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Dedicated to the memory of Corresponding Member of the Russian Academy of Sciences Gennady Alekseevich Leonov

Theory of Hidden Oscillations and Stability of Control Systems

N. V. Kuznetsov

St. Petersburg State University, Institute for Problems in Mechanical Engineering, Russian Academy of Sciences, St. Petersburg, Russia

e-mail: n.v.kuznetsov@spbu.ru

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Abstract—The development of the theory of absolute stability, the theory of bifurcations, the theory of chaos, theory of robust control, and new computing technologies has made it possible to take a fresh look at a number of well-known theoretical and practical problems in the analysis of multidimensional control systems, which led to the emergence of the theory of hidden oscillations, which represents the genesis of the modern era of Andronov's theory of oscillations. The theory of hidden oscillations is based on a new classification of oscillations as self-excited or hidden. While the self-excitation of oscillations can be effectively investigated analytically and numerically, revealing a hidden oscillation requires the development of special analytical and numerical methods and also it is necessary to determine the exact boundaries of global stability, to analyze and reduce the gap between the necessary and sufficient conditions for global stability, and distinguish classes of control systems for which these conditions coincide. This survey discusses well-known theoretical and engineering problems in which hidden oscillations (their absence or presence and location) play an important role.

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INTRODUCTION

History of the theory of hidden oscillations. The mathematical modeling of the dynamics and determination of stability in technical systems is the most relevant direction in the scientific and technological development of any state that seeks to occupy a leading position in the modern world. The study of limiting dynamic regimes (attractors) and stability is necessary both in classical theoretical and in actual practical problems. One of the primary tasks of this kind is related to designing control systems: automatic regulators, in the XVIII–XIX centuries. Regulators had to ensure the transition of the dynamics of the control object to the operating mode and the stability of the operating mode relative to the initial deviations and external perturbations. A classic example is the Watt regulator, used to maintain a given constant speed of rotation of a turbine shaft. In Great Britain alone in 1868, about 75000 steam engines equipped with Watt regulators worked at industrial enterprises [1]. However, in the middle of the XIX century many of the steam engines produced were inoperative due to the phenomenon of self-oscillation, when the regulator could not cope with maintaining the alignment of the moments. The high accident rate of these machines posed new engineering requirements for their design and thereby stimulated the development of the mathematical theory of the stability of control systems.

The famous paper of I.A. Vyshnegradsky on the Watt regulator [2], published in 1877, is an example of a mathematical formulation and solution of such problems. This publication investigated the closed dynamic steam engine–regulator model, for which the phenomenon of self-oscillation was reformulated in mathematical terms as an instability of the solution of a differential equation. Vyshnegradsky determined the stability conditions for a stationary solution (a trivial attractor) corresponding to the required operating mode for an approximate linear model, excluding dry friction.

However, the important question remained open after Vyshnegradsky's paper, namely, the question of the rigorous proof of global stability and the admissibility of linearizing the model by discarding dry friction. In 1885, M. Leaute first showed that in engineering control systems with dry friction, limiting periodic oscillations—limit cycles (periodic attractors)—can occur [3]. This led to criticism of Vyshnegradsky's approach and his findings, including criticism by the outstanding Russian scientist N.E. Zhukovsky [4].

A systematic study of limit cycles and the criteria for their absence in applied dynamical systems is related to the work of A.A. Andronov, who created the mathematical theory of oscillations, explaining the behavior of many applied systems. This theory combined the mathematical ideas of the analysis of local stability according to A.M. Lyapunov [5] and limit oscillations of A. Poincare [6] for smooth dynamical systems with engineering needs to take into account discontinuous nonlinearities.

Andronov's theory of oscillations made it possible to analytically study the occurrence of limit oscillations, as well as obtain the necessary and sufficient conditions for the absence of oscillations and global stability for low-order systems. The foundations of this theory are presented in the famous monograph "Theory of oscillations", the first edition of which was published in 1937 [7] and is devoted to the analysis of two-dimensional mathematical models of various oscillatory systems.[1]

Since 1944, Andronov was actively involved in applying the theory of oscillations to problems of automatic control and organized the famous seminar at the Institute of Automation and Telemechanics (now the Institute of Control Sciences) in Moscow, where he founded the world-famous scientific school on the theory of automatic control—a section of modern control theory [10, 11]. The first results of Andronov in this direction is a rigorous nonlocal analysis of the nonlinear model of the Watt regulator with dry friction and proof of the sufficiency of Vyshnegradsky's conditions for the absence of oscillations and global stability of the operating mode (the existence of a rest segment that attracts trajectories from any initial data) [12, 13]. The significance of these results was noted during the election of Andronov in 1946 as a full member of the USSR Academy of Sciences in the Department of Technical Sciences, where he became the first academician in control theory [14, p. 56]. In the postwar years, participants of Andronov's seminar included M.A. Aizerman, V.V. Petrov, and Ya.Z. Tsypkin, who became prominent representatives of the Moscow school of control theory.

Further development of the results of Andronov–Vyshnegradsky and obtaining sufficient conditions for stability and the absence of oscillations for multidimensional automatic control systems with discontinuous characteristics is related to the works of A.Kh. Gelig, G.A. Leonov and their students [15–18]. Note that such problems still remain relevant for modern turbine regulators, as shown by the recent accident at the Sayano-Shushenskaya hydroelectric power station and an analysis of its possible causes [19–22]. The motivation for the further development of methods for analyzing control systems with discontinuous characteristics is the importance of considering modern tribology models [23–25] and control models [26–34].

In the general case, for the practical use of regulators, it is necessary to identify in a closed system all stable stationary and oscillatory regimes, as well as their basins of attraction. In the works of L.I. Mandelstam, N.D. Papaleksi, and A.A. Andronov in the description of oscillatory systems, a stable stationary mode is characterized as *a non-self-excited state*, and the transition from an unstable stationary mode to a periodic oscillatory mode (limit cycle) is described as *self-excitation of self-oscillations*[2] [7, 35]. In 1963, the American meteorologist E. Lorenz showed that a similar self-excitation can lead not only to periodic but also to chaotic limit regimes (chaotic attractors) in smooth multidimensional dynamical systems [36]. For smooth systems with a periodic perturbation, a similar effect was discovered by the Japanese scientist J. Ueda in 1961; however, this result became known only in the 1970s [37]. For a piecewise linear system, self-excitation of chaotic oscillations was first discovered in Chua electronic circuit models in 1984 [38].

The tasks of analyzing multidimensional control systems and obtaining the necessary and sufficient conditions for global stability, including those guaranteeing the absence of chaotic oscillations, showed the need for further development of the theory of oscillations of Andronov and the creation of new analytical and numerical methods for the analysis of stability and oscillations.

At the present stage of the study of oscillations, the engineering concepts of *the transition process* and *retention* are closely related to the possibility of a numerical analysis of limiting oscillations. In the general case, an oscillation can generally be easily numerically localized if the initial data from its vicinity in the

[1] A.A. Witt was a coauthor of the first edition of the book [8], published in 1937, but his name was removed from the first edition and restored only in the second edition of the book, published in 1959. In 1941, a monograph by I. Rockard, one of the creators of the atomic bomb in France, was published with a similar name and similar ideas in French [9] without reference to the works of Andronov.

[2] The concept of self-excitation of oscillations in Andronov's works was also used to describe *the bifurcation process* of the transition of the system's state to the oscillation mode with changing parameters [8].

phase space lead to a long-term behavior that approaches the oscillation. Such an oscillation (or set of oscillations) in the phase space of dynamical system is called an attractor.[3] The classical engineering analysis of stability and oscillations in a system consists in determining stationary states, analytically determining their local stability, and then numerically analyzing the behavior of the system with the initial data in the vicinity of unstable stationary states. Such an analysis allows us to show the attraction of trajectories from a neighborhood of unstable equilibrium states to stable equilibrium states: to trivial attractors (such as the attraction of trajectories to a trivial attractor in the mathematical pendulum model), or to reveal ontrivial (oscillatory) attractors. Due to the self-excitation property, modern computational tools make it easy to detect attractors in the van der Pol,[4] Lorentz, and Chua models by numerically integrating the trajectories with the initial data from a neighborhood of unstable equilibrium states [39].

