the control curve. As for Remark 8, it replaces the check for the absence of extra intersections of the control curve with the straight line A_1A_2 in n-dimensional space. The following fact is a natural generalization of Remarks 7 and 8.

Remark 9. If at some value of T system (2.2) admits a set of roots with condition (4.4), then the corresponding control (with k half-trajectory switchings, $k = n - j$) at $u(0) = 1$ will satisfy the necessary optimality conditions in the case when from the numbers ω_i, $i = \overline{1,n}$, we can choose k (renumbered) values ω_j, $j = \overline{1,k}$, for which the corresponding curve $\overline{r} = (\psi(\omega_1\rho), \ldots, \psi(\omega_k\rho))^{\mathrm{T}}$, $\rho \in [0,\ T]$ in k-dimensional space has only k common points with the plane $A_1A_2\ldots A_k$ drawn through the points taken on the curve with parameters γ_j, $j = \overline{1,k}$.

Next, we will consider varieties of nontrivial scenarios, relating them to the completion of not only the first layer of the diagram (with $T = T_1$) but also any subsequent layer with a certain number k (at $T = T_k$). Let us list some possible types of completion of the continuous evolution of kth layer due to a change in the number of roots of system (2.2).

Type 1. Assume that at $T = T_k$ the number of roots has become $(n-1)$. This is possible, for example, in a situation where the function $\gamma_n(T)$ goes to zero when the control curve in the n-dimensional space representing point A_n coincides with the origin K_0. According to Remark 5, the further continuous evolution of variables is possible only with an increase in their number. For the optimal control function $u(t)$, this is possible only by adding two new, arbitrarily close switching moments [12], which corresponds to a "needle" variation in the graph. With this value of parameter $T = T_k$ on the control curve, the hyperplane must pass through the points A_i, $i = \overline{1,n-1}$, but should not be a secant to this curve; otherwise there would be sudden changes in the graph $u(t)$. Therefore, the hyperplane will pass as a tangent to the control curve. Parameter γ_M of the desired point M of tangency can be determined from the condition that not only does the point belong to the hyperplane A_M but also the tangential vector calculated in it $\mathbf{v} = (\varphi(\omega_1\gamma_M), \ldots, \varphi(\omega_k\gamma_M))^{\mathrm{T}}$, expressed through the function $\varphi(\omega\rho) = [\cos(\omega\rho) - \psi(\omega\rho)]/\rho$. The tangency point M on the control curve will receive the "double" status, as will the corresponding switching points $\tau = T - \gamma_M$ on the graph of the control function $u(t)$. On the diagram on the horizontal $T = T_k$ a point with a parameter γ_M will be added, from which two new curves will emerge in the next layer, so that the evolution of the renumbered functions $\gamma_i(T)$, $i = \overline{1,n+1}$, will be described by a system with $n+1$ differential equations.

Type 2. Assume at $T = T_k$ the number of roots decreased to $(n-2)$. This is possible, for example, in a situation where the neighboring values $\gamma_i(T)$ and $\gamma_{i+1}(T)$ converge, up to coincidence, when the representing points A_i and A_{i+1} coincide on the control curve. To continue continuous evolution, according to Remark 6, at least two more variables $\gamma_k(T)$ will be required, which corresponds to the appearance of a double point on the control curve. Unlike the previous case (Type 1), where such a point was determined by constructing a tangential hyperplane, now there are an insufficient number of available points A_i for such a procedure. Although through $(n-1)$ points, it was possible to draw no more than two tangential hyperplanes to the control curve, through $(n-2)$ there are an infinite number of points that can be drawn. Therefore, to find the double point (let us call it L) on the horizontal $T = T_k$, it is more convenient to turn to system (4.1). Its coefficients are calculated for the known values of $T = T_k$ and $(n-2)$ "old" variables γ_i. Let us denote the parameter of the desired double point by γ_L, considering it common to two new variables γ_j and γ_{j+1}. There will be exactly n unknowns in the system, including parameter γ_L, the difference $x = \dot{\gamma}_j - \dot{\gamma}_{j+1}$, and derivatives $(n-2)$ of the old variables. By eliminating the derivatives from the system, we obtain a trigonometric equation with one unknown γ_L. The required root of γ_L must satisfy two conditions, verified by integrating the set of functions $\gamma_i(T)$, $i = \overline{1,n}$, over an arbitrarily small interval $[T_k, T_k + \Delta T]$:

(a) compliance with inequality (2.3),

(b) the absence of unnecessary points of intersection of the control curve and the hyperplane drawn through the points with parameters $\gamma_i(T)$, $i = \overline{1,n}$.

