ISSN 1064-2307, Journal of Computer and Systems Sciences International, 2017, Vol. 56, No. 1, pp. 1–18. © Pleiades Publishing, Ltd., 2017.
Original Russian Text © N.E. Zubov, E.Yu. Zybin, E.A. Mikrin, M.Sh. Misrikhanov, V.N. Ryabchenko, 2017, published in Izvestiya Akademii Nauk, Teoriya i Sistemy Upravleniya, 2017, No. 1, pp. 3–20.

SYSTEMS THEORY
AND GENERAL CONTROL THEORY

General Analytical Forms for the Solution of the Sylvester and Lyapunov Equations for Continuous and Discrete Dynamic Systems

N. E. Zubov*, E. Yu. Zybin, E. A. Mikrin, M. Sh. Misrikhanov, and V. N. Ryabchenko

Bauman State Technical University, Moscow, 105005 Russia

Korolev Rocket and Space Corporation Energia, Korolev, Moscow oblast, 141070 Russia

**e-mail: nik.zubov@gmail.com*

Received June 23, 2015; in final form, April 14, 2016

Abstract—An approach to forming analytical solutions of the discrete and continuous Sylvester and Lyapunov linear algebraic matrix equations is described. The approach is based on reducing the square matrix to the Jordan normal form. Examples, algorithms, and implementations in Matlab are presented.

DOI: 10.1134/S1064230717010130

0. INTRODUCTION

The Sylvester and Lyapunov linear algebraic matrix equations are key equations in the theory of controlled dynamic systems. The Lyapunov equation is used to analyze the stability, controllability, and observability of dynamic systems. The Sylvester equation is widely used for the stabilization of dynamic systems.

The majority of methods for solving these equations are numerical. The most practically important among them are the methods based on the orthogonal transformations of the original matrices because such methods are numerically stable. Presently, there are two such algorithms for solving the Sylvester and Lyapunov matrix equations based on reducing their matrices to the real Schur or Hessenberg form—these are the Bartels–Stuart (BS) algorithm [1] and Golub–Nash–Van Loan (GNL) algorithm [2].

In practical applications of system theory, numerical methods are often insufficient. For example, the methods mentioned above make it possible to obtain a unique solution only if the equation is uniquely solvable (consistent). They do not take into consideration various insolvability conditions and the cases when equations have multiple solutions, which is needed for solving sets of equations arising in systems theory.

In this paper, we develop a method for the analytical solution of the Sylvester and Lyapunov equations based on normal forms of numerical matrices. To obtain an analytical form of the sets of solutions (and conditions for their existence), we reduce the original matrices to the Jordan normal form by analogy with [3].

It is well known [3–5] that any square $n \times n$ matrix A over the number field $\Re$ ($\Re = \mathbb{R}$ or $\Re = \mathbb{C}$) can be represented by the decomposition

$$A_{n\times n} = R_{n\times n} J_{n\times n} L_{n\times n}, \tag{0.1}$$

where J is the Jordan normal form or just the Jordan form of the matrix A, R is the invertible matrix of the right eigenvectors

$$R = [r_1 \ \cdots \ r_n],$$

and L is the inverse of the matrix of right eigenvectors (in a special case, this is the matrix of left eigenvectors)

$$L = (R)^{-1} = \begin{bmatrix} l_1 \\ \vdots \\ l_n \end{bmatrix}.$$

The Jordan form of a matrix is a quasi-diagonal (block diagonal) matrix

$$J = \left[\begin{array}{c:c:c} J_1 & 0 & 0 \\ \hdashline 0 & \ddots & 0 \\ \hdashline 0 & 0 & J_u \end{array}\right] = \operatorname{diag}(J_1, \ldots, J_u) = \operatorname{diag}\left(J_k\big|_{k=\overline{1,u}}\right),$$

where u is the number of elementary divisors of the matrix.

The diagonal elements J_k are called Jordan blocks; they have a special structure with the eigenvalues on the main (principal) diagonal and ones on the superdiagonal (or subdiagonal). Depending on the location of the ones, the upper

$$\widehat{J}_k = \begin{bmatrix} \lambda_k & 1 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \lambda_k \end{bmatrix}$$

and the lower

$$\breve{J}_k = \begin{bmatrix} \lambda_k & 0 & 0 & 0 \\ 1 & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & \lambda_k \end{bmatrix}$$

Jordan blocks are distinguished. Up to a permutation of the Jordan blocks, any matrix can be reduced to a unique Jordan normal form using similarity transformations.

To represent the elementwise structure of the Jordan form of a matrix in the general form, we define the *generalized notation of the Jordan form* of a matrix, namely, the upper form

$$\widehat{J} = \begin{bmatrix} \lambda_1 & \widehat{J}_{1,2} & 0 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & 0 & 0 & 0 \\ 0 & 0 & \lambda_i & \widehat{J}_{i,i+1} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{i+1} & \ddots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \widehat{J}_{n-1,n} \\ 0 & 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix} \tag{0.2}$$

and the lower form

$$\breve{J} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ \breve{J}_{2,1} & \ddots & 0 & 0 & 0 & 0 \\ 0 & \ddots & \lambda_i & 0 & 0 & 0 \\ 0 & 0 & \breve{J}_{i+1,i} & \lambda_{i+1} & 0 & 0 \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \breve{J}_{n,n-1} & \lambda_n \end{bmatrix}, \tag{0.3}$$

where the superdiagonal $\widehat{J}_{i,i+1}$ and subdiagonal $\breve{J}_{i+1,i}$ elements are ones or zeros, depending on the size of the Jordan blocks corresponding to the ith eigenvalue; we have the following obvious equalities:

$$\lambda_{i+1} = \lambda_i\big|_{\widehat{J}_{i,i+1}=1} \text{ and } \lambda_{i+1} = \lambda_i\big|_{\breve{J}_{i+1,i}=1}.$$

Based on these facts, we formulate and prove theorems on the analytical solution of the Sylvester and Lyapunov equations.

1. CONTINUOUS SYLVESTER AND LYAPUNOV EQUATIONS

Theorem 1. The analytical solution of the continuous Sylvester equation

$$A_{n\times n}X_{n\times m} \pm X_{n\times m}B_{m\times m} = C_{n\times m} \tag{1.1}$$

is given by the formula

$$X = R^A Y L^B. \tag{1.2}$$

Here $R^A = (L^A)^{-1}$ and $R^B = (L^B)^{-1}$ are the matrices of the right eigenvectors, and the elements $y_{i,j}$ are determined by the formulas

$$y_{i,j} = \frac{l_i^A C r_j^B - \breve{J}^A_{i,i-1} y_{i-1,j} \mp \hat{J}^B_{j-1,j} y_{i,j-1}}{\lambda_i^A \pm \lambda_j^B} \text{ if } \lambda_i^A \pm \lambda_j^B \neq 0, \tag{1.3}$$

$$y_{i,j} = \mu_{i,j} \in \Re \text{ if } \lambda_i^A \pm \lambda_j^B = 0 \text{ and } l_i^A C r_j^B - \breve{J}^A_{i,i-1} y_{i-1,j} \mp \hat{J}^B_{j-1,j} y_{i,j-1} = 0, \tag{1.4}$$

$$Y = \varnothing \text{ if } \lambda_i^A \pm \lambda_j^B = 0 \text{ and } l_i^A C r_j^B - \breve{J}^A_{i,i-1} y_{i-1,j} \mp \hat{J}^B_{j-1,j} y_{i,j-1} \neq 0, \tag{1.5}$$

where $i = \overline{1,n}$ and $j = \overline{1,m}$, $\mu_{i,j}$ are arbitrary parameters, $\breve{J}^A$ and $\hat{J}^B$ are the lower and the upper Jordan normal forms of the corresponding matrices, λ^A and λ^B are the eigenvalues on the main diagonal of the Jordan normal forms, and $y_{i,j}$ is an element of the matrix Y. If the equation has no solutions, we write $Y = \varnothing$.