In this approach, the problem of the existence of attractors in the system to which trajectories from the vicinity of equilibrium states are not attracted remained open. In 2009 G.A. Leonov and N.V. Kuznetsov, the author of this review, proposed a new classification of attractors of dynamic systems, which became the basis for the theory of hidden oscillations: an attractor is called hidden if its basin of attraction does not intersect with a neighborhood of all equilibria; otherwise, it is called a self-excited attractor [39–43]. The concept of an attractor allows us to generalize the ideas of Andronov on the correspondence between steady periodic oscillations and limit cycles of dynamical systems also for steady non-periodic oscillations (arbitrary oscillations, depending on their basins of attraction, can similarly be classified as self-excited or hidden relative to unstable stationary states).Thus, for the first time the terms *hidden attractor* and *hidden oscillation* were introduced and their mathematical definitions were given. The proposed classification is consistent with the experimental approach to the analysis of the occurrence of oscillations and numerical procedures for searching for attractors, reflects the difficulties of solving a number of actual engineering problems and well-known scientific problems, and has also become a catalyst for the discovery of new attractors in well-known systems. While *self-excited*[5] *attractors* can be easily detected and visualized by trajectories in numerical experiments with the initial data from the neighborhoods of unstable equilibrium states, *hidden attractors* are not related to equilibrium states and their basins of attraction are hidden in the phase space of the system. Therefore, the numerical search for hidden attractors and determining the initial data for their visualization in the general case turns out to be a nontrivial problem (which scientists from different countries[6] are currently trying to solve).

The theory of hidden oscillations [45–47] is a modern stage in the development of the theory of oscillations of Andronov. It is in demand in many theoretical and relevant engineering problems in which hidden attractors (their absence or presence and location) play an important role.

The importance of identifying hidden attractors for control systems is related to the classical problems of determining the exact boundaries of global stability, analyzing the gap between the necessary and sufficient conditions for global stability and their convergence, and identifying classes of control systems for which these conditions coincide. In practice, the transition of the state of the control system to the hidden attractor mode caused by external perturbations leads to undesirable operating regimes and is often the cause of accidents and catastrophes.

For the analysis of hidden oscillations, effective analytical and numerical methods have been developed that allow us to determine the stability boundaries and detect hidden oscillations in various urgent problems. The main results obtained in this direction are the subject of the doctoral thesis of the author of this review: "Analytical and numerical methods for the analysis of hidden oscillations"[7] [43]. In recent years, the theory of hidden oscillations has been recognized by the scientific community both in Russia and abroad: the first fundamental publications on this subject were included in 2016 in 1% of the most cited papers of the Web of Science base, with two publications [39, 41] becoming the most cited papers in the journals[8] in which they were published.

This review is based on plenary lectures at the 11th Russian Multi-Conference on Control Problems (St. Petersburg, 2018) and the 5th IFAC Conference on the Analysis and Control of Chaotic Systems

[3] An attractor of a dynamical system is a bounded closed invariant set in phase space that is locally attractive (i.e., having an open neighborhood—a basin of attraction, all trajectories with the initial data from which tend to the attractor over time).

[4] Several paragraphs in the monograph [7] are devoted to the corresponding analytical analysis.

[5] G. Barkhausen used the similar German term *Selbsterregte Schwingungen* in his papers [44].

[6] https://scholar.google.com/scholar?q=hidden+attractor

[7] This doctoral thesis was defended at St. Petersburg State University in 2016 (reviewers: I.M. Burkin, N.G. Kuznetsov, G.A. Leonov, E.A. Mikrin, V.G. Peshekhonov, R.M. Yusupov, V.I. Nekorkin, and A.M. Sergeev (Leading Organization–Institute of Applied Physics of the Russian Academy of Sciences)).

[8] RAS, news 11.12.2016 (Russian Highly Cited Researchers Award, 2016): http://www.ras.ru/news/shownews.aspx?id=036a64c2-32f2-4624-bc32-8f0e4d138e7d. See the citation of the review on latent oscillations in the translated version of Izvestia RAN. Control theory and systems: http://citations.springer.com/search?query=Computer+and+Systems+Sciences+International.

(Eindhoven, 2018). The review presents theoretical and engineering problems in the framework of the theory of hidden oscillations: the Andronov-Vyshnegradsky problem on the nonlinear analysis of the centrifugal governor of steam engine the Keldysh problem on the nonlinear analysis of flutter suppression models, the Aiserman and Kalman conjectures on the absolute stability of control systems in the Lurie form, Hilbert's 16th problem, the Sommerfeld effect of frequency jamming, Chua circuits, phase-locked loops and the Kapranov conjecture, and detection of hidden oscillations in dynamic models of drilling rigs.

1. HIDDEN OSCILLATIONS IN COUNTEREXAMPLES TO THE AIZERMAN AND KALMAN CONJECTURES

In the middle of the 20th century M.A. Aizerman and R.E. Kalman studied the coincidence of the global stability condition in nonlinear control systems with the stability condition of linear approximations for control systems with a single equilibrium state and nonlinearities from a given sector.

The criteria for the absence of oscillations in automatic control systems began to be developed in the 1940s. In 1944, the famous work of A.I. Lurie and V.N. Postnikov [48] was published, in which for the first time an effective technique was proposed for obtaining sufficient conditions for the absence of oscillations and global stability, based on a special class of Lyapunov functions,[9] for a mathematical model of a control system of the following form:

$$\dot{z} = Pz + qf(r^{\mathrm{T}}z), \tag{1.1}$$

where z is the state vector of an object, P is a constant matrix, q and r are constant vectors, $f(\cdot)$ is a scalar continuous nonlinearity from a given class, and T is a transpose operation. Such a control system in the literature is often called the Lurie system. In the same year, A.A. Andronov and A.G. Meyer [12, 13] for the Watt regulator model from the work of Vyshnegradsky [2] in the form of the Lurie system (1.1) with discontinuous nonlinearity,

$$P = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ -\alpha & 1 & -\beta \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad r = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad f(\sigma) = \frac{1}{2}\operatorname{sign}(\sigma), \tag{1.2}$$

taking into account dry friction, demonstrated the coincidence of the conditions of the asymptotic stability of the linearized model without dry friction and the global stability of the nonlinear model when Vyshnegradsky's conditions—$\alpha > 0, \beta > 0$, and $\alpha\beta > 1$—were satisfied.

These works led to the problem of distinguishing classes of Lurie systems for which the necessary conditions for stability and the absence of self-excited oscillations coincide with sufficient conditions for global stability. In 1957, Kalman formulated the following criterion [50]: *If $f(\sigma)$ is replaced in Fig. 1 is replaced by constants k corresponding to all possible values of $f'(\sigma)$ and it is found that the closed-loop system is stable for all such k, then it intuitively clear that the system must be monostable; i.e., all transient solutions will converge to a unique, stable critical point.* From the absence of self-excited oscillations, the necessary condition for stability, a conclusion is drawn here on global stability. It is interesting that, having great engineering intuition, Kalman formulated a criterion that is justified (as it turned out later) in 3-dimensional space [51, 52]. However, in the general case, e.g. in 4-dimensional space[10], the Kalman criterion is incorrect, and counterexamples were constructed for it [39]. For example, the Lurie system (1.1) with

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad r = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \tag{1.3}$$

and $a_0 = (m_1^2 + \beta^2)(m_2^2 + \beta^2)$, $a_1 = 2\beta(m_1^2 + m_2^2 + 2\beta^2)$, $a_2 = m_1^2 + m_2^2 + 6\beta^2$, $a_3 = 4\beta$, $m_1 = 0.9$, $m_2 = 1.1$, and $\beta \geq 0$ has an infinite sector of linear stability, and for nonlinearities

$$f(\sigma) = \tanh(100\sigma) \quad \text{and} \quad f(\sigma) = \operatorname{sign}(\sigma)$$

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[9] The corresponding rigorous mathematical statements were first formulated later for the general case of continuous systems by E.A. Barbashin and N.N. Krasovsky [49].

[10] Note that we can construct counterexamples to the discrete analogue of the Kalman conjecture for 2-dimensional systems [53]. Here, the difference in the phase space dimension needed for constructing counterexamples for continuous and discrete cases coincides with the difference in the dimension of the spaces of dynamical systems in which chaos occurs: 3 and 1, respectively.

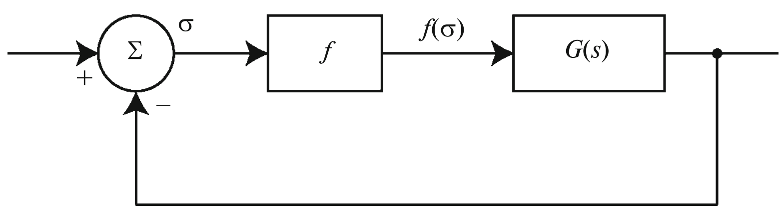

Fig. 1. $G(s)$ is the transfer function and $f(\sigma)$ is the scalar function.