Note that Types 1 and 2 considered above provide an exhaustive list of possible deficiencies of roots (4.4) only for the case $n = 3$. A generalization can be situations where, in order to continue contin-

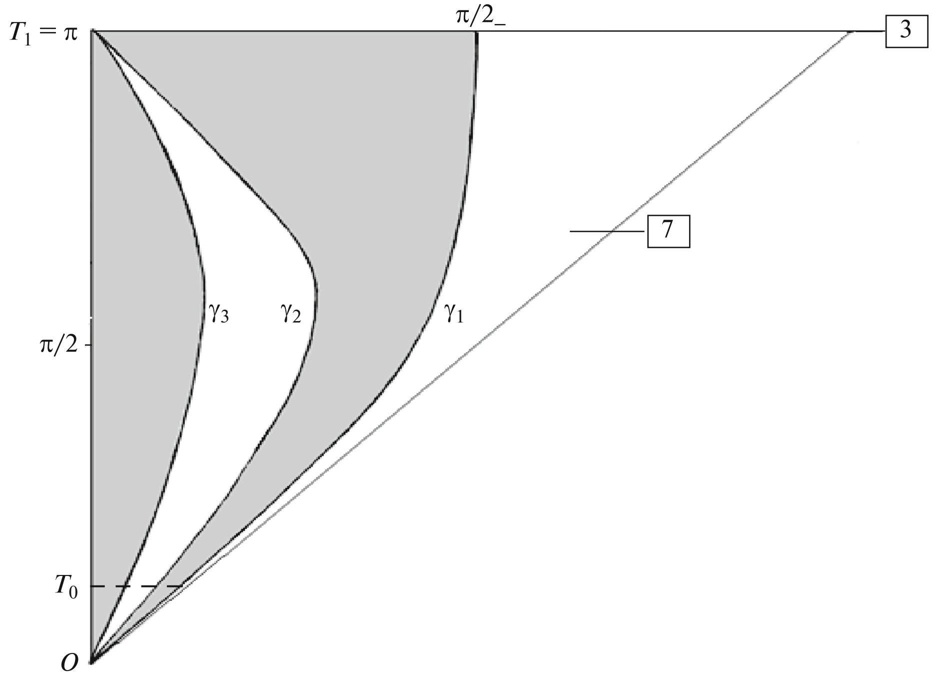

Fig. 7. The right (mirror) part of the diagram for $n = 3$ at $\omega_1 = 1$, $\omega_2 = 3$, $\omega_3 = 5$.

uous evolution, according to Remark 6, three or more missing points A_i will be needed. The graph of the function $u(t)$ may correspond to two or more needle variations. The number of possible scenarios (with different combinations of parameters ω_i, $i = \overline{1,n}$) is infinite; therefore, instead of a finite algorithm, we can only contemplate a general approach to continue the diagram of the optimal control functions. It consists of the joint consideration of the diagram and the geometric properties of the control curve.

Type 3. Assume that at $T \geq T_k$ system (2.2) has $j = n + m$, $m \in N$, roots. Then the evolution of functions $\gamma_i(T)$, $i = \overline{1, n+m}$, will be described by the set $(n + m)$ of differential equations, of which the first n have form (4.1), and the remaining m equations are obtained by differentiation with respect to parameter T of the relations of form (2.5), where instead of $(T - t)$ each of the m variables $\gamma_i(T)$, $i = \overline{n+1, n+m}$, is substituted.

5. RESULTS OF NUMERICAL EXPERIMENTS FOR A PLATFORM WITH THREE OSCILLATORS

Example 1. In Fig. 7, for values $\omega_1 = 1$, $\omega_2 = 3$, and $\omega_3 = 5$, the right part of the diagram of the optimal control functions is shown (its left part is mirrored, but with the white and gray colors interchanged). From the vertex O, the curves $\gamma_1(T)$, $\gamma_2(T)$, and $\gamma_3(T)$ emerge with the angular coefficients calculated using formulas (3.9) (for case 2) in the form

$$k_1 = \sqrt{2+\sqrt{2}}/2, \quad k_2 = \sqrt{2}/2, \quad k_3 = \sqrt{2-\sqrt{2}}/2. \tag{5.1}$$