A proof of Theorem 1 is given in the Appendix.

Let us analyze the results of Theorem 1 and formulate practically important corollaries.

It is known (see [4, 6]) that, if the matrices A and B are diagonalizable, then their Jordan forms coincide with the matrices of the eigenvalues and all the elements of $\breve{J}^A_{i+1,i}$ are $\hat{J}^B_{i,i+1}$ zero.

Corollary 1. The analytical solution of the continuous Sylvester equation

$$A_{n\times n}X_{n\times m} \pm X_{n\times m}B_{m\times m} = C_{n\times m}$$

with simple (diagonalizable) matrices A and B is given by the formula

$$X = R^A Y L^B.$$

Here $R^* = (L^*)^{-1}$ are the matrices of the right eigenvectors and the elements $y_{i,j}$ are given by the formulas

$$y_{i,j} = \frac{l_i^A C r_j^B}{\lambda_i^A \pm \lambda_j^B} \text{ if } \lambda_i^A \pm \lambda_j^B \neq 0, \tag{1.6}$$

$$y_{i,j} = \mu_{i,j} \in \Re \text{ if } \lambda_i^A \pm \lambda_j^B = 0 \text{ and } l_i^A C r_j^B = 0, \tag{1.7}$$

$$Y = \varnothing \text{ if } \lambda_i^A \pm \lambda_j^B = 0 \text{ and } l_i^A C r_j^B \neq 0, \tag{1.8}$$

where $i = \overline{1,n}$ and $j = \overline{1,m}$, $\mu_{i,j}$ are arbitrary parameters, and λ^A and λ^B are the eigenvalues of the matrices.

If, in addition to the diagonalizability of A and B, conditions (A.9) hold for all i and j, then Eq. (1.1) has a unique solution, which is obtained from the following corollary [6].

Corollary 2. In the case of the unique solvability of the continuous Sylvester equation

$$A_{n\times n}X_{n\times m} \pm X_{n\times m}B_{m\times m} = C_{n\times m}$$

with simple (diagonalizable) matrices A and B, its solution is determined by the formula

$$X = R^A(\Lambda^{AB} \odot (L^A C R^B))L^B, \tag{1.9}$$

where $i = \overline{1,n}$, $j = \overline{1,m}$, $R^* = (L^*)^{-1}$ are the matrices of the right eigenvalues, $\Lambda_{i,j}^{AB} = \lambda_i^A \pm \lambda_j^B\Big|_{i=\overline{1,n};\, j=\overline{1,m}}$, and $\odot$ denotes the elementwise multiplication of matrices.

Since the Lyapunov [5, 6] equation is a special case of the Sylvester equation, we have the following theorem.

Theorem 2. The analytical solution of the continuous Lyapunov equation

$$A_{n\times n}^{\mathrm{T}} X_{n\times n} \pm X_{n\times n} A_{n\times n} = C_{n\times n} \tag{1.10}$$

is given by the formula

$$X = L^{\mathrm{T}} Y L. \tag{1.11}$$

Here $L = (R)^{-1}$ is the matrix of the left eigenvectors and the elements $y_{i,j}$ are determined by the formulas

$$y_{i,j} = \frac{r_i^{\mathrm{T}} C r_j - J_{i-1,i} y_{i-1,j} \mp J_{j-1,j} y_{i,j-1}}{\lambda_i \pm \lambda_j} \text{ if } \lambda_i \pm \lambda_j \neq 0, \tag{1.12}$$

$$y_{i,j} = \mu_{i,j} \in \Re \text{ if } \lambda_i \pm \lambda_j = 0 \text{ and } r_i^{\mathrm{T}} C r_j - J_{i-1,i} y_{i-1,j} \mp J_{j-1,j} y_{i,j-1} = 0, \tag{1.13}$$

$$Y = \varnothing \text{ if } \lambda_i \pm \lambda_j = 0 \text{ and } r_i^{\mathrm{T}} C r_j - J_{i-1,i} y_{i-1,j} \mp J_{j-1,j} y_{i,j-1} \neq 0, \tag{1.14}$$

where $i = \overline{1,n}$, $j = \overline{1,n}$, r_i is the ith column of the matrix R and $\mu_{i,j}$ are arbitrary parameters, J is the upper Jordan normal form of the matrix A, and λ_i are the eigenvalues which are located on the main diagonal of the Jordan normal form.

The proof of Theorem 2 is based on the assertion of Theorem 1 taking into account the obvious change of notation and the decompositions

$$B = R\hat{J}L = RJL = R^B \hat{J}^B L^B, \tag{1.15}$$

$$A^{\mathrm{T}} = L^{\mathrm{T}} \hat{J}^{\mathrm{T}} R^{\mathrm{T}} = L^{\mathrm{T}} J^{\mathrm{T}} R^{\mathrm{T}} = R^A \breve{J}^A L^A, \tag{1.16}$$

which are similar to (0.1). It is easy to verify that, for stable (Hurwitz) matrices [4–6], Theorem 2 gives a unique positive definite solution.

2. THE DISCRETE SYLVESTER AND LYAPUNOVE EQUATIONS

By analogy with the continuous case, we formulate and prove theorems on the solution of the discrete Sylvester and Lyapunov equations.

Theorem 3. The analytical solution of the discrete Sylvester equation

$$A_{n\times n} X_{n\times m} B_{m\times m} \pm X_{n\times m} = C_{n\times m} \tag{2.1}$$

is given by the formula

$$X = R^A Y L^B.$$

Here $R^* = (L^*)^{-1}$ are the matrices of the right eigenvectors, and the elements $y_{i,j}$ are determined by the formulas

$$y_{i,j} = \frac{l_i^A C r_j^B - \breve{J}_{i,i-1}^A \lambda_j^B y_{i-1,j} - \hat{J}_{j-1,j}^B \lambda_i^A y_{i,j-1} - \breve{J}_{i,i-1}^A \hat{J}_{j-1,j}^B y_{i-1,j-1}}{\lambda_i^A \lambda_j^B \pm 1} \text{ if } \lambda_i^A \lambda_j^B \pm 1 \neq 0,$$

$$y_{i,j} = \mu_{i,j} \text{ if } \lambda_i^A \lambda_j^B \pm 1 = 0 \text{ and } l_i^A C r_j^B - \breve{J}_{i,i-1}^A \lambda_j^B y_{i-1,j} - \hat{J}_{j-1,j}^B \lambda_i^A y_{i,j-1} - \breve{J}_{i,i-1}^A \hat{J}_{j-1,j}^B y_{i-1,j-1} = 0,$$

$$Y = \varnothing \text{ if } \lambda_i^A \lambda_j^B \pm 1 = 0 \text{ and } l_i^A C r_j^B - \breve{J}_{i,i-1}^A \lambda_j^B y_{i-1,j} - \hat{J}_{j-1,j}^B \lambda_i^A y_{i,j-1} - \breve{J}_{i,i-1}^A \hat{J}_{j-1,j}^B y_{i-1,j-1} \neq 0,$$

where $i = \overline{1,n}$ and $j = \overline{1,m}$, $\mu_{i,j}$ are arbitrary parameters, $\breve{J}^*$ and $\hat{J}^*$ the lower and upper Jordan normal forms of the corresponding matrices, and λ^A and λ^B are the eigenvalues on the main diagonals of the Jordan normal forms.

A proof of Theorem 3 can be found in the Appendix.

This theorem immediately entails the following corollaries.