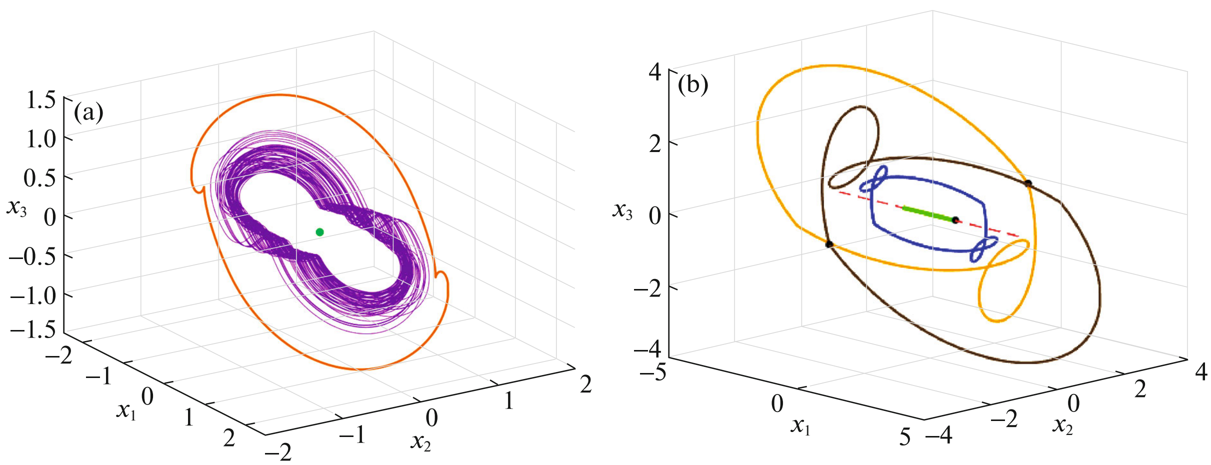

Fig. 2. (a) $f(\sigma) = \tanh(\sigma)$, $\beta = 0.1$ ($a_0 = 1.0004$, $a_1 = 0.408$, $a_2 = 2.08$, $a_3 = 0.4$): a hidden chaotic attractor and a hidden periodic attractor coexist with a stable state of equilibrium; (b) $f(\sigma) = \mathrm{sign}(\sigma)$, $\beta = 0.03$ ($a_0 = 0.98191881$, $a_1 = 0.121308$, $a_2 = 2.0254$, $a_3 = 0.12$): three coexisting periodic attractors: two symmetric hidden periodic attractors and one self-excited (relative to the resting segment) attractor.

oscillations are found in it [54, 55][11] (see Fig. 2, $z = (x_1, x_2, x_3, x_4)$). Similar counter-examples to the Kalman conjecture and hidden oscillations are also brought about by taking into account the dynamics of the servo drive $\dot{z} + z = x$ in the turbine control model (1.2) (for example, for $\alpha = 1.3$ and $\beta = 1.2$) when smoothing nonlinearity to approximate the solution of the discontinuous system in the Aizerman–Pyatnitsky definition [57]. Note that it is possible to determine the stability and search for hidden periodic oscillations for piecewise linear nonlinearities and systems of a small dimension by Andronov's point-transformation method, while the search for chaotic hidden oscillations in such systems requires the use of special analytical and numerical methods developed in the theory of hidden oscillations.

Aizerman, who was an active participant in Andronov's seminar in the post-war years, in 1949 formulated the problem of substantiating an assertion similar to the Kalman criterion, where only the nonlinearity itself, and not its derivative, was required to belong to the linear stability sector [58]. Counterexamples to Aizerman's problem exist in 2-dimensional space[12] [60] (Fig. 2) [39, 54].

Advancement of the conjectures of Aizerman and Kalman can also be explained by the fact that the method of harmonic balance and linearization, developed by that time in the papers of van der Pol [63], N.M. Krylov, and N.N. Bogolyubov [64] and widely used in practice [65, 66], showed the absence of oscillations in the conditions of these conjectures. Thus, the conditions of Vyshnegradsky for model (1.2) correspond to an infinite sector of linear stability (i.e., system (1.2) with $f(\sigma) = k\sigma$ is stable for all $k \in [0, +\infty)$), and therefore the harmonic balance method shows the absence of hidden oscillations. Currently, using the methods of the theory of differential inclusions, which were developed after Andronov's research, Vyshnegradsky's conditions can be strictly obtained from considering a special discontinuous

[11] Here the solution of the discontinuous system is understood in the terms of Filippov [56]; the function $\tanh(100\sigma)$ is used as a smooth approximation $\mathrm{sign}(\sigma)$.

[12] In the two-dimensional case, the departure of the trajectories is possible only to infinity, and in the three-dimensional case, the existence of limit periodic trajectories is possible [59].

Lyapunov function or frequency conditions of absolute stability [18]. However, in the general case, a rigorous analysis of the absence and occurrence of hidden oscillations to obtain the necessary and sufficient conditions for the global stability of control systems is a nontrivial problem [39, 41].

In the 1960s, Kalman, working on the justification of his criterion, came to strict sufficient conditions for the global stability of control systems (which guarantee the absence of hidden attractors[13]). Shortly before this, similar results were obtained by V.M. Popov and V.A. Yakubovich [17, 68]. These results were subsequently disseminated by A.Kh. Gelig and G.A. Leonov to control systems with discontinuous nonlinearities (differential inclusions) [16–18].

Note that for discontinuous systems with piecewise constant nonlinearity and hysteresis, the results of the analysis of periodic oscillations by the harmonic balance method can be refined in some cases by the Tsypkin hodograph method [69, 70] but counterexamples can also be constructed to its method in the general case (for example, in system (1.3), when $\beta = 0.03$, trajectories from n.h. $z = (\pm 0.6252, \pm 3.7324, 0.0, \pm 3.4754)$ are attracted to hidden periodic oscillations that are not detected from the analysis of the Tsypkin hodograph).

Later, in the development of the theory of hidden oscillations in [39, 41, 67, 71], an effective analytical-numerical method was proposed for constructing counterexamples to the Aizerman and Kalman conjectures for various classes of nonlinearities in which hidden periodic and chaotic attractors coexist with a single stable equilibrium state. The practical significance of revealing hidden oscillations for control systems and constructing counterexamples is related to the problems of determining the exact boundaries of global sustainability, analyzing the gap between the necessary and sufficient conditions for global sustainability, and also identifying the classes of control systems for which these conditions coincide.

We also note that for nonautonomous control systems and nonstationary linearization, even the conjecture of determining local stability as a first approximation is not true in the general case: in [72–75], various counterexamples of the Perron type were constructed and sufficient stability criteria were proposed. For example [76, 74], for the Lurie system (1.1) with a nonautonomous linear part

$$\begin{gathered} P(t) = \begin{pmatrix} -a & 0 \\ 0 & (\sin(\ln(t+1) + \cos(\ln(t+1) - 2a) \end{pmatrix}, \\ q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad r = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f(\sigma) = \sigma^2, \end{gathered} \tag{1.4}$$

for $1 < 2a < 1 + e^{-\pi}/2$ nonzero solutions of the first approximation system, they have negative Lyapunov exponents $-a$ or $-2a + 1$ (i.e., the zero solution is stable), and the solutions of a nonlinear system with nonzero initial data have one positive indicator $1 - 2a + e^{-\pi}/2 > 0$ and tend to infinity (i.e., the zero solution is unstable).

2. HIDDEN OSCILLATIONS AND STABILITY OF AIRCRAFT CONTROL SYSTEMS

In 1944, simultaneously with the works of Andronov–Mayer [12] and Lurie–Postnikov [48], the work of M.V. Keldysh [77] devoted to the study of flutter aircraft controls was published. In it Keldysh formulates the problem of the nonlinear analysis of flutter suppression models with a hydraulic friction damper with dry friction. These models are described by the Lurie system (1.1) with discontinuous nonlinearity. For an approximate analysis of the occurrence of oscillations and an assessment of the stability region of the rest segment, Keldysh used the method of harmonic balance and wrote that he did not give a rigorous mathematical proof of all the provisions there but drew a number of conclusions based on intuition. The practical significance at that time of Keldysh's work in the field of flutter was noted in 1946 when he was elected a full member of the USSR Academy of Sciences in the Department of Technical Sciences [14, p. 72]. Thanks to the work of Keldysh and his school at the Central Aerohydrodynamic Institute (TsAGI) in the 1930–1940s, the former Soviet Union managed to avoid numerous accidents that accompanied the development of aviation abroad (for example, it is known that in Germany in 1935–1943 there were more than 150 accidents and catastrophes of experimental aircraft from flutter).