Equations (4.2), $n = 3$, were integrated numerically while simultaneously checking the absence of extra intersections of the control curve with the plane drawn through the points with parameters $\gamma_1(T)$, $\gamma_2(T)$, and $\gamma_3(T)$. The process ended at $T_1 = \pi$, $\gamma_1 = \pi/2$, and $\gamma_2 = \gamma_3 = 0$. According to Remark 1, for integer frequencies, the quantity $T_* = 2\pi$ is the period of repetition of the shape of the diagram, mirror-doubled relative to the straight line $T = \pi$ and generating a square. A horizontal line $T = 2\pi$ will pass through its vertex, symbolizing a mode with one moment of control switching. Thus, Fig. 7 contains all possible fragments of an infinitely extendable diagram. For any values of T, here only modes with numbers of 1, 3, and 7 of the optimal control switchings are possible.

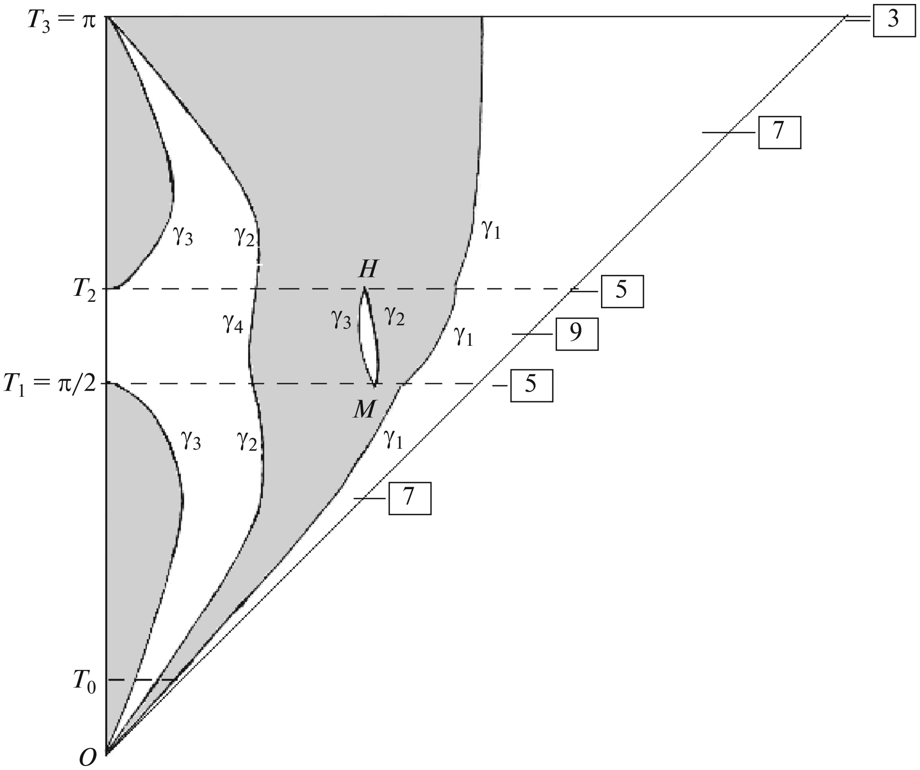

Fig. 8. The right (mirror) part of the diagram for $n = 3$ at $\omega_1 = 1$, $\omega_2 = 3$, $\omega_3 = 7$.

Example 2. Figure 8 shows the right (mirror) part of the diagram for the case $\omega_1 = 1$, $\omega_2 = 3$, and $\omega_3 = 7$. Curves $\gamma_1(T)$, $\gamma_2(T)$, and $\gamma_3(T)$ emerge from the vertex O with the same angular coefficients (5.1), but the integration of the system is completed at $T_1 = \pi/2$ due to the function $\gamma_3(T)$ vanishing. The values $\gamma_1 = 2\pi/5$ and $\gamma_2 = \pi/5$ at $T_1 = \pi/2$ satisfy not only system (2.2) but also the necessary optimality conditions in the form of Remark 8. Indeed, the spatial control curve (Fig. 4) has a projection onto the plane (y_1, y_2) in the form of Fig. 2, on which the straight line drawn through points A_1 and A_2 (with parameters $2\pi/5$ and $\pi/5$) has no other intersections with the control curve. According to Remark 8, this is sufficient, despite the fact that, for example, the projection onto the plane (y_1, y_3) (Fig. 3) does not satisfy this property.