Corollary 3. The solution of the discrete Sylvester equation

$$A_{n\times n}X_{n\times m}B_{m\times m} \pm X_{n\times m} = C_{n\times m}$$

with simple (diagonalizable) matrices A and B is given by the formula

$$X = R^A Y L^B.$$

Here $R^* = (L^*)^{-1}$ are the matrices of the right eigenvectors, and the elements $y_{i,j}$ are determined by the formulas

$$y_{i,j} = \frac{l_i^A C r_j^B}{\lambda_i^A \lambda_j^B \pm 1} \text{ if } \lambda_i^A \lambda_j^B \pm 1 \neq 0, \tag{2.2}$$

$$y_{i,j} = \mu_{i,j} \in \Re \text{ if } \lambda_i^A \lambda_j^B \pm 1 = 0 \text{ and } l_i^A C r_j^B = 0, \tag{2.3}$$

$$Y = \varnothing \text{ if } \lambda_i^A \lambda_j^B \pm 1 = 0 \text{ and } l_i^A C r_j^B \neq 0,$$

where $i = \overline{1,n}$ and $j = \overline{1,m}$, $\mu_{i,j}$ are arbitrary parameters, and λ^A and λ^B are the eigenvalues of the matrices.

Corollary 4. In the case of the unique solvability of the discrete Sylvester equation

$$A_{n\times n}X_{n\times m}B_{m\times m} \pm X_{n\times m} = C_{n\times m}$$

with simple (diagonalizable) matrices A and B, the solution is given by the formula

$$X = R^A(\Lambda^{AB} \odot (L^A C R^B))L^B, \tag{2.4}$$

where $R^* = (L^*)^{-1}$ are the matrices of the right eigenvectors,

$$\Lambda_{i,j}^{AB} = \frac{1}{\lambda_i^A \lambda_j^B \pm 1}, \; i = \overline{1,n}, \; j = \overline{1,m},$$

and $\odot$ denotes the elementwise (Hadamard) multiplication of matrices.

The reasoning above implies the following result.

Theorem 4. The analytical solution to the discrete Lyapunov equation

$$A_{n\times n}^{\mathrm{T}} X_{n\times n} A_{n\times n} \pm X_{n\times n} = C_{n\times n} \tag{2.5}$$

is given by the formula

$$X = L^{\mathrm{T}} Y L. \tag{2.6}$$

Here $L = (R)^{-1}$ is the matrix of the left eigenvalues and the elements $y_{i,j}$ are determined by the formulas

$$y_{i,j} = \frac{r_i^{\mathrm{T}} C r_j - J_{i-1,i}\lambda_j y_{i-1,j} - J_{j-1,j}\lambda_i y_{i,j-1} - J_{i-1,i}J_{j-1,j}y_{i-1,j-1}}{\lambda_i \lambda_j \pm 1} \text{ if } \lambda_i\lambda_j \pm 1 \neq 0, \tag{2.7}$$

$$y_{i,j} = \mu_{i,j} \text{ if } \lambda_i\lambda_j \pm 1 = 0 \text{ and } r_i^{\mathrm{T}} C r_j - J_{i-1,i}\lambda_j y_{i-1,j} - J_{j-1,j}\lambda_i y_{i,j-1} - J_{i-1,i}J_{j-1,j}y_{i-1,j-1} = 0,$$

$$Y = \varnothing \text{ if } \lambda_i\lambda_j \pm 1 = 0 \text{ and } r_i^{\mathrm{T}} C r_j - J_{i-1,i}\lambda_j y_{i-1,j} - J_{j-1,j}\lambda_i y_{i,j-1} - J_{i-1,i}J_{j-1,j}y_{i-1,j-1} \neq 0,$$

where $i = \overline{1,n}$ and $j = \overline{1,n}$, $\mu_{i,j}$ are arbitrary parameters, J is the upper Jordan normal form of A, and λ^A and λ^B are the eigenvalues on the main diagonal of the Jordan normal form.

Theorem 4 is an alternative form of the solution to the Lyapunov equation obtained in [7], where this solution was obtained (under certain restrictions) based on the algebraic spectral approach.

3. GENERALIZATION AND ANALYSIS OF THE RESULTS

To facilitate the analysis of the results obtained above, we summarize the assertions of Theorems 1–4 in Table 1.

It is seen from Table 1 that, in addition to the known unique solvability conditions of the Sylvester and Lyapunov equations [1–3, 5, 6], we found the conditions for the existence of a set of solutions and insolvability conditions.

Table 1. Solution of the continuous and discrete Sylvester and Lyapunov equations

Solution	Condition
Continuous Sylvester equation $AX \pm XB = C,\ X = R^A Y L^B$	
$y_{i,j} = \varnothing$	$\lambda_i^A \pm \lambda_j^B = 0,\ l_i^A C r_j^B - \breve{J}_{i,i-1}^A y_{i-1,j} \mp \hat{J}_{j-1,j}^B y_{i,j-1} \neq 0$
$y_{i,j} = \mu_{i,j}$	$\lambda_i^A \pm \lambda_j^B = 0,\ l_i^A C r_j^B - \breve{J}_{i,i-1}^A y_{i-1,j} \mp \hat{J}_{j-1,j}^B y_{i,j-1} = 0$
$y_{i,j} = \dfrac{l_i^A C r_j^B - \breve{J}_{i,i-1}^A y_{i-1,j} \mp \hat{J}_{j-1,j}^B y_{i,j-1}}{\lambda_i^A \pm \lambda_j^B}$	$\lambda_i^A \pm \lambda_j^B \neq 0$
Continuous Lyapunov equation $A^{\mathrm{T}} X \pm XA = C,\ X = L^{\mathrm{T}} Y L$	
$y_{i,j} = \varnothing$	$\lambda_i \pm \lambda_j = 0,\ r_i^{\mathrm{T}} C r_j - J_{i-1,i} y_{i-1,j} \mp J_{j-1,j} y_{i,j-1} \neq 0$
$y_{i,j} = \mu_{i,j}$	$\lambda_i \pm \lambda_j = 0,\ r_i^{\mathrm{T}} C r_j - J_{i-1,i} y_{i-1,j} \mp J_{j-1,j} y_{i,j-1} = 0$
$y_{i,j} = \dfrac{r_i^{\mathrm{T}} C r_j - J_{i-1,i} y_{i-1,j} \mp J_{j-1,j} y_{i,j-1}}{\lambda_i \pm \lambda_j}$	$\lambda_i \pm \lambda_j \neq 0$
Discrete Sylvester equation $AXB \pm X = C,\ X = R^A Y L^B$	
$y_{i,j} = \varnothing$	$\lambda_i^A \lambda_j^B \pm 1 = 0,\ l_i^A C r_j^B - \breve{J}_{i,i-1}^A \lambda_j^B y_{i-1,j} - \hat{J}_{j-1,j}^B \lambda_i^A y_{i,j-1} - \breve{J}_{i,i-1}^A \hat{J}_{j-1,j}^B y_{i-1,j-1} \neq 0$
$y_{i,j} = \mu_{i,j}$	$\lambda_i^A \lambda_j^B \pm 1 = 0,\ l_i^A C r_j^B - \breve{J}_{i,i-1}^A \lambda_j^B y_{i-1,j} - \hat{J}_{j-1,j}^B \lambda_i^A y_{i,j-1} - \breve{J}_{i,i-1}^A \hat{J}_{j-1,j}^B y_{i-1,j-1} = 0$
$y_{i,j} = \dfrac{l_i^A C r_j^B - \breve{J}_{i,i-1}^A \lambda_j^B y_{i-1,j} - \hat{J}_{j-1,j}^B \lambda_i^A y_{i,j-1} - \breve{J}_{i,i-1}^A \hat{J}_{j-1,j}^B y_{i-1,j-1}}{\lambda_i^A \lambda_j^B \pm 1}$	$\lambda_i^A \lambda_j^B \pm 1 \neq 0$
Discrete Lyapunov equation $A^{\mathrm{T}} XA \pm X = C,\ X = L^{\mathrm{T}} Y L$	
$y_{i,j} = \varnothing$	$\lambda_i \lambda_j \pm 1 = 0,\ r_i^{\mathrm{T}} C r_j - J_{i-1,i} \lambda_j y_{i-1,j} - J_{j-1,j} \lambda_i y_{i,j-1} - J_{i-1,i} J_{j-1,j} y_{i-1,j-1} \neq 0$
$y_{i,j} = \mu_{i,j}$	$\lambda_i \lambda_j \pm 1 = 0,\ r_i^{\mathrm{T}} C r_j - J_{i-1,i} \lambda_j y_{i-1,j} - J_{j-1,j} \lambda_i y_{i,j-1} - J_{i-1,i} J_{j-1,j} y_{i-1,j-1} = 0$
$y_{i,j} = \dfrac{r_i^{\mathrm{T}} C r_j - J_{i-1,i} \lambda_j y_{i-1,j} - J_{j-1,j} \lambda_i y_{i,j-1} - J_{i-1,i} J_{j-1,j} y_{i-1,j-1}}{\lambda_i \lambda_j \pm 1}$	$\lambda_i \lambda_j \pm 1 \neq 0$

Of special interest is the case of the existence of a set of solutions when the size of the Jordan blocks is greater than one. Consider how the elements $y_{i,j}$ are formed in the general case when, for example, the continuous Sylvester equation is solved.