Note that while in control theory the problem of completely suppressing oscillations is often posed, Keldysh in his work allowed, due to the design features of the regulator, the operation of the regulatory

[13]Regarding counterexamples with hidden attractors [41, 67] in 2011, R.E. Kalman wrote to the author of the article: "I was far too young and lacking technical mathematical knowledge to go more deeply into the matter."

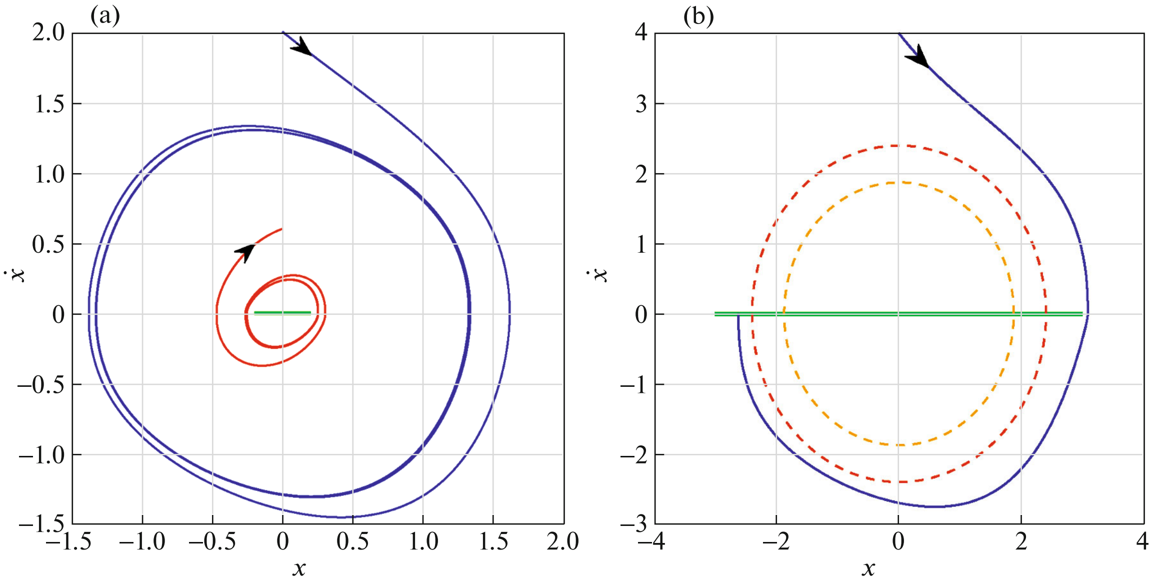

Fig. 3. A numerical analysis of the Keldysh model for $J = k = \kappa = 1$, when the Keldysh conditions $\mu < -\delta_{cr}$ for the existence of two limit cycles are satisfied: (a) at $\Phi = 0.2$, $\mu = -1.3967\delta_{cr} \approx -1.2987$, the external path is twisted to a stable limit cycle, and the internal path is wound off from an unstable limit cycle that limits the basin of attraction of the rest segment; and (b) at $\Phi = 3$, $\mu = -1.0076\delta_{cr}$, the limit cycles have disappeared (the harmonic balance method gives the limit cycles with amplitudes $a_{\pm} = \frac{3}{8}\left(\pi\mu \pm \sqrt{\pi^2\mu^2 - \frac{32}{3}\Phi}\right)$ shown by the dotted line) and we have the stiffness in the system's behavior.

system in a limited area of stability of the operating mode (this meant the possibility of the coexistence of undesirable limit regimes—hidden attractors). For example, for the two-dimensional model (1.1) with

$$P = \begin{pmatrix} 0 & J^{-1} \\ -k & -\mu J^{-1} \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad r = \begin{pmatrix} 0 \\ J^{-1} \end{pmatrix}, \quad f(\sigma) = (\Phi + \kappa\sigma^2)\,\mathrm{sign}(\sigma), \tag{2.1}$$

Keldysh, describing the behavior of the discontinuous model on the plane (based on the physical properties of dry friction) and applying the harmonic balance method to search for approximate periodic solutions, obtained the following statement: *when* $\mu < -\delta_{cr} = -(8/\pi)\sqrt{2\Phi\kappa/3}$, *the system has two periodic solutions and the region bounded by an internal closed solution is filled with phase-converging trajectories ending in one of the possible equilibrium positions; in the case of the reverse inequality, the system does not have periodic solutions and all movements tend to equilibrium positions*.

In the analysis of Keldysh, the outer limit cycle corresponds to stable flutter and, in the theory of hidden oscillations, it is a hidden attractor, since its basin of attraction does not touch the stationary set—the rest segment. Note that in the case of the close proximity of two limit cycles, numerical analysis may not detect a stable limit cycle, a hidden attractor, due to discretization and, skipping both cycles, show a tendency to the rest segment. Analysis by Keldysh to determine the boundary of the basin of attraction of the rest segment in the general case should be supplemented by the conditions of nonintersection of cycles with the rest segment [78, 79], which are not taken into account in the classical method of harmonic balance [65, 66]. Figure 3 [78–80] shows the qualitative behavior of trajectories ($z = (x, \dot{x})$) on the phase plane in the presence of two limit cycles and their "destruction" by the rest segment. The subsequent development of the theory of differential inclusions, the theory of hidden oscillations, and analytical and numerical methods of analysis not available to Keldysh at the time, have now allowed [78–80] conducting a rigorous analysis of the stability and the occurrence of hidden oscillations in the Keldysh models (including, for the 2-dimensional model (2.1), a strict estimate of the global stability region $\mu < -2\sqrt{\Phi\kappa}$ was established, which refines the Keldysh condition with $\mu < -\delta_{cr} \approx -2.08\sqrt{\Phi\kappa}$; and the 4-dimensional model can be reduced by choosing the parameters to model (1.3) with hidden oscillations).

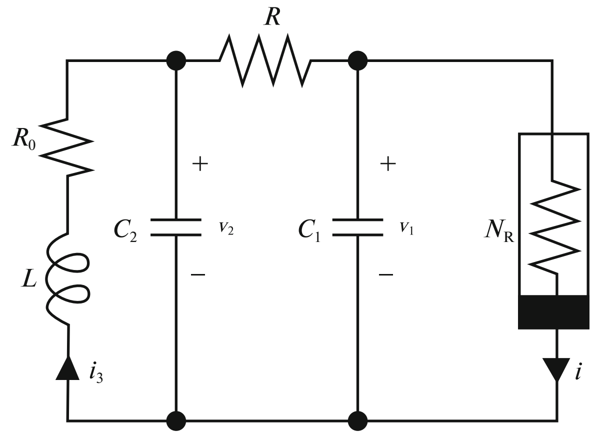

Fig. 4. Chua circuit is the electronic generator of chaotic oscillations.

The problems of the numerical analysis of the occurrence of hidden oscillations and the stability of control systems remain relevant in the design of modern aircraft [81, 82]. Thus, at the leading IEEE conference on control, Conference on Decision and Control (California, United States, 1997), a report was presented on the simulation of control systems and accidents of the 5th generation multipurpose fighter YF-22 Lockheed/Boeing due to the occurrence of aircraft oscillations during landing. It concludes that *"since stability in simulations does not imply stability of the physical control system* (*an example is the crash of the YF22*)*, stronger theoretical understanding is required"* [83]. Such a study is currently, in particular, the subject of the theory of hidden oscillations and the stability of control systems. The need for the validation and further improvement of the existing analytical and numerical methods and their software implementations used to design aircraft, was pointed out by TsAGI's director, S.L. Chernyshev, in a plenary report at the International Scientific Conference on Mechanics "VIII Polyakhov Readings" in 2018 [84]. The analysis of the stability and hidden oscillations in aircraft control systems is also considered in the work [85–89].

3. HIDDEN CHAOTIC OSCILLATIONS IN THE CHUA ELECTRONIC GENERATOR

In the 1980s L. Chua (Chua, University of Berkeley, United States) proposed the first electronic oscillation control circuit that generates a chaotic signal [90] (Fig. 4).