On the horizontal $T_1 = \pi/2$, the presence of only two points out of three corresponds to Type 1 described above. The parameter $\gamma_M \approx 1.150466$ of the third point M is found from the condition of tangency of the plane A_1A_2M with the control curve (these points are shown in Fig. 4). As parameter T increases, the tangential plane turns into a secant plane, and two new points appear on the control curve from point M. The second layer of the diagram begins with the renumbered values of variables $\gamma_1 = 2\pi/5$, $\gamma_2 = \gamma_3 \approx 1.150466$, and $\gamma_4 = \pi/5$, and a new system of equations is integrated. It corresponds to Type 3: to the three equations (4.1), we add a fourth equation, obtained by differentiating relations (2.5) at $n = 3$ with respect to parameter T, where γ_4 is substituted instead of $(T - t)$. The resulting system is solved with respect to the derivatives in the form

$$\frac{d\gamma_i}{dT} = \frac{R_i}{R} \quad i = \overline{1,4},$$

$$R = 2(\mu_1 V_{234} + \mu_2 V_{134} + \mu_3 V_{124} + \mu_4 V_{123})/T, \quad \mu_i = G_i/\gamma_i^2, \quad i = \overline{1,4}, \tag{5.2}$$

$$R_1 = (\mu_2 V_{034} + \mu_3 V_{024} + \mu_4 V_{023})/\gamma_1, \quad R_2 = (-\mu_1 V_{034} + \mu_3 V_{014} + \mu_4 V_{013})/\gamma_2,$$

$$R_3 = (-\mu_1 V_{024} - \mu_2 V_{014} + \mu_4 V_{012})/\gamma_3, \quad R_4 = (-\mu_1 V_{023} - \mu_2 V_{013} - \mu_3 V_{012})/\gamma_4.$$

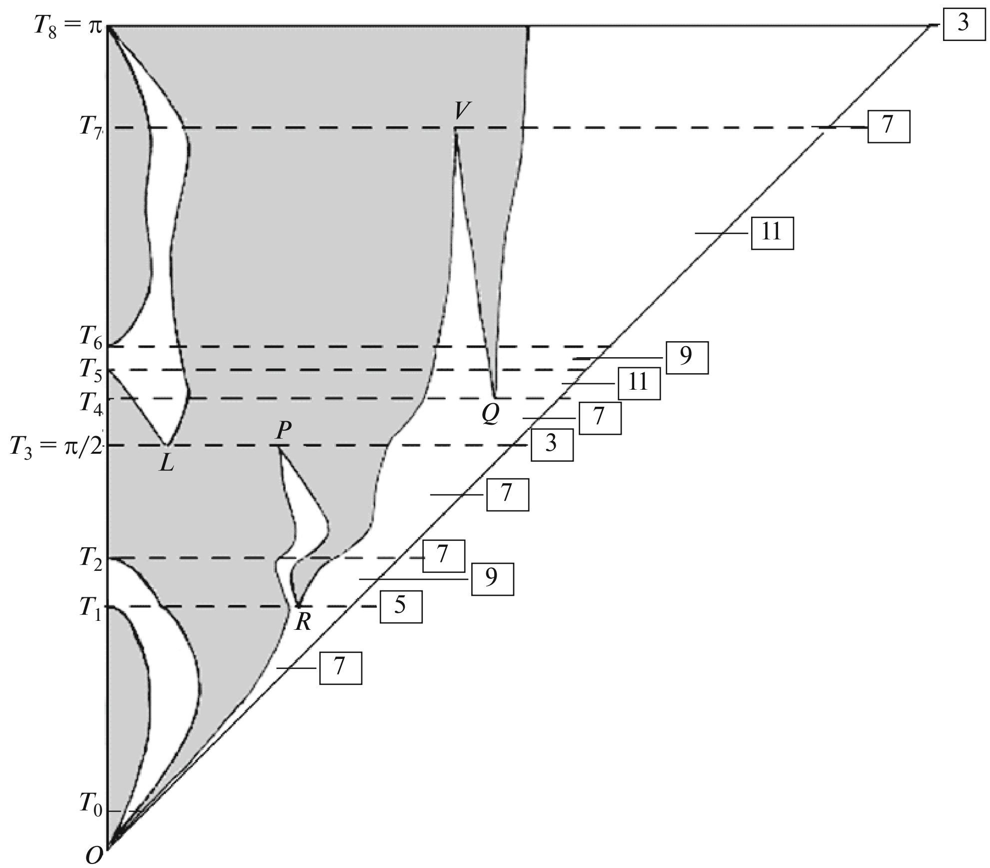

Fig. 9. The right (mirror) part of the diagram for $n = 3$ at $\omega_1 = 1$, $\omega_2 = 7$, $\omega_3 = 13$.