Theorem 1 states that, if the conditions

$$\begin{cases} \lambda_i^A \pm \lambda_j^B = 0, \\ l_i^A C r_j^B - \breve{J}_{i,i-1}^A y_{i-1,j} \mp \hat{J}_{j-1,j}^B y_{i,j-1} = 0 \end{cases} \tag{3.1}$$

(we call them the *nonuniqueness solvability conditions of the continuous Sylvester equation*) hold, then the element $y_{i,j}$ takes the value of the arbitrary parameter $\mu_{i,j} \in \Re$. However, if there are Jordan blocks of a size greater than one, then additional constraints in the form of the value of the arbitrary parameter $\mu_{i,j}$ can be imposed for the solvability of the next equation. We demonstrate this fact by way of example.

Let, in the case $i = 3, j = 3$, conditions (3.1) hold:

$$\begin{cases} \lambda_3^A \pm \lambda_3^B = 0, \\ l_3^A C r_3^B - \breve{J}_{32}^A y_{23} \mp \hat{J}_{23}^B y_{32} = 0. \end{cases} \tag{3.2}$$

Then, Theorem 1 implies that $y_{i,j}$ takes the value of an arbitrary parameter, e.g., μ_{33}:

$$y_{33} = \mu_{33}. \tag{3.3}$$

However, already at the next step ($i = 3, j = 4$), when it holds that

$$\lambda_3^A \pm \lambda_4^B = 0,$$

we must satisfy the condition $l_3^A C r_4^B - \breve{J}_{32}^A y_{24} \mp \hat{J}_{24}^B y_{33} = 0$; hence, taking into account (3.3), we obtain constraints on the arbitrary parameter μ_{33}, which, actually, takes a quite definite value of

$$\mu_{33} = \pm l_3^A C r_4^B \mp \breve{J}_{32}^A y_{24}.$$

Under the corresponding conditions on the Jordan blocks, a similar constraint can be imposed when the row index i is increased.

The analysis of the second condition in (3.1) shows that the element $y_{i,j}$ must take its final value taking into account the solvability of the following equations.

1. In the case $\breve{J}_{i+1,i}^A \neq 0$,

$$l_{i+1}^A C r_j^B - \breve{J}_{i+1,i}^A y_{i,j} \mp \hat{J}_{j-1,j}^B y_{i+1,j-1} = 0, \tag{3.4}$$

and, therefore, $y_{i,j}$ is found by the formula

$$y_{i,j} = l_{i+1}^A C r_j^B \mp \hat{J}_{j-1,j}^B y_{i+1,j-1}. \tag{3.5}$$

2. In the case $\hat{J}_{j,j+1}^B \neq 0$,

$$l_i^A C r_{j+1}^B - \breve{J}_{i,i-1}^A y_{i-1,j+1} \mp \hat{J}_{j,j+1}^B y_{i,j} = 0, \tag{3.6}$$

and, therefore, $y_{i,j}$ is found by the formula

$$y_{i,j} = \pm l_i^A C r_{j+1}^B \mp \breve{J}_{i,i-1}^A y_{i-1,j+1}. \tag{3.7}$$

3. In the case $\breve{J}_{i+1,i}^A \neq 0$ and, $\hat{J}_{j,j+1}^B \neq 0$, both formula (3.5) and (3.7) can be used; however, additional constraints on the matrices $L^A, C,$ and R^B are imposed, which are determined by the conditions for the equivalence of transformations. For example, if

$$\hat{J}_{j-1,j}^B = \breve{J}_{i,i-1}^A = 0,$$

then the equation has a solution only if the condition

$$l_{i+1}^A C r_j^B = \pm l_i^A C r_{j+1}^B \tag{3.8}$$

is satisfied. We believe that this feature is the cause of the fact that the issues concerning multiplicity of solutions (existence conditions and formulas for the solutions) are poorly studied and not implemented in computer software.

Based on Table 1, we can write algorithms for solving the continuous and discrete Sylvester equations (by analogy, algorithms for solving the Lyapunov equations can be easily written).

3.1. Algorithm for Solving the Continuous Sylvester Equation

1. Reduce the matrices A and B to the Jordan normal form:

$$A \to (L^A, \breve{J}^A, R^A), \quad B \to (L^B, \hat{J}^B, R^B). \tag{3.9}$$

2. Redefine the right-hand side:

$$\tilde{C} = L^A C R^B. \tag{3.10}$$

3. Solve the equation (the block diagram in Fig. 1)

$$\breve{J}^A Y + Y\hat{J}^B = \tilde{C}. \tag{3.11}$$

3.1. For each i, j, calculate $\lambda_i^A + \lambda_j^B$.

3.2. If $\lambda_i^A + \lambda_j^B \neq 0$, then

$$y_{i,j} = \frac{\tilde{C}_{i,j} - \breve{J}^A_{i,i-1} y_{i-1,j} - \hat{J}^B_{j-1,j} y_{i,j-1}}{\lambda_i^A + \lambda_j^B}. \tag{3.12}$$

3.3. If $\lambda_i^A \pm \lambda_j^B = 0$, then do the following:

3.3.1. Check the solvability condition

$$\tilde{C}_{i,j} - \breve{J}^A_{i,i-1} y_{i-1,j} - \hat{J}^B_{j-1,j} y_{i,j-1} = 0; \tag{3.13}$$

if (3.13) is not satisfied, then the equation is not solvable.

3.3.2. If $\breve{J}^A_{i+1,i} = \hat{J}^B_{j,j+1} = 0$, then the element $y_{i,j}$ takes the value of the arbitrary variable.

3.3.3. If $\breve{J}^A_{i+1,i} \neq 0$, then

$$y_{i,j} = \tilde{C}_{i+1,j} - \hat{J}^B_{j-1,j} y_{i+1,j-1}. \tag{3.14}$$

3.3.4. If $\hat{J}^B_{j,j+1} \neq 0$, then

$$y_{i,j} = \tilde{C}_{i,j+1} - \breve{J}^A_{i,i-1} y_{i-1,j+1}. \tag{3.15}$$

4. Calculate the desired matrix:

$$X = R^A Y L^B. \tag{3.16}$$

3.2. Algorithm for Solving the Discrete Sylvester Equation

1. Reduce the matrices A and B to the Jordan normal form:

$$A \to (L^A, \breve{J}^A, R^A), \quad B \to (L^B, \hat{J}^B, R^B). \tag{3.17}$$

2. Redefine the right-hand side:

$$\tilde{C} = L^A C R^B. \tag{3.18}$$

3. Solve the equation (the block diagram in Fig. 2)

$$\breve{J}^A Y \hat{J}^B + Y = \tilde{C}. \tag{3.19}$$

3.1. For each i, j, calculate $\lambda_i^A \lambda_j^B + 1$.

3.2. If $\lambda_i^A \lambda_j^B + 1 \neq 0$, then

$$y_{i,j} = \frac{\tilde{C}_{i,j} - \breve{J}^A_{i,i-1} \lambda_j^B y_{i-1,j} - \hat{J}^B_{j-1,j} \lambda_i^A y_{i,j-1} - \breve{J}^A_{i,i-1} \hat{J}^B_{j-1,j} y_{i-1,j-1}}{\lambda_i^A \lambda_j^B \pm 1}. \tag{3.20}$$