The mathematical model of the circuit can be written in the form of a three-dimensional Lurie system (1.1) with piecewise-linear nonlinearity saturation (which is the typical nonlinearity for control problems), where

$$P = \begin{pmatrix} -\alpha(m_1+1) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{pmatrix}, \quad q = \begin{pmatrix} -\alpha \\ 0 \\ 0 \end{pmatrix}, \quad r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$
$$f(\sigma) = (m_0 - m_1)\mathrm{sat}(\sigma) = \frac{1}{2}(m_0 - m_1)(|\sigma+1| - |\sigma-1|). \tag{3.1}$$

Here $z = (\mathrm{v}_1, \mathrm{v}_2, \mathrm{Ri}_3)$, $\alpha = (\mathrm{RC}_1)^{-1}$, $\beta = \mathrm{RL}^{-1}$, $\gamma = \mathrm{R}_0\mathrm{L}^{-1}$, $\mathrm{RC}_2 = 1$, and $\mathrm{i}(\mathrm{v}_1) = (m_1 c^{\mathrm{T}} z + f(c^{\mathrm{T}} z))\mathrm{R}^{-1}$ is the current–voltage characteristic of a nonlinear resistor $\mathrm{N_R}$.

In physical experiments, where the electronic circuit was triggered by the inclusion of a nonlinear resistor at zero initial data (initial voltage across the capacitors and current through the coil) corresponding to an unstable zero equilibrium state,[14] only self-excited attractors could be observed. Hundreds of such var-

[14]In physical experiments, the system's state leaves the unstable stationary mode due to external perturbations (an example is the impossibility of observing the upper position of the physical pendulum without additional stabilization). When analyzing the corresponding mathematical dynamic models, it is necessary to take into account that the unstable equilibrium states themselves do not fall into the basin of attraction of self-excited attractors. Therefore, in numerical modeling, the system state can remain in an unstable equilibrium state, and to study the dynamics in its vicinity, one has to choose the initial data that are different from the equilibrium state itself.

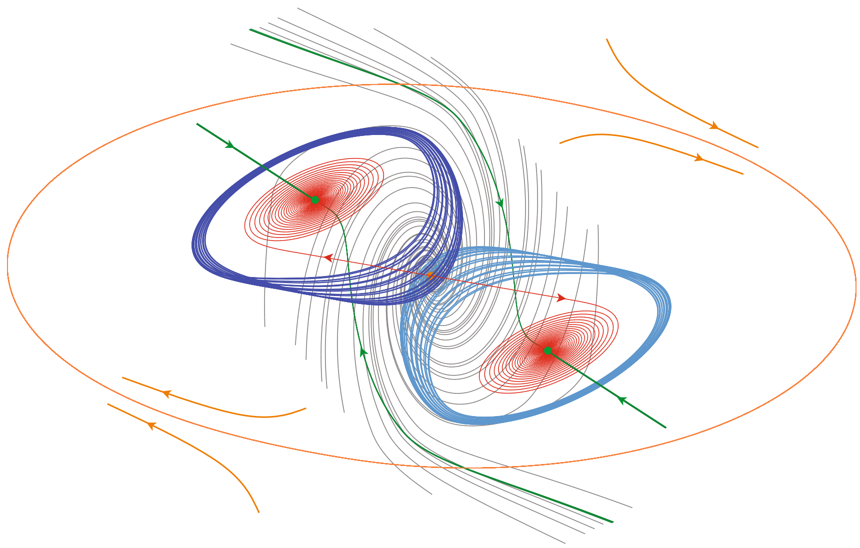

Fig. 5. Five coexisting attractors in the three-dimensional phase space of the Chua chain for $\alpha = 8.4$, $\beta = 12$, $\gamma = -0.005$, $m_0 = -1.2$, and $m_1 = -0.05$: two stable equilibrium states, a hidden periodic attractor, and two symmetric hidden chaotic attractors. The basins of attraction of hidden attractors do not touch the unstable zero equilibrium state, from the vicinity of which the trajectories are attracted to two symmetric stable equilibrium states.

ious self-excited attractors have been discovered in the Chua chain [91]. Chua himself hypothesized that only self-excited chaotic attractors can be found in his chain [90].

In 2009, as part of the development of the theory of hidden oscillations, the idea of constructing a hidden Chua chaotic attractor was proposed for the first time [40, 92, 93] and in 2011 the first hidden chaotic attractor in the classical Chua chain was discovered[15] [94, 95]. This hidden attractor has a very thin basin of attraction and coexists with a stable zero state of equilibrium, thus turning out to be "hidden" from the standard physical experiments.

After the discovery of the hidden attractor, Chua set the task of determining the maximum possible number of coexisting attractors in the chain, including hidden attractors. Currently, five coexisting attractors have been found in the Chua chain for the first time, three of which are hidden [96, 97] (Fig. 5).[16]

Note that data transmission systems using auto-tuning and synchronization of chaotic signals (for example, Chua circuits) may not work correctly in the presence of hidden attractors, which was shown in the publications [98, 99].

4. HIDDEN OSCILLATIONS AND STABILITY IN ELECTROMECHANICAL SYSTEMS

One of the striking examples of hidden oscillations in electromechanical systems is the effect of the rotor speed jamming when starting an electric machine mounted on an elastic base. In this system, for some parameters, a significant part of the energy can go into the oscillations of the base. In this case, the rotor speed is significantly less than the frequency during normal operation. This effect was first noticed in 1902 by the famous German scientist A. Sommerfeld [100], and then analytically described in the work of I.I. Blekhman [101]. In the theory of hidden oscillations [102, 103], it was shown that this effect corre-

[15] In 2009, the plenary report [92] needed an example of applying the harmonic balance method to the Chua electronic circuit. For the parameters that I randomly selected, the attractor found in the Chua chain turned out to be weakly connected to the equilibrium states. A small additional control made it possible to disconnect the attractor found from the equilibrium states. In 2010, this example was finalized and presented at the IFAC conference "Periodic Control Systems" [40], and also published in the journal "Proceedings of the Academy of Sciences" [93]. Then, using the hidden attractor in the modified circuit, using the parameter continuation method, we managed to get rid of the additional control and get the hidden attractor in the classical Chua chain in 2011.

[16] This phase portrait with three Chua hidden attractors was selected for the cover of the International Journal of Bifurcation and Chaos in Applied Sciences and Engineering: https://www.worldscientific.com/na101/home/literatum/publisher/wspc/journals/content/ijbc/2017/ijbc.27.issue-12/ijbc.27.issue-12/20171218/ijbc.27.issue-12.cover.jpg. The initial data for visualizing hidden attractors in system (3.1): $z = (9.2942, 5.5013, -31.4277)$, $z = \pm(1.5179, 0.2875, -1.7414)$.

sponds to the coexistence of two hidden attractors in the mathematical model that describes the dynamics of the rotor speed, speed, and base deflection. Moreover, in physical experiments, when the motor starts naturally from the zero initial states (which do not correspond to the stationary mode here), only one of the attractors is observed, which corresponds to an undesirable operating mode. Determining the basin of attraction of the second (hidden) attractor corresponding to the desired operating mode, and controlling the state of the system for transferring it into the basin of attraction of tis hidden attractor is an urgent modern engineering problem [104, 105].

A similar Sommerfeld effect was recently described in the papers of the science group from the Technical University of Eindhoven (the Netherlands) on drilling rigs, where the electric energy was spent on exciting torsional oscillations of the drill, which ultimately led to costly breakdowns of the drill [106]. In the theory of hidden oscillations, it was shown in [107–109] that such undesirable regimes of drilling rigs correspond to hidden attractors, into the basin of attraction of which the system goes over with a sharp change in the friction force during drilling. Here, a long drill that allows bending-torsional oscillations plays the role of an elastic base in the Sommerfeld effect.

5. HIDDEN CHAOTIC ATTRACTORS IN PHYSICAL MODELS

The development of the theory of hidden oscillations made it possible to discover previously unknown hidden attractors in various known physical models. Many physical systems have built-in regulation mechanisms that are realized in the form of attraction basins—regions of the initial states of the system, for each of which all the corresponding regimes are attracted to a single attractor (stable equilibrium, periodic, quasi-periodic, or chaotic attractor). The interest in chaotic attractors was largely due to attempts to describe turbulence in models of fluid dynamics. One of the striking results in this direction is the proof of O.A. Ladyzhenskaya of the existence of a finite-dimensional global attractor for the two-dimensional Navier–Stokes model [110]. From the point of view of control theory, in this case, the internal mechanisms of regulating the system make all trajectories attracted to the global attractor. The first book devoted to analytical methods for localizing global attractors in finite-dimensional dynamical systems was the book by G. Leonov and V. Reitmann, published in German in 1987 [111].