Here the columns $\mathbf{g}_i = (\psi(\omega_1\gamma_i), \psi(\omega_2\gamma_i), \psi(\omega_3\gamma_i))^{\mathrm{T}}$, $i = \overline{1,4}$, and column $\mathbf{g}_0 = (\psi(\omega_1 T), \psi(\omega_2 T), \psi(\omega_3 T))^{\mathrm{T}}$ are used. The determinant of the third order G_i, $i = \overline{1,4}$, is obtained from the determinant $G = \|(\mathbf{g}_1 - \mathbf{g}_2)(\mathbf{g}_2 - \mathbf{g}_3)(\mathbf{g}_3 - \mathbf{g}_4)\|$ by the formal replacement of $\mathbf{g}_i$ by column $\mathbf{h}_i = (\cos(\omega_1\gamma_i), \cos(\omega_2\gamma_i), \cos(\omega_3\gamma_i))^{\mathrm{T}}$, $i = \overline{1,4}$, and all determinants $V_{ijk} = \|(\mathbf{g}_i)(\mathbf{g}_j)(\mathbf{g}_k)\|$ composed of columns with increasing numbers.

The integration of system (5.2) is completed when $T_2 \approx 1.982313$ due to the coincidence of the values of functions $\gamma_3(T)$ and $\gamma_2(T)$. Their graphs on the diagram (Fig. 8) intersect at the vertex H of the "lentil" starting upward from point M. On the control curve, the vanishing double point $\gamma_2 = \gamma_3 \approx 1.0760698$ will undergo a moment of tangency with the hyperplane, after which the remaining point A_1 ($\gamma_1 \approx 1.459590$) will keep its name and A_4 ($\gamma_4 \approx 0.624976$) will be renamed A_2. The role of the third point A_3 will further be taken by K_0 ($\gamma_3 = 0$), since, as the numerical verification shows, only such a plane $A_1A_2A_3$ does not intersect the control curve at other points. At $T_2 > 1.982313$, the evolution of functions $\gamma_1(T)$, $\gamma_2(T)$, and $\gamma_3(T)$ is found by integrating equations (4.2) at $n = 3$. This third layer ends with the values $T_3 = \pi$, $\gamma_1 = \pi/2$, and $\gamma_2 = \gamma_3 = 0$, and it is the last one in the diagram (Fig. 8). In each layer of the diagram, the number indicates the corresponding number of control switchings $u(t)$, $t \in [0, 2T]$.

Example 3. Figure 9 shows the right (mirror) part of the diagram for the case $\omega_1 = 1$, $\omega_2 = 7$, and $\omega_3 = 13$. The plot was chosen to illustrate a situation of Type 2, since system (2.2) obviously admits the solution $T = \pi/2$, $\gamma_1 = \pi/3$, $\gamma_2 = \gamma_3 = 0$.

Here again the curves $\gamma_1(T)$, $\gamma_2(T)$, and $\gamma_3(T)$ emerge from the vertex O with the same angular coefficients (5.1), but the integration of system (4.2) is completed at $T_1 \approx 0.946912$, where $\gamma_3(T_1) = 0$. At this moment on the control curve, the point A_3 is compatible with K_0. To the two remaining points A_1 and A_2 (Type 1), a third one will be added after finding the parameter $\gamma_R \approx 0.690766$ of point R touching the reference curve and the plane A_1A_2R. In the second layer of the diagram, the integration of system (5.2) is completed at $T_2 \approx 1.146912$ in view of $\gamma_4(T_2) = 0$. In the third layer, system (4.2) is integrated at $n = 3$ until the curves $\gamma_2(T)$ and $\gamma_3(T)$ intersect at point P. On the horizontal $T_3 = \pi/2$, only one point ($\gamma_1 = \pi/3$) remains instead of three. To find a new point L, according to the scheme described above (see Type 2), we compose the system of equations (4.1) with the replacement of $x = \dot{\gamma}_2 - \dot{\gamma}_3$ in the form

$$\dot{\gamma}_1 \sin\left(\frac{\omega_i \pi}{3}\right) - x\sin(\omega_i \gamma_L) = \frac{1}{2}\sin\left(\frac{\omega_i \pi}{2}\right), \quad i = \overline{1,3}. \tag{5.3}$$