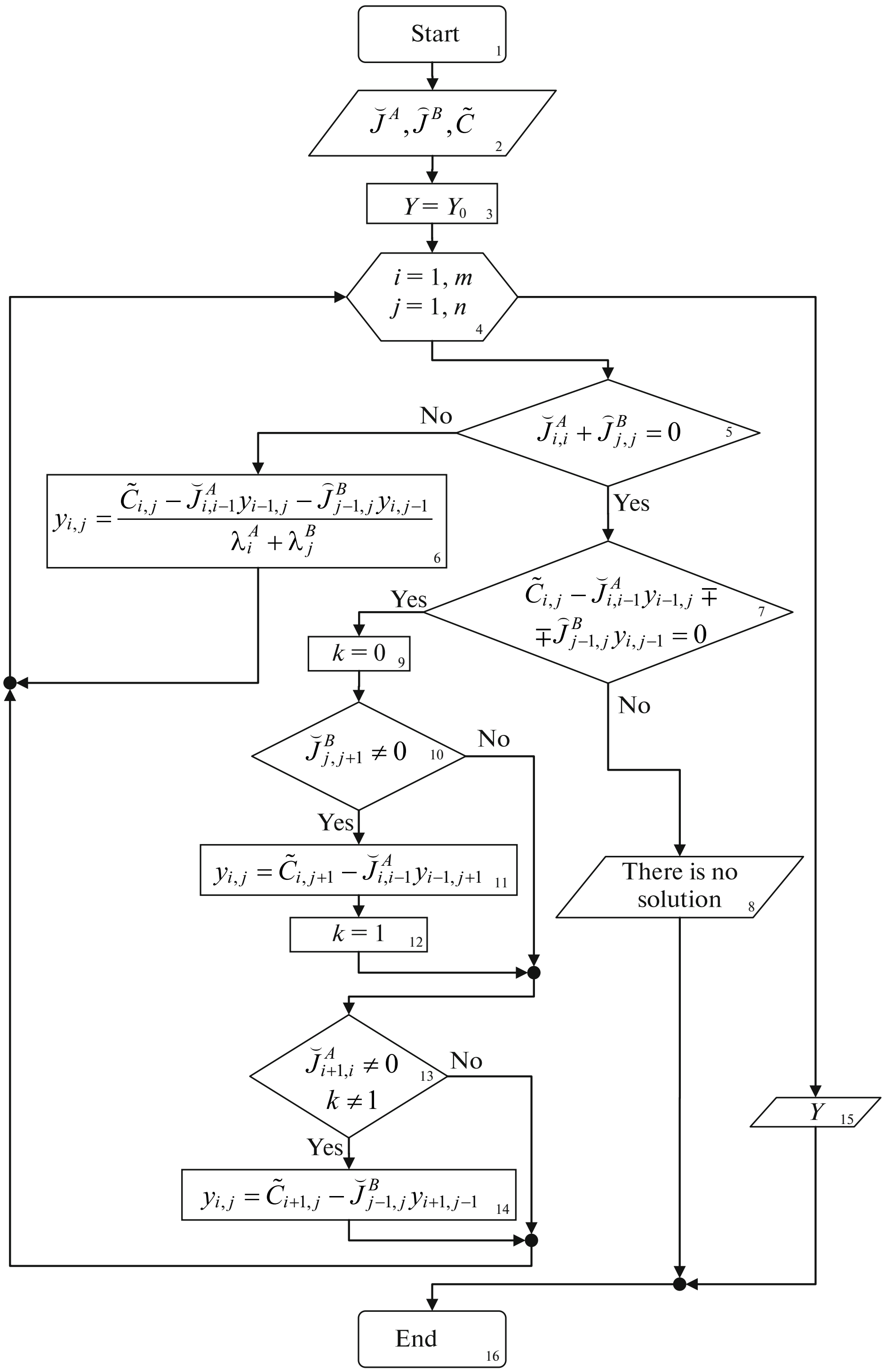

Fig. 1. Algorithm for solving the continuous matrix equation $\breve{J}^{A}Y + Y\hat{J}^{B} = \tilde{C}$.

3.3. If $\lambda_i^A\lambda_j^B + 1 = 0$, then:

3.3.1. Check the solvability condition

$$\tilde{C}_{i,j} - \breve{J}^A_{i,i-1}\lambda^B_j y_{i-1,j} - \hat{J}^B_{j-1,j}\lambda^A_i y_{i,j-1} - \breve{J}^A_{i,i-1}\hat{J}^B_{j-1,j} y_{i-1,j-1} = 0; \quad (3.21)$$

if (3.21) is not satisfied, then the equation is not solvable.

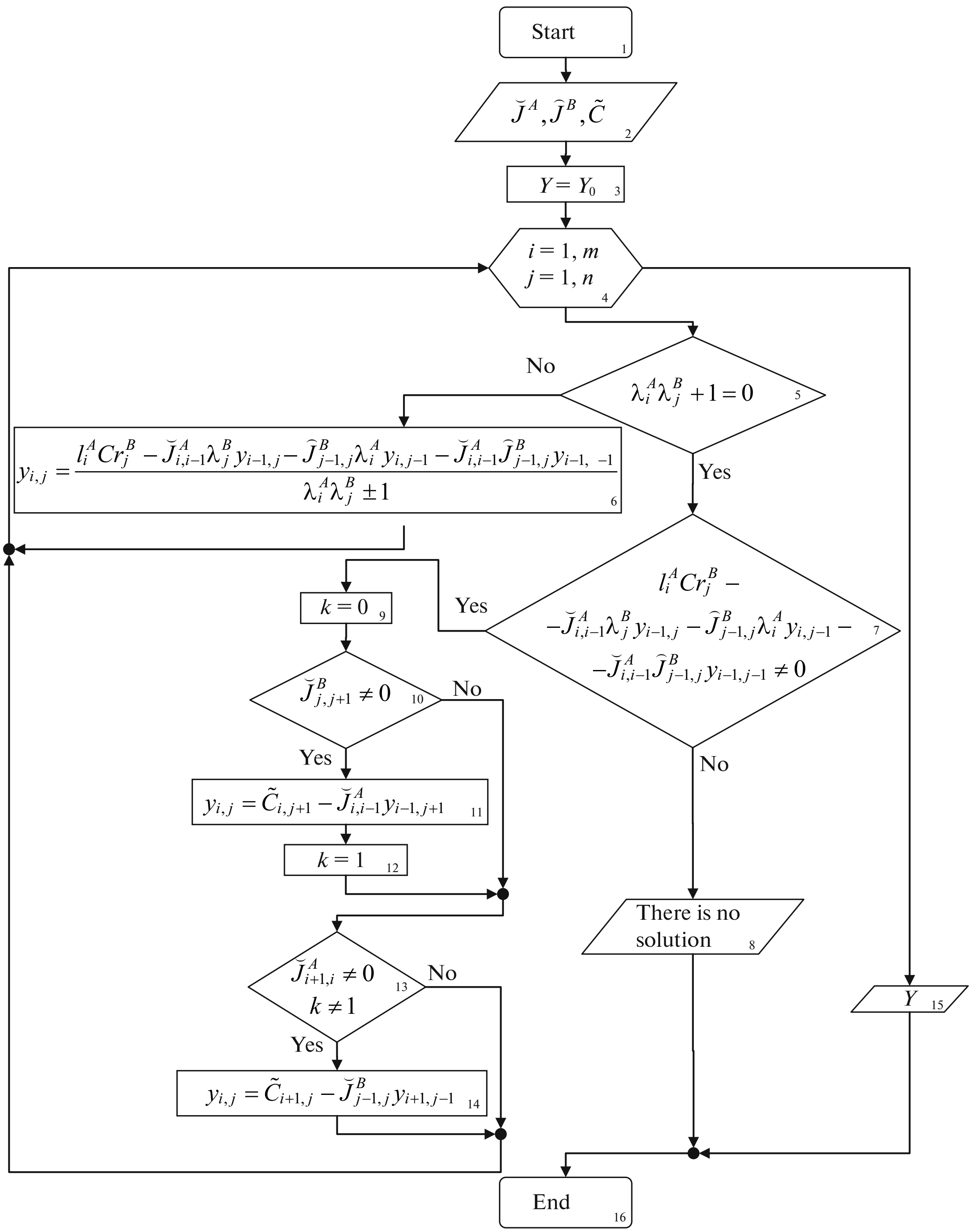

Fig. 2. Algorithm for solving the discrete matrix equation $\breve{J}^A Y \widehat{J}^B + Y = \widetilde{C}$.