In the general case, a global attractor can contain several coexisting local attractors (self-excited or hidden attractors). For example, in the famous Lorentz model [36], depending on the parameters, a global attractor may contain one or three local attractors: a self-excited chaotic attractor can coexist with two trivial attractors (stable equilibrium states). In the general case, the problem of determining all the coexisting local attractors is important [112]. If self-excited attractors can be easily visualized in numerical experiments by trajectories with the initial data from a neighborhood of unstable equilibrium states (the first chaotic Lorentz attractor [36] was discovered), then the search for hidden attractors requires special methods. The development and application of such methods in the theory of hidden oscillations [39, 43] made it possible for the first time to detect hidden attractors in the well-known physical models of Rabinovich, Glukhovsky–Dolzhansky, Rabinovich–Fabrikant, etc. [42, 113–119].

The problems of controlling the system's state and transferring the state to the basin of attraction of the given attractor are studied in cybernetic physics and in chaos control theory [120, 121]. For the Rössler model [122] in the form of the Lurie system (1.1) with

$$P = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & \alpha & -\beta \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ 0 \\ -a \end{pmatrix}, \quad r = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad f(\sigma) = \sigma^2, \tag{5.1}$$

using delay feedback stabilization (the Pyragas method),

$$qf(r^{\mathrm{T}}z(t)) \to qf(r^{\mathrm{T}}z(t)) + 0.1rr^{\mathrm{T}}(z(t-\tau) - z(t)), \tag{5.2}$$

it is possible [123, 124] (see Fig. 6) to calculate and stabilize an unstable periodic trajectory with period T embedded in a chaotic self-excited attractor (for discrete systems, similar ideas were developed, for example, in [125]). Also in [126–129] some results are presented in these directions. The problems of identifying complex oscillatory regimes and hidden attractors arise in various models of decentralized and network control [113, 130–133]. Note that stabilization of unstable periodic trajectories made it possible to obtain lower estimates of the Lyapunov dimension and topological entropy in the study of Eden's conjecture and its various refinements [114, 124, 134–136], including those important for the problems of estimating the intensity of information exchange necessary and sufficient for monitoring/stabilizing systems with nonlinear dynamics through communication channels with a limited bandwidth [137].

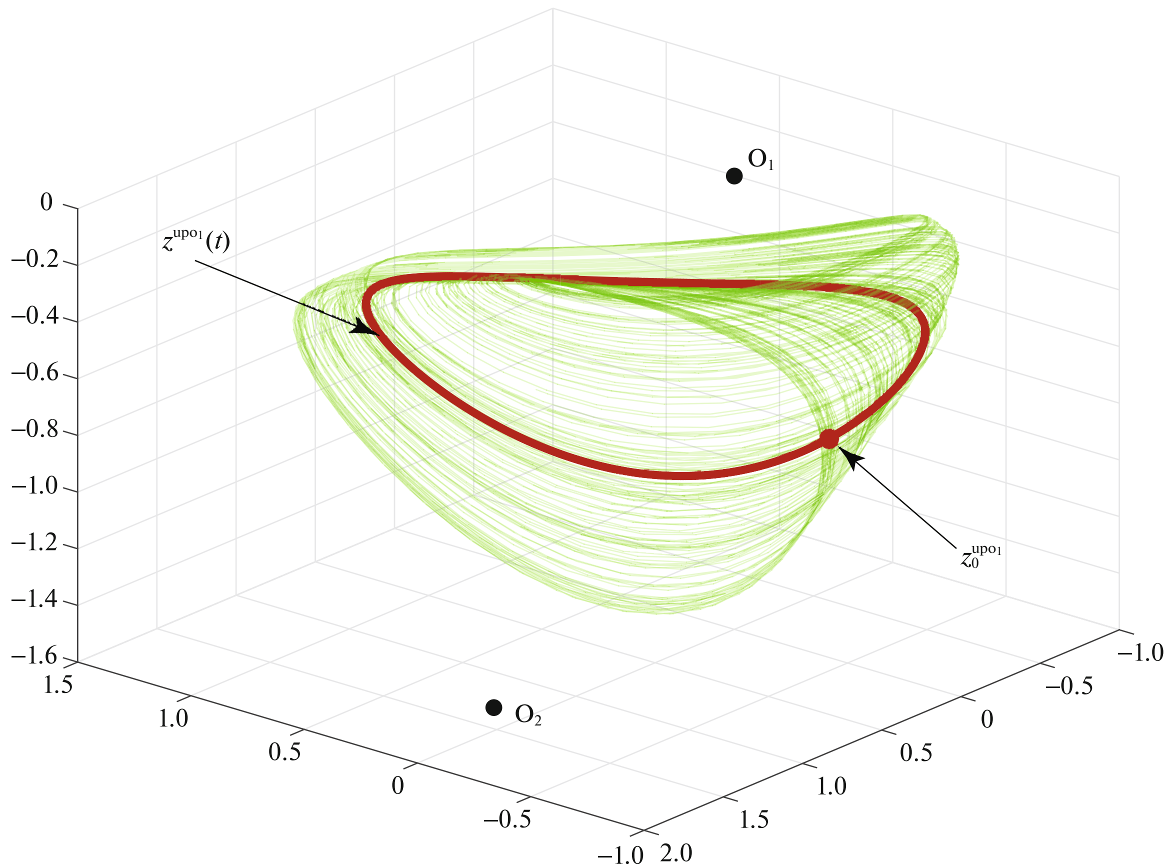

Fig. 6. Chaos control in the Rössler model with $\alpha = 0.386$ and $\beta = 0.2$. Stabilization and calculation in the model (5.1) of periodic oscillation $z^{upo_1}(t) = z^{upo_1}(t, z_0^{upo_1})$ with initial data $z_0^{upo_1} \approx (0.9189, 0.7152, 0.5388)$ and period $\tau \approx 6.3245$ built into a self-excited attractor using the Pyragas method (stabilization by feedback with delay (5.2)). The finite-time Lyapunov dimension on the trajectory $z^{upo_1}(t, z_0^{upo_1})$ is equal to 2.3204 and on the chaotic pseudo-trajectory $\tilde{z}(t, z_0^{upo_1})$ obtained by standard numerical integration of model (5.1) (without replacement (5.2)) from point $z_0^{upo_1}$ is 2.2641; similar values for topological entropy: 0.1365 and 0.1035, respectively.

We note that the transition of the state of the system from an unstable state of equilibrium with an arbitrarily small perturbation of the state of the system to a self-excited attractor (spontaneous excitation of oscillations) corresponds to a *soft mode of excitation* of oscillations, and the transition from a stable state of equilibrium to a hidden attractor for some initial deviation of the system state from the state of equilibrium corresponds to a hard mode of excitation of oscillations [7].

6. DIFFICULTIES OF NUMERICAL SEARCH FOR HIDDEN OSCILLATIONS AND STABILITY ANALYSIS

The development of the concept of hidden and self-excited oscillations began in the scientific school of G.A. Leonov in 2008 as part of a study of the construction of nested limit cycles in Hilbert's 16th problem. In it, internal nested cycles are not connected with equilibrium states and are hidden oscillations.

After the works of Andronov (second half of the 20th century), the bifurcation theory [138] was actively developed to study the scenarios of the birth of oscillations and structural changes in the phase portraits of dynamical systems [138], which allowed significant progress in the study of scenarios of stability loss and attractos birth; however, many tasks concerning these well-known problems of global analysis, including Hilbert's 16th problem, remain unresolved and require further development of the theory and numerical methods.