Substituting values $\omega_1 = 1$, $\omega_2 = 7$, and $\omega_3 = 13$, we exclude variables from the system $\dot{\gamma}_1$ and x, by obtaining the equation $\sin(\gamma_L) = \sin(13\gamma_L)$. In the current range $(0 < \gamma_L < \pi/2)$, it has five roots, of which only $\gamma_L \approx 0.2243995$ simultaneously satisfies conditions (a) and (b) mentioned in the Remarks (see Type 2).

In the fourth layer of the diagram (Fig. 9), curves $\gamma_2(T)$ and $\gamma_3(T)$ emerge from point L, which together with $\gamma_1(T)$ are determined by integrating system (4.2) at $n = 3$. The process is interrupted when $T_4 \approx 1.736850$, when the control curve touches the plane $A_1A_2A_3$ at an additional point Q with parameter $\gamma_Q \approx 1.466848$. On the diagram in the fifth layer from point R, two new curves emerge; therefore, to the three equations (4.1), the fourth and fifth equations will be added, obtained by differentiation with respect to parameter T of relation (2.5) at $n = 3$, where, respectively, γ_4 and γ_5 are substituted instead of $(T - t)$. The resulting system is reduced to the form

$$\begin{gathered}
\frac{d\gamma_i}{dT} = \frac{H_i}{H}, \quad i = \overline{1,5}, \\
H = 2[V_{345}(\mu_2\lambda_1 - \mu_1\lambda_2) + V_{245}(\mu_3\lambda_1 - \mu_1\lambda_3) + V_{145}(\mu_3\lambda_2 - \mu_2\lambda_3) \\
+ \mu_4(\lambda_1 V_{235} + \lambda_2 V_{135} + \lambda_3 V_{125}) - \lambda_5(\mu_1 V_{234} + \mu_2 V_{134} + \mu_3 V_{124} + \mu_4 V_{123})]/T, \\
\mu_i = G_i/\gamma_i^2, \quad i = \overline{1,4}, \quad \lambda_i = F_i/\gamma_i^2, \quad i = 1,2,3,5, \\
H_1 = [V_{045}(\mu_3\lambda_2 - \mu_2\lambda_3) + \mu_4(\lambda_2 V_{035} + \lambda_3 V_{025}) - \lambda_5(\mu_2 V_{034} + \mu_3 V_{024} + \mu_4 V_{023})]/\gamma_1, \\
H_2 = [V_{045}(\mu_1\lambda_3 - \mu_3\lambda_1) + \mu_4(-\lambda_1 V_{035} + \lambda_3 V_{015}) - \lambda_5(-\mu_1 V_{034} + \mu_3 V_{014} + \mu_4 V_{013})]/\gamma_2, \\
H_3 = [V_{045}(\mu_2\lambda_1 - \mu_1\lambda_2) + \mu_4(-\lambda_1 V_{025} - \lambda_2 V_{015}) - \lambda_5(-\mu_1 V_{024} - \mu_2 V_{014} + \mu_4 V_{012})]/\gamma_3, \\
H_4 = [V_{035}(\mu_2\lambda_1 - \mu_1\lambda_2) + V_{025}(\mu_3\lambda_1 - \mu_2\lambda_2) + V_{015}(\mu_3\lambda_2 - \mu_2\lambda_3) - \lambda_5(-\mu_1 V_{023} - \mu_2 V_{013} - \mu_3 V_{012})]/\gamma_4, \\
H_5 = [V_{034}(\mu_2\lambda_1 - \mu_1\lambda_2) + V_{024}(\mu_3\lambda_1 - \mu_1\lambda_3) + V_{014}(\mu_3\lambda_2 - \mu_2\lambda_3) + \mu_4(\lambda_1 V_{023} + \lambda_2 V_{013} + \lambda_3 V_{012})]/\gamma_5.
\end{gathered} \tag{5.4}$$