3.3.2. If $\breve{J}^A_{i+1,i} = \widehat{J}^B_{j,j+1} = 0$, then the element $y_{i,j}$ takes the value of the arbitrary variable.

3.3.3. If $\breve{J}^A_{i+1,i} \neq 0$, then

$$y_{i,j} = \widetilde{C}_{i+1,j} - \widehat{J}^B_{j-1,j} y_{i+1,j-1}. \tag{3.22}$$

3.3.4. If $\hat{J}^{B}_{j,j+1} \neq 0$, then

$$y_{i,j} = \widetilde{C}_{i,j+1} - \breve{J}^{A}_{i,i-1} y_{i-1,j+1}. \quad (3.23)$$

4. Calculate the desired matrix:

$$X = R^{A} Y L^{B}. \quad (3.24)$$

In conclusion of the theoretical part of this paper, we would like to note the main directions of future research related to the topics discussed above.

1. The use of the Matlab *jordan* function is a significant drawback of the software implementation of the algorithms described above. This function is resource-hungry because it uses symbolic computations for constructing the Jordan decomposition. The cause of using the symbolic computations is that the algorithms for finding the eigenvectors are not numerically stable. We plan to develop a special algorithm for finding the eigenvectors.

2. Another drawback of the use of the Jordan decomposition, which was mentioned in [5], is the deterioration of the numerical stability of the solution when nonorthogonal transformations are used. The analysis of numerical stability of the proposed algorithms and the comparison of their stability with that of the known methods, as well as the investigation of ways for improving the stability of the algorithms (some of them were described in [6]) is another direction of research.

3. In addition, we are going to use similar approaches with the necessary improvements for finding the analytical solution of the two-term matrix equation

$$AXC \pm BXD = Q, \quad (3.25)$$

which is a natural generalization of the continuous and discrete Sylvester and Lyapunov equations.

4. EXAMPLES OF SOLVING EQUATIONS

To verify the efficiency of the proposed algorithms, we discuss a few examples. In addition to test examples, we also compare the solutions obtained using the implementations of the proposed algorithms (the solvers *silv* and *dsilv*) with those produced by the solvers of the continuous and discrete Sylvester and Lyapunov equations *lyap* and *dlyap* included in Matlab in the 2011 version.[1]

Example 1. The set of solutions of a continuous Sylvester equation. Suppose we need to find all the solutions to the equation of form (1.1)

$$AX + XB = C \quad (4.1)$$

with the given matrices $A, B,$ and C:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 2 & 5 \\ 3 & 0 & 3 \\ 2 & 1 & 2 \end{bmatrix}.$$

To solve Eq. (4.1), we use the algorithm for solving the continuous Sylvester equation.

1. Reduce the matrices A and B to the Jordan normal form:

$$L^{A} = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -1 & -1 \\ -1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \breve{J}^{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad R^{A} = \begin{bmatrix} 0 & -\frac{1}{2} & -1 \\ -1 & -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & 0 \end{bmatrix},$$

$$L^{B} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \quad \hat{J}^{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad R^{B} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

[1] The modern versions of Matlab do not significantly differ from the version of 2011 in the part concerning the equations considered in this paper.

Table 2. Computational results

i	1	1	1	2	2	2	3	3	3
j	1	2	3	1	2	3	1	2	3
λ_i^A	−1	−1	−1	1	1	1	1	1	1
λ_j^B	0	1	1	0	1	1	0	1	1
$\lambda_i^A + \lambda_j^B$	−1	0	0	1	2	2	1	2	2

2. Redefine the right-hand side of the equation:

$$\widetilde{C} = L^A C R^B = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -1 & -1 \\ -1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 2 & 5 \\ 3 & 0 & 3 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 6 & -1 \\ \frac{3}{2} & 4 & -\frac{7}{2} \end{bmatrix}.$$

3. Solve the equation $\breve{J}^A Y + Y\widehat{J}^B = \widetilde{C}$.

For each i and j, calculate $\lambda_i^A + \lambda_j^B$. For convenience, the results of the calculations are summarized in Table 2.

Since

$$\lambda_1^A + \lambda_1^B = -1 \neq 0,$$

the element y_{11} is found by formula (3.12):

$$y_{11} = \frac{\widetilde{C}_{11}}{\lambda_1^A + \lambda_1^B} = \frac{1}{2}.$$

Since

$$\lambda_1^A + \lambda_2^B = 0,$$

we check the solvability condition (3.13):

$$\widetilde{C}_{i,j} - \breve{J}^A_{i,i-1} y_{i-1,j} - \widehat{J}^B_{j-1,j} y_{i,j-1} = \widetilde{C}_{12} = 0.$$

Therefore, the equation is solvable for y_{12}.

Since

$$\widehat{J}^B_{j,j+1} = \widehat{J}^B_{2,3} = 1 \neq 0,$$

the element y_{12} is found by formula (3.15):

$$y_{12} = \widetilde{C}_{13} = \frac{1}{2}. \tag{4.2}$$

The identity

$$\lambda_1^A + \lambda_3^B = 0$$

requires us to check the solvability condition (3.13):

$$\widetilde{C}_{i,j} - \breve{J}^A_{i,i-1} y_{i-1,j} - \widehat{J}^B_{j-1,j} y_{i,j-1} = \widetilde{C}_{12} - y_{12} = \frac{1}{2} - \frac{1}{2} = 0.$$

Therefore, the equation is solvable for y_{13}.

Since

$$\breve{J}^A_{i+1,i} = 0$$

and the element $\widehat{J}^{B}_{j,j+1}$ does not exist, the element y_{12} takes the value of the arbitrary variable (e.g., x_{13}):

$$y_{13} = x_{13}. \tag{4.3}$$

For all other combinations of i and j, it holds that

$$\lambda_i^A + \lambda_j^B \neq 0;$$

therefore, the elements $y_{i,j}$ are found by formula (3.12):

$$y_{21} = \frac{\widetilde{C}_{21}}{\lambda_2^A + \lambda_1^B} = \frac{1}{1} = 1,\ y_{22} = \frac{\widetilde{C}_{22}}{\lambda_2^A + \lambda_2^B} = \frac{6}{2} = 3,\ y_{23} = \frac{\widetilde{C}_{23} - \widehat{J}^{B}_{23} y_{22}}{\lambda_2^A + \lambda_3^B} = \frac{-1-3}{2} = -2,$$

$$y_{31} = \frac{\widetilde{C}_{31} - \breve{J}^{A}_{32} y_{21}}{\lambda_3^A + \lambda_1^B} = \frac{3/2-1}{1} = \frac{1}{2},\ y_{32} = \frac{\widetilde{C}_{i,j} - \breve{J}^{A}_{32} y_{22}}{\lambda_3^A + \lambda_2^B} = \frac{4-3}{2} = \frac{1}{2},$$

$$y_{33} = \frac{\widetilde{C}_{33} - \breve{J}^{A}_{i,i-1} y_{23} - \widehat{J}^{B}_{23} y_{32}}{\lambda_3^A + \lambda_3^B} = \frac{-7/2 + 2 - 1/2}{2} = -1.$$

4. Calculate the desired matrix by formula (3.16):

$$X = R^A Y L^B = \begin{bmatrix} 0 & -\frac{1}{2} & -1 \\ -1 & -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & x_{13} \\ 1 & 3 & -2 \\ \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ x_{13}+1 & 1 & 1 \\ -x_{13} & 0 & 1 \end{bmatrix}.$$

Substitution of the result into the original equation (4.1) transforms it to the identity

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ x_{13}+1 & 1 & 1 \\ -x_{13} & 0 & 1 \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ x_{13}+1 & 1 & 1 \\ -x_{13} & 0 & 1 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}}_{B} \equiv \underbrace{\begin{bmatrix} 3 & 2 & 5 \\ 3 & 0 & 3 \\ 2 & 1 & 2 \end{bmatrix}}_{C}. \tag{4.4}$$

Example 2. The unique solution of a discrete Sylvester equation. Suppose we need to solve the equation of form (2.1)

$$AXB + X = C \tag{4.5}$$

with the given matrices $A, B,$ and C:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

To solve Eq. (4.5), we use the algorithm for solving the discrete Sylvester equation.