In the middle of the 20th century N.N. Bautin, developing the ideas of A.A. Andronov and L.S. Pontryagin on the robustness and structural stability of systems [7, 139] and studying the dangerous and safe boundaries of linear stability, proposed a method for the theoretical construction of systems with nested limit cycles [140]. However, Bautin's method, based on successive relatively small perturbations, allowed one to construct only nested cycles with a very small amplitude, which were not visible in the numerical experiments [39, 141, 142]. In 2008, in the defense by the author of this review of his doctoral dissertation [74] at the University of Jyväskylä (Finland) and at the plenary report [143] of the international seminar "Stability and oscillations of nonlinear control systems" named after E.S. Pyatnitsky at the Institute of Control Sciences of the Russian Academy of Sciences (Moscow), the difficulties of computing such small limit cycles were discussed: they are hidden from the standard computational methods. The difficulties of identifying systems with limit cycles and their calculation are clearly illustrated by the experiment of A.N. Kolmogorov, a description of which is given in the book by V.I. Arnold [144]: "*To estimate the number of limit cycles of quadratic vector fields on the plane, he distributed several hundreds of such fields (with randomly selected coefficients of polynomials of the second degree) to several hundred students of the Mechanics-Mathematical Faculty of Moscow State University as a mathematical practical course. Each student had to find the number of limit cycles of his field. The result of this experiment was completely unexpected: not a single field had a single limit cycle. With a small change in the field coefficients, the limit cycle is preserved. Therefore, systems with one, two, three (and even, as it became known later, four) limit cycles form open sets in the coefficient space, so that the probabilities of getting into them with a random choice of polynomial coefficients are positive.*"

The creation of special analytical and numerical procedures in the theory of hidden oscillations for two-dimensional polynomial systems made it possible in [39, 145] to solve the Hilbert–Kolmogorov problem: construct a quadratic system and visualize four limit cycles in it (this is the largest known number of coexisting cycles for two-dimensional quadratic systems). Such examples with four limit cycles are counterexamples to the statement in the famous work of I.G. Petrovsky[17] and E.M. Landis [146] on the possibility of the existence of only three limit cycles in two-dimensional polynomial quadratic systems. The main difficulties in the numerical analysis of nested limit cycles are related to the flattening of the trajectories, the bifurcations of the mergering of limit cycles, and the birth of semistable cycles.

The difficulties related to the presence of hidden oscillations also arise in the engineering design of various control systems, for example, classical phase synchronization control systems (phase-locked loop, PLL). PLL systems were first used at the beginning of the 20th century to adjust the phase/frequency of the local generator's electrical signal to the input signal (master-slave synchronization).

The frequency difference ranges for which the generators are tuned, which corresponds to the stability of the PLL mathematical models in the signal phase space, with the required transient properties, are important engineering characteristics of the PLL operation. The possibility of the effective nonlinear analysis of mathematical models of the simplest PLL systems was first shown in the work of 1933 by F. Tricomi [147], in which the qualitative behavior of two-dimensional pendulum-type systems (describing the dynamics of PLL and electric machine models under load) by the phase plane method was studied. These ideas were then developed by Andronov and his followers (see [148–152] and others). In 1956 M.V. Kapranov published a work in which he supposed that, similarly to the conjectures of Vyshnegradsky and Aizerman, the conditions for global stability of the two-dimensional model of the PLL in the Lurie form (1.1) with

$$P = \begin{pmatrix} \dfrac{-1}{\tau_1 + \tau_2} & 0 \\ \dfrac{-K_{\text{vco}}\tau_2}{\tau_1 + \tau_2} & 0 \end{pmatrix}, \quad q = \begin{pmatrix} \dfrac{1}{\tau_1 + \tau_2} \\ \dfrac{-K_{\text{vco}}\tau_1}{\tau_1 + \tau_2} \end{pmatrix}, \quad r = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad f(\sigma) = \sin\sigma - \frac{\omega_e^{\text{free}}}{K_{\text{vco}}} \tag{6.1}$$

are determined by the conditions for the absence of self-excited oscillations [153] (the trivial boundary of global stability related to the local bifurcations of the loss of stability). The model reviewed by Kapranov describes, for example, the dynamics of the generator control in the modern two-phase phase-locked loop model [154] (see Fig. 7) with respect to the shifted state of the filter and phase distortion $z(t) = (x(t) - (\tau_1\omega_e^{\text{free}})/K_{\text{vco}},\ \theta_{\text{ref}}(t) - \theta_{\text{vco}}(t))$ at the constant frequency of the input signal $\dot{\theta}_{\text{ref}}(t) \equiv \omega_{\text{ref}}$, $\omega_e^{\text{free}} = \omega_{\text{ref}} - \omega_{\text{vco}}^{\text{free}}$, and the transfer function of the filter $H(s) = (1 + \tau_1 s)/(1 + (\tau_1 + \tau_2)s)$.

Later, Kapranov's conjecture was refuted and examples were constructed where the condition for the loss of global stability was determined by the nonlocal birth of a hidden oscillation from the flattening of the trajectories [39, 149] (the hidden boundary of global stability related to nonlocal bifurcations). We

[17]Academician of the Academy of Sciences of the USSR (since 1946), rector of Moscow State University (1951–1973).

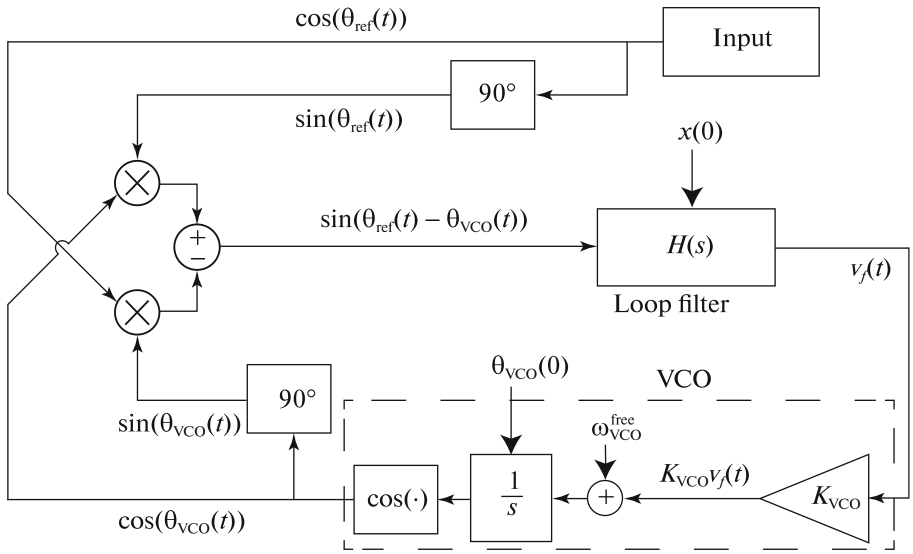

Fig. 7. Phase synchronization control system in a two-phase PLL model.

note that the harmonic balance method does not allow one to determine the exact stability boundary here, predicting the existence of a periodic solution within the global stability region, and gives only a conservative estimate [155]. Thus, for model (6.1) with $\tau_1 = 0.0185$, $\tau_2 = 0.0448$, and $K_{vco} = 250$, the estimate of the capture range (corresponding to the attraction of all trajectories to the stationary set) obtained by the method of constructing the Lyapunov periodic function, $\omega_e^{free} \in [0,126.3)$, and by the method of analyzing the birth of a hidden oscillation: $\omega_e^{free} \in [0,178.1)$. In this case, a numerical estimate of the range of fast capture without slipping cycles (F. Gardner's problem): $\omega_e^{free} \in [0,101.5)$.

In [156–159], special examples of modeling PLL systems in SPICE design packages were implemented.[18] These examples demonstrate the difficulties of the reliable numerical analysis of PLL in the presence of hidden oscillations (see Fig. 8), and can also be used to validate engineering packages of the PLL design. This problem was set by the famous American engineer S. Goldman (Texas Instruments, United States) [160].

The difficulties in the numerical analysis of hidden oscillations have shown the need to develop and apply analytical criteria for the stability of control systems and the absence of hidden oscillations. For phase-locked loop systems, such mathematical methods of nonlinear analysis were developed by G.A. Leonov[19] and his students [161–165]. In these works, the difficulties related to the analysis of multidimensional PLL models were overcome and the classical results of control theory were generalized to systems with periodic continuous and discontinuous nonlinearities in a cylindrical phase space (similar models and difficulties also arise when studying angular orientation control systems [166]). For example, the criteria for the global stability of Barbashin–Krasovsky and La Salle require unboundedness of the Lyapunov functions in all coordinates, while in a cylindrical phase space, we consider the phase change within the period and periodic Lyapunov functions [167]. These fundamental mathematical differences are often not taken into account in practice, for example, [168], which can lead to incorrect conclusions.

Currently, various PLL schemes are implemented in the form of compact electronic circuits and software algorithms, and are widely used in modern telecommunication equipment, distributed computer architectures, global navigation systems (GPS, GLONASS), energy and electrical networks, as well as in many other applications [169–172]. It is sufficient to note that the frequency synthesizers used in every modern computer have in their design such phase synchronization control systems. In recent years, due to the significant increase in the frequencies of oscillators and requirements for transient conditions, as

[18]Designing PLL systems for the analysis of stability and oscillations, various software simulators of electronic systems, Simulation Program with Integrated Circuit Emphasis (SPICE), are used by engineers, giving the illusion of virtual reality—observing real physical processes.

[19]In 1986, G.A. Leonov (as part of a team of researchers led by V.V. Shahgildyan) was awarded the USSR State Prize for these works.