Here the new column $\mathbf{g}_5 = (\psi(\omega_1\gamma_5), \psi(\omega_2\gamma_5), \psi(\omega_3\gamma_5))^{\mathrm{T}}$ is added to the columns previously introduced in (5.2) $\mathbf{g}_i$, $i = \overline{0,4}$. The determinant of the third order F_i, $i = 1,2,3,5$, is obtained from the determinant $F = \|(\mathbf{g}_1 - \mathbf{g}_2)(\mathbf{g}_2 - \mathbf{g}_3)(\mathbf{g}_3 - \mathbf{g}_5)\|$ by the formal replacement of $\mathbf{g}_i$ by column $\mathbf{h}_i = (\cos(\omega_1\gamma_i), \cos(\omega_2\gamma_i), \cos(\omega_3\gamma_i))^{\mathrm{T}}$, $i = 1,2,3,5$, and all the determinants $V_{ijk} = \|(\mathbf{g}_i)(\mathbf{g}_j)(\mathbf{g}_k)\|$ are composed of columns with increasing numbers.

The integration of system (5.4) is completed when $T_5 \approx 1.853483$ due to the variable $\gamma_5(T_5) = 0$ vanishing. In the sixth layer of the diagram, the fourth order system (5.2) is integrated until the moment when at $T_6 \approx 1.933983$ the control curve intersects the plane passing through A_1, A_2, A_3, and A_4 at the point K_0 ($\gamma_5 = 0$). Next, in the seventh layer of the diagram, the fifth-order system (5.4) is integrated until $T_7 \approx 2.756850$ at the point V the curves $\gamma_2(T)$ and $\gamma_3(T)$ intersect. The remaining three curves in the eighth layer are determined by integrating system (4.2) at $n = 3$. The diagram (Fig. 9) ends with a horizontal line $T_8 = \pi$.

In examples 1–3, the numbers ω_i, $i = \overline{1,3}$, are odd integers; thus, when $T = \pi$, system (2.2) admits the solution $\gamma_1 = \pi/2$ and $\gamma_2 = \gamma_3 = 0$. It can be shown (by expanding the terms of this system into a Taylor series) that in the diagram in the vicinity of the horizontal $T = \pi$ in all three examples the values of the derivatives will be $\dot{\gamma}_1 = 0$, $\dot{\gamma}_2 = -\sqrt{3}/2$, and $\dot{\gamma}_3 = -1/2$. The small neighborhoods of vertex O also look

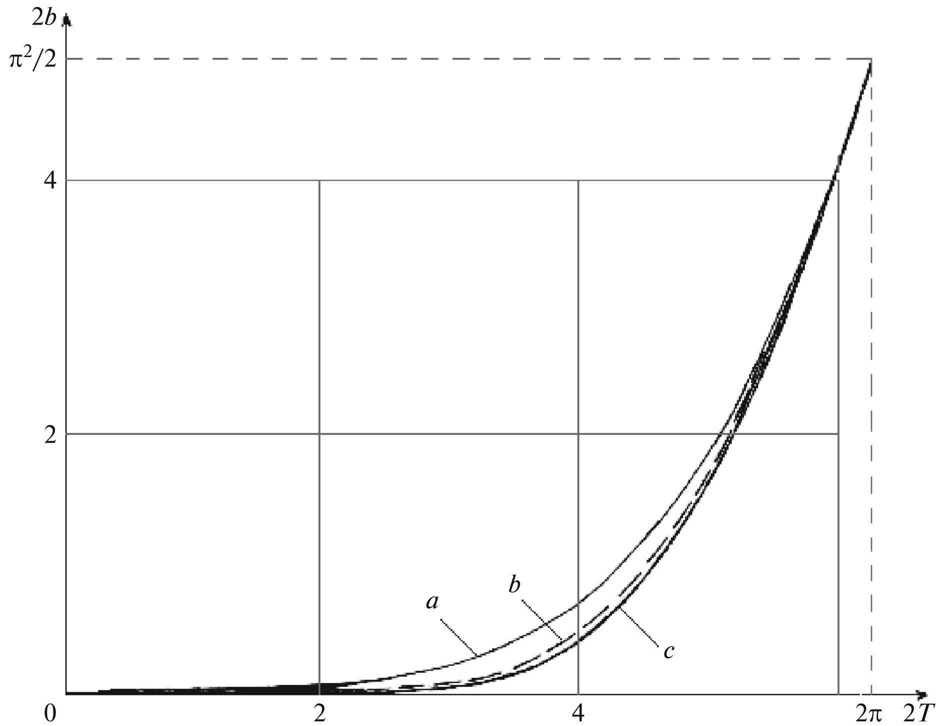

Fig. 10. Dependence of range $2b$ on time $2T$ for different combinations of parameters.

almost identical, since the angular coefficients of the lines emanating from it coincide. In each diagram, the dotted line marks its own horizontal line at level T_0 (2.8), which gave a (clearly underestimated) estimate of the range where regimes with n control switchings on a half-trajectory were guaranteed.