1. Reduce the matrices A and B to the Jordan normal form:

$$L^A = \begin{bmatrix} 0 & 0 & -1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \breve{J}^A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R^A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 0 \end{bmatrix},$$

$$L^B = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}, \quad \hat{J}^B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R^B = \begin{bmatrix} 1 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

2. The analysis of the eigenvalues of the matrices A and B shows that the equation is uniquely solvable because

$$\lambda_i^A \lambda_j^B + 1 \neq 0.$$

Then, by Corollary 4, the solution can be written by formula (2.4) as

$$X = R^A(\Lambda^{AB} \odot (L^A C R^B))L^B,$$

where in this case

$$\Lambda^{AB} = \begin{bmatrix} \frac{1}{\lambda_1^A\lambda_1^B+1} & \frac{1}{\lambda_1^A\lambda_2^B+1} & \frac{1}{\lambda_1^A\lambda_3^B+1} \\ \frac{1}{\lambda_2^A\lambda_1^B+1} & \frac{1}{\lambda_2^A\lambda_2^B+1} & \frac{1}{\lambda_2^A\lambda_3^B+1} \\ \frac{1}{\lambda_3^A\lambda_1^B+1} & \frac{1}{\lambda_3^A\lambda_2^B+1} & \frac{1}{\lambda_3^A\lambda_3^B+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1/3 & 1 \end{bmatrix}.$$

Hence, we have

$$X = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 0 \end{bmatrix}}_{R^A} \left(\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{3} & 1 \\ 1 & \frac{1}{3} & 1 \end{bmatrix}}_{\Lambda^{AB}} \odot \left(\underbrace{\begin{bmatrix} 0 & 0 & -1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}}_{L^A} \underbrace{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}}_{R^B} \right) \right) \underbrace{\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}}_{L^B}$$

and, upon simplification, we obtain

$$X = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The substitution of the matrix X into Eq. (4.5) confirms the validity of the result:

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{B} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{X} \equiv \underbrace{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{C}. \tag{4.6}$$

Example 3. A continuous Sylvester equation (comparison of results). Let us solve Eq. (4.1) with the coefficient matrices

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

using the Matlab *lyap* solver and the *silv* solver, which implements the algorithms proposed in this paper and was developed by the authors.

1. The Matlab solver $X_{lyap} = lyap(A, B, -C)$ produces the solution

$$X_{lyap} = \begin{bmatrix} \frac{5656}{195} & -\frac{45891}{58} & 1 \\ \frac{13969}{42} & -1507148789265460000 & -\frac{2633}{195} \\ -\frac{17702}{51} & 1507148789265460500 & \frac{2828}{195} \end{bmatrix}.$$

The verification of this result shows that the solution is inaccurate because

$$AX_{lyap} + X_{lyap}B - C \approx \begin{bmatrix} 0 & 29 & 0 \\ 0 & 38 & 0 \\ 0 & 179 & 0 \end{bmatrix}. \tag{4.7}$$

2. The solver proposed in this paper $X_{silv} = silv(A, B, C', x')$ yields the set of solutions

$$X_{silv} = \begin{bmatrix} 0 & 1 - 2x_{22} & 1 \\ x_{22} & \frac{1}{4} + x_{22} + x_{23} & 1 \\ x_{22} & -\frac{1}{4} - x_{23} & 0 \end{bmatrix}, \quad x_{22} \in \Re \tag{4.8}$$

in which each element turns Eq. (4.1) into the identity

$$AX_{silv} + X_{silv}B - C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.9}$$

Note that, in addition to producing the incorrect result, the Matlab solver *lyap* failed to detect the existence of a multiplicity of solutions, which it usually reports about.

Example 4. A continuous Lyapunov equation (the stability of solution). Suppose we want to solve the equation of form (1.10)

$$A^{\mathrm{T}}X + XA = C \tag{4.10}$$

with the coefficient matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}.$$

1. The solver $X_{lyap} = lyap(A, -C)$ outputs the message *Warning: Matrix is close to singular or badly scaled. Results may be inaccurate. RCOND = 5.114818e–018*:

$$X_{lyap} \approx \begin{bmatrix} 0 & 0 & 0 \\ 1 & -11.7 & 12.7 \\ 0 & 12.7 & -10.7 \end{bmatrix}. \tag{4.11}$$

This message implies that, in Matlab's opinion, the result is inaccurate due to the poor conditioning of the matrices.

2. The solver proposed in this paper $X_{silv} = silv(A, C', x')$ produces the exact solution formula in the form of the set of matrices

$$X_{silv} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & x_{11} + \frac{1}{2} & -x_{11} + \frac{1}{2} \\ 0 & -x_{11} + \frac{1}{2} & x_{11} + \frac{3}{2} \end{bmatrix}, \quad x_{11} \in \Re. \tag{4.12}$$

Here we may set

$$X_{lyap} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -11.7 & 12.7 \\ 0 & 12.7 & -10.7 \end{bmatrix},$$

because, at $x_{11} = -12.2$, solution (4.11) belongs to set (4.12), i.e.,

$$X_{silv}\big|_{x_{11}=-12.2} = X_{lyap}.$$

Thus, in addition to the set of solutions, the *silv* solver produced the exact result even though the matrix A is indeed ill conditioned:

$$\mathrm{cond}(A) = \|A\| \cdot \|A^{-1}\| = 8.1931 \times 10^{16}.$$

Here $\|\cdot\|$ is the 2-norm of the matrix [4].

CONCLUSIONS

A method for the analytical solution of the Sylvester and Lyapunov equations is developed; this method is based on the representation of numerical matrices in normal forms. To obtain an analytical form of the set of solutions, it is proposed to reduce the given matrices to the Jordan normal form.

APPENDIX

Proof of Theorem 1. Taking into account decomposition (0.1), we can write the original equation (1.1) in the form

$$R^A \breve{J}^A L^A X \pm X R^B \hat{J}^B L^B = C. \tag{A.1}$$

Multiply Eq. (A.1) by the invertible matrices L^A and R^B on the left and on the right, respectively:

$$\breve{J}^A \underbrace{L^A X R^B}_{Y} \pm \underbrace{L^A X R^B}_{Y} \hat{J}^B = L^A C R^B. \tag{A.2}$$

Define the intermediate variable

$$Y = L^A X R^B; \tag{A.3}$$

then, Eq. (A.2) can be written as

$$\breve{J}^A Y \pm Y \hat{J}^B = L^A C R^B. \tag{A.4}$$

Therefore, the continuous Sylvester equation (1.1) is reduced to Eq. (A.4) of the same form in which the given matrices are represented in the Jordan normal form.