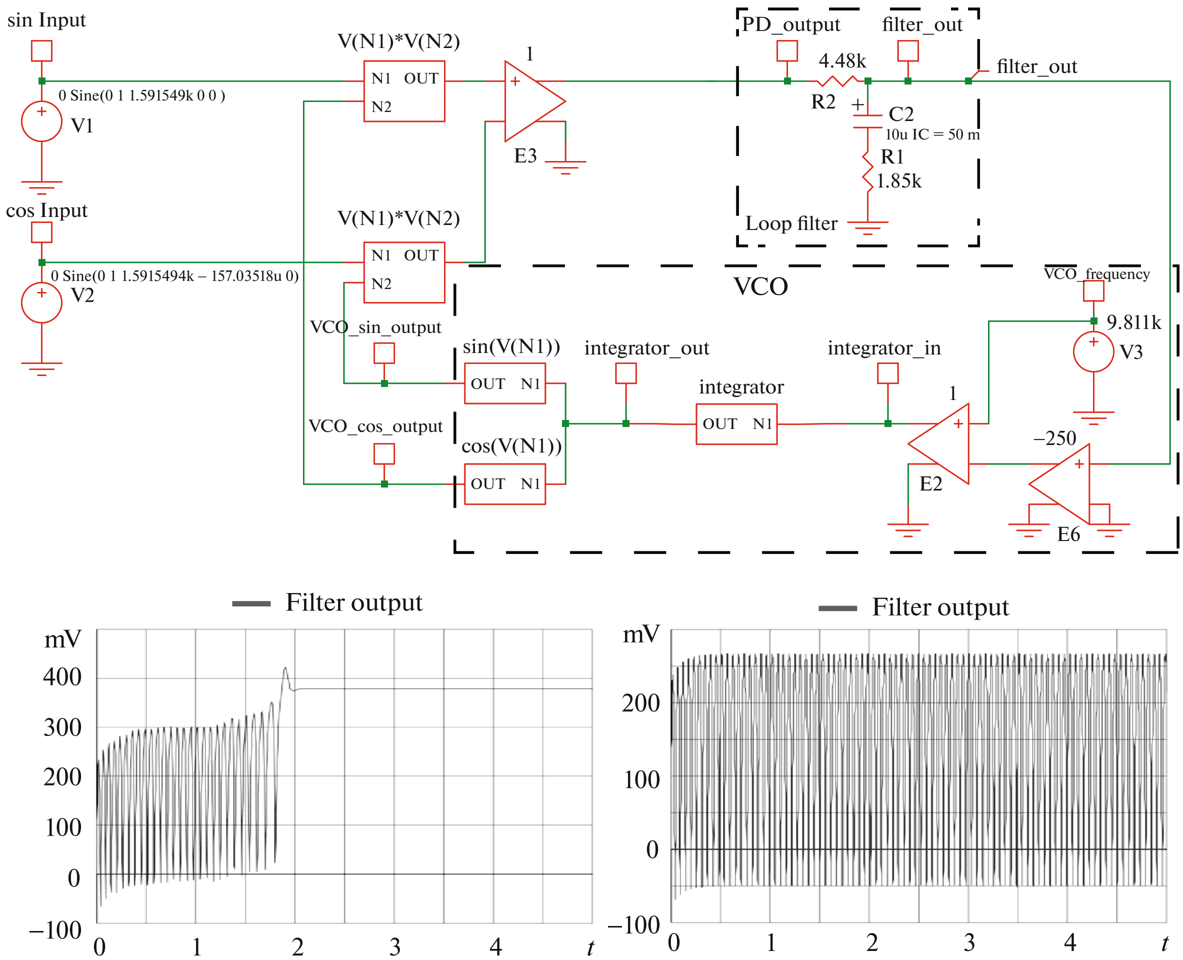

Fig. 8. Simulation example in SIMetrix (SPICE) of a two-phase PLL system: $\tau_1 = R1C2 = 1.85\times10^{-2}$, $\tau_2 = R2C2 = 4.48\times10^{-2}$, $\omega_{vco}^{free} = 2\pi 9.811\times10^3$, $\omega_{ref} = 2\pi 1.5915494\times10^3$, $K_{vco} = 250$, $x(0) = 0.5$, $\theta_{vco}(0) = 0$ with a hidden attractor, limiting the area of attraction of the operating mode. With the sampling step set by default to 10 m, the simulation shows the achievement of a synchronous operation mode—the control signal $v_f(t)$ is stabilized (lower left plot); for the reduced sampling step 1 m with the same initial data, the simulation shows the beat mode and lack of synchronization (lower right plot).

well as the development of digital data transmission systems based on PLL, the further development of new mathematical methods for the analysis and synthesis of mathematical models of synchronizing phase control systems was required and was realized [156, 173–183].

CONCLUSIONS

This review is devoted to the theory of hidden oscillations and its relevant applications, such as the Andronov-Vyshnegradsky problem on the nonlinear analysis of the centrifugal governor of steam engine, the Keldysh problem on the nonlinear analysis of flutter suppression systems, the Aiserman and Kalman conjectures on the absolute stability of control systems in the form of Lurie, Chua circuits, the Sommerfeld effect of frequency jamming, the Hilbert–Kolmogorov problem, phase-locked loops and the Kapranov conjecture, and detection of hidden oscillations in dynamic models of drilling rigs and aircraft control systems.

The rapid development of modern computer technology has made mathematical modeling and numerical analysis of dynamics readily available and has led to the creation of a large number of specialized modeling packages. However, it turned out that the capabilities of such packages and standard numerical methods are often limited, and their application can significantly distort the characteristics of the real dynamics and lead to incorrect qualitative conclusions on the stability and oscillatory conditions.

To conduct reliable mathematical modeling of technical systems, it is important to pay special attention to the rigorous derivation of the mathematical models used and the consideration of the limits of their applicability, the development of effective analytical and numerical methods for studying dynamics, taking into account the possibilities and limitations of the existing analytical methods for studying the stability and the occurrence of oscillations. The self-excitation of oscillations can be effectively detected analytically and numerically. The analysis of hidden oscillations (their absence or presence and location) requires the development of special analytical and numerical methods and is key to determining the exact boundaries of global stability, analyzing the gap between the necessary and sufficient conditions for global stability and their convergence, as well as identifying the classes of control systems for which these conditions match. In practice, the transition of the state of the control system to a hidden attractor caused by external perturbations leads to undesirable operating regimes and often turns out to be the cause of accidents and catastrophes.

The theory of hidden oscillations is a modern stage in the development of Andronov's theory of oscillations. It is in demand in many theoretical and relevant engineering problems, in which hidden attractors (their absence or presence and location) play an important role, and, in particular, allows us to solve the stability problems of control systems. In recent years, the theory of hidden oscillations has attracted a great deal of attention from the world's scientific community. In 2016, the first publications on this subject were included in 1% of the most cited articles of the Web of Science base and became the most cited articles in journals [39] and a translated version of the journal [41]. One of the prestigious magazines, Physics Reports (a review Section of Physics Letters) published a review article on hidden oscillations, prepared by specialists from Russia together with their colleagues from Poland, Iran, and India [113]. A large number of scientists in different countries are engaged in the tasks of identifying hidden attractors and determining the boundaries of the stability of control systems.

Obtaining the results presented in the review would be impossible without the training and participation of young specialists from the Department of Applied Cybernetics of St. Petersburg State University, which has been repeatedly recognized as the Leading Scientific School of the Russian Federation.[20] In recent years, more than ten dissertations on the theory of hidden oscillations were defended at the department under the direction of Leonov and the author of this review, and many of their students continue their work at the department: http://apcyb.spbu.ru/.

The material of this review was discussed at a meeting of the Bureau of the Department of Energy, Engineering, Mechanics and Control Processes of the Russian Academy of Sciences (March 3, 2020);[21] and at seminars of Acad. V.M. Fomin (ITAM SB RAS, October 23, 2019), Acad. F.L. Chernousko (IPMech RAS, April 18, 2019), President of International Federation of Automatic Control (IFAC) F. Allgöwer (Institute for System theory and Automatic Control, Stuttgart, February 7, 2019), and Acad. N.F. Morozov (IPMash RAS, May 14, 2018)—the author thanks all the participants for their interest, discussions, and valuable comments. Also, the materials of this review on the theory of hidden oscillations and stability of control systems were presented and discussed in review and plenary reports at a number of leading Russian and international conferences conferences, including the conferences of the International Federation of Automatic Control (IFAC) [45–47, 184–188].

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