Since in Figs. 7–9, the diagrams end in the same way at $T = \pi$, $\gamma_1 = \pi/2$, and $\gamma_2 = \gamma_3 = 0$, the values of the corresponding half distance $b = \pi^2/4$ are also the same according to formula (3.17).

Figure 10 shows the dependencies $2b$ of the range of motion of the platform on the time $2T$ of motion. Examples 1, 2, and 3 correspond to curves *c*, *b,* and *a*.

CONCLUSIONS

The problem of the optimal speed of motion (at the given distance) of a platform with n oscillators from one state of rest to another is considered. The proposed scheme for constructing a diagram of the optimal control functions has become a natural generalization of the case $n = 2$ studied in [12]. The general approach is to consider the diagram and the geometric properties of the control curve together in n-dimensional space. For small movements of the platform, the optimal control function turned out to be piecewise constant with n moments of switching on a half-trajectory. It turned out that it was not obvious that they were almost independent of the ratio of the natural frequencies of the oscillators. For example, on charts with different combinations of parameters ω_1, ω_2, and ω_3 (Figs. 7–9), the neighborhoods of the vertex O were almost identical in range $T < T_0$, calculated using formula (2.8) for $\omega_3 = 13$ (Fig. 9). Therefore, in particular, the solution of the original problem in the formulation [1], where the platform with n pendulums is considered in a linear approximation, and therefore, under the assumption of small displacements, can be considered complete. For it, the result of the study asserts that the number of optimal control switchings is $(2n + 1)$, and the control functions are taken from the bottom of the corresponding diagram, according to Statement 3.

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CONFLICT OF INTEREST

The author of this work declare that he has no conflicts of interest.

REFERENCES

1. F. L. Chernous'ko, L. D. Akulenko, and B. N. Sokolov, *Oscillation Control* (Nauka, Moscow, Nauka, 1980) [in Russian].
2. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *Mathematical Theory of Optimal Processes* (Nauka, Moscow, 1969) [in Russian].
3. V. M. Mamalyga, "On optimal control of a single oscillatory system," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 3, 8–17 (1978).
4. O. R. Kayumov, "Global controllability of certain Lagrangian systems," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 6, pp. 16–23 (1986).
5. A. I. Ovseevich and A. K. Fedorov, "Asymptotically optimal feedback control for a system of linear oscillators," Dokl. Math. **88** (2), 613–617 (2017).
6. I. M. Anan'evskii, N. V. Anokhin and A. I. Ovseevich, "A synthesis of the control of dynamical systems on the basis of the method of Lyapunov functions," J. Math. Sci. **199** (4), 597–603 (2010).
7. A. A. Ovseevich, "Local feedback control bringing a linear system to equilibrium," J. Optim. Theory Appl. **165** (2), 532–544 (2015).
8. I. M. Anan'evskii and T. A. Ishkhanyan, "Control of a rigid body carrying dissipative oscillators under perturbations," J. Comput. Syst. Sci. Int. **58** (1), 40–49 (2019).
9. I. M. Anan'evskii, "Motion control for platforms bearing elastic links with unknown phase states," J. Comput. Syst. Sci. Int. **58** (6), 844–851 (2019).
10. I. M. Anan'evskii and A. I. Ovseevich, "Controlled motion of a linear chain of oscillators," J. Comput. Syst. Sci. Int. **60** (5), 684–694 (2021).
11. O. R. Kayumov, "Time-optimal movement of platform with oscillators," Mech. Solids **56** (8), 1622–1637 (2021).
12. O. R. Kayumov, "Diagrams of optimal control functions in the problem of the fastest platform movement with two oscillators," J. Comput. Syst. Sci. Int. **61** (5), 776–792 (2022).
13. O. R. Kayumov, "On the properties of the time-optimal movement of a platform with oscillators," in *Proceedings of 16th International Conference on Stability and Oscillations of Nonlinear Control Systems (Pyatnitskiy's Conference), STAB 2022* (Moscow, 2022), p. 9807541.