Taking into account the generalized notation of the Jordan normal forms (0.2) and (0.3) of the matrices A and B

$$\breve{J}^A = \begin{bmatrix} \lambda_1^A & 0 & 0 & 0 & 0 & 0 \\ \breve{J}_{2,1}^A & \ddots & 0 & 0 & 0 & 0 \\ 0 & \ddots & \lambda_i^A & 0 & 0 & 0 \\ 0 & 0 & \breve{J}_{i+1,i}^A & \lambda_{i+1}^A & 0 & 0 \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \breve{J}_{m,m-1}^A & \lambda_m^A \end{bmatrix}, \tag{A.5}$$

$$\widehat{J}^B = \begin{bmatrix} \lambda_1^B & \widehat{J}_{1,2}^B & 0 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & 0 & 0 & 0 \\ 0 & 0 & \lambda_j^B & \widehat{J}_{j,j+1}^B & 0 & 0 \\ 0 & 0 & 0 & \lambda_{j+1}^B & \ddots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \widehat{J}_{n-1,n}^B \\ 0 & 0 & 0 & 0 & 0 & \lambda_n^B \end{bmatrix}, \tag{A.6}$$

consider the equation corresponding, e.g., to the ith row and jth column of Eq. (A.4):

$$\breve{J}_{i,i-1}^A y_{i-1,j} + \lambda_i^A y_{i,j} \pm \widehat{J}_{j-1,j}^B y_{i,j-1} \pm \lambda_j^B y_{i,j} = l_i^A C r_j^B; \tag{A.7}$$

hence, we explicitly write the equation for $y_{i,j}$:

$$(\lambda_i^A \pm \lambda_j^B) y_{i,j} = l_i^A C r_j^B - \breve{J}_{i,i-1}^A y_{i-1,j} \mp \widehat{J}_{j-1,j}^B y_{i,j-1}. \tag{A.8}$$

Depending on the eigenvalues λ_i^A and λ_j^B, and the right-hand side of Eq. (A.8), three cases are possible.

1. The equation has no solutions:

$$y_{i,j} = \varnothing$$

if

$$\lambda_i^A \pm \lambda_j^B = 0$$

and

$$l_i^A C r_j^B - \breve{J}_{i,i-1}^A y_{i-1,j} \mp \widehat{J}_{j-1,j}^B y_{i,j-1} \neq 0.$$

In this case we conclude that the given equation is unsolvable.

2. The equation has a unique solution

$$y_{i,j} = \frac{l_i^A C r_j^B - \breve{J}_{i,i-1}^A y_{i-1,j} \mp \widehat{J}_{j-1,j}^B y_{i,j-1}}{\lambda_i^A \pm \lambda_j^B}$$

if

$$\lambda_i^A \pm \lambda_j^B \neq 0. \tag{A.9}$$

3. The equation has multiple solutions

$$y_{i,j} = \mu_{i,j} \in \Re$$

(the element $y_{i,j}$ takes the value of an arbitrary parameter) if

$$\lambda_i^A \pm \lambda_j^B = 0$$

and

$$l_i^A C r_j^B - \breve{J}_{i,i-1}^A y_{i-1,j} \mp \widehat{J}_{j-1,j}^B y_{i,j-1} = 0. \tag{A.10}$$

In this case, condition (A.10) simultaneously provides a restriction on the choice of the preceding elements $y_{i-1,j}$ and $y_{i,j-1}$.

Having found the elements of the matrix Y, we find the desired solution to the continuous Sylvester equation from Eq. (A.3):

$$X = R^A Y L^B. \tag{A.11}$$

Proof of Theorem 3. The proof of Theorem 3 is similar to the proof of Theorem 1 (A.1)–(A.11). In this case, we have the sequence of transformations

$$R^A \breve{J}^A L^A X R^B \widehat{J}^B L^B \pm X = C, \tag{A.12}$$

$$\breve{J}^A \underbrace{L^A X R^B}_{Y} \hat{J}^B \pm \underbrace{L^A X R^B}_{Y} = L^A C R^B, \tag{A.13}$$

$$Y = L^A X R^B, \tag{A.14}$$

$$\breve{J}^A Y \hat{J}^B \pm Y = L^A C R^B, \tag{A.15}$$

$$\breve{J}^A_{i,i-1}\lambda^B_j y_{i-1,j} + \lambda^A_i \lambda^B_j y_{i,j} + \hat{J}^B_{j-1,j}\lambda^A_i y_{i,j-1} + \breve{J}^A_{i,i-1}\hat{J}^B_{j-1,j} y_{i-1,j-1} \pm y_{i,j} = l^A_i C r^B_j, \tag{A.16}$$

$$(\lambda^A_i \lambda^B_j \pm 1) y_{i,j} = l^A_i C r^B_j - \breve{J}^A_{i,i-1}\lambda^B_j y_{i-1,j} - \hat{J}^B_{j-1,j}\lambda^A_i y_{i,j-1} - \breve{J}^A_{i,i-1}\hat{J}^B_{j-1,j} y_{i-1,j-1}. \tag{A.17}$$

Here, $\breve{J}^A$ and $\hat{J}^B$ are matrices (A.5) and (A.6), respectively.

1. The equation has no solutions, i.e.,

$$y_{i,j} = \varnothing$$

if

$$\lambda^A_i \lambda^B_j \pm 1 = 0$$

and

$$l^A_i C r^B_j - \breve{J}^A_{i,i-1}\lambda^B_j y_{i-1,j} - \hat{J}^B_{j-1,j}\lambda^A_i y_{i,j-1} - \breve{J}^A_{i,i-1}\hat{J}^B_{j-1,j} y_{i-1,j-1} \neq 0.$$

2. The equation has the unique solution

$$y_{i,j} = \frac{l^A_i C r^B_j - \breve{J}^A_{i,i-1}\lambda^B_j y_{i-1,j} - \hat{J}^B_{j-1,j}\lambda^A_i y_{i,j-1} - \breve{J}^A_{i,i-1}\hat{J}^B_{j-1,j} y_{i-1,j-1}}{\lambda^A_i \lambda^B_j \pm 1} \tag{A.18}$$

if $\lambda^A_i \pm \lambda^B_j \neq 0$.

3. The equation has multiple solutions

$$y_{i,j} = \mu_{i,j} \in \Re$$

if

$$\lambda^A_i \lambda^B_j \pm 1 = 0$$

and

$$l^A_i C r^B_j - \breve{J}^A_{i,i-1}\lambda^B_j y_{i-1,j} - \hat{J}^B_{j-1,j}\lambda^A_i y_{i,j-1} - \breve{J}^A_{i,i-1}\hat{J}^B_{j-1,j} y_{i-1,j-1} = 0. \tag{A.19}$$

The desired solution is obtained from (A.14):

$$X = R^A Y L^B. \tag{A.20}$$

ACKNOWLEDGMENTS

This work was supported by the Russian Science Foundation, project no. 14-11-00046.

REFERENCES

1. R. H. Bartels and G. W. Stewart, "Solution of the matrix equation $AX + XB = C$," Commun. ACM **15**, 820–826 (1972).
2. G. H. Golub, S. Nash, and C. van Loan, "A Hessenberg–Schur method for the problem $AX + XB = C$," IEEE Trans. Automat. Control **24**, 909–913 (1979).
3. F. R. Gantmakher, *The Theory of Matrices* (Nauka, Moscow, 1988; Chelsea, New York, 1959).
4. V. V. Voevodin and Yu. A. Kuznetsov, *Matrices and Computations* (Nauka, Moscow, 1984) [in Russian].
5. Kh. D. Ikramov, *Numerical Solution of Matrix Equations,* Ed. by D. K. Faddeev (Nauka, Moscow, 1984) [in Russian].
6. M. Sh. Misrikhanov, *Invariant Control of Multidimensional Systems. Algebraic Approach* (Nauka, Moscow, 2007).
7. M. Sh. Misrikhanov, "Analytical solution of matrix Lyapunov equation. Algebraic approach," Vestn. Ivan. Energet. Univ., No. 4, 30–36 (2002).

Translated by A. Klimontovich