=== SOIL PHYSICS ===

Functional Description of Water-Retention Capacity and Relative Hydraulic Conductivity of the Soil Taking into Account Hysteresis

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Abstract—The aim of this study is to model the water-retention capacity and the relative hydraulic conductivity of the soil as a capillary-porous medium, as well as to verify the proposed models in comparison with the well-known world analogues. This aim is achieved by solving the following tasks: (1) description of the soil hydrophysical properties in the form of three systems of functions with corresponding sets of common parameters, (2) verification of these systems by evaluating the relative hydraulic conductivity and scanning branches of the water-retention capacity using parameters identified from the literature data on the main branches of the water-retention capacity hysteresis for the 3501 Rubicon sandy loam; (3) application of the equality of exponential parameter values in the calculations of the drying and wetting branches of hysteresis loop to eliminate the artificial "pump effect"; (4) study of the influence of the additive parameter on the errors of the point approximation of data on the main branches of the water-retention capacity, as well as on the errors of estimating the relative hydraulic conductivity and scanning branches of the water-retention capacity hysteresis; and (5) identification of significant differences between the errors of these estimates according to the Williams–Kloot criterion for choosing the best system of functions. Application of the proposed models seems to be preferable for solving the problems of irrigation farming, such as forecasting the water availability of crops and calculating precision irrigation rates.

Keywords: mathematical model, point approximation, "pump effect", hydrophysical soil properties

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INTRODUCTION

The soil water saturation is estimated by the effective water saturation $S_e = (\theta - \theta_r)/(\theta_s - \theta_r)$, where θ (cm^3 cm^{-3}) is the volumetric water content; θ_s (cm^3 cm^{-3}) is the saturated volumetric water content; and θ_r (cm^3 cm^{-3}) is the residual water content corresponding to the minimum specific volume of water as a liquid in the soil. The hydrophysical properties of the soil include its water-retention capacity and hydraulic conductivity. Water-retention capacity is described as the dependence of S_e or θ on the capillary pressure (matric potential) of soil moisture ψ (cm H$_2$O) ($\psi < 0$ in unsaturated soils). To describe the hydraulic conductivity, the dependence of the hydraulic conductivity coefficient k (cm day^{-1}) on the values of S_e, θ, or ψ is used. The maximum k value is equal to the saturated hydraulic conductivity coefficient k_s in the soil (cm day^{-1}). The k/k_s ratio is called the relative hydraulic conductivity of the soil. An ambiguous character of the dependences describing the hydrophysical properties of the soil as functions, the arguments of which are ψ values, is due to the hysteresis phenomenon [22, 25, 27, 28].

The dependences $\theta(\psi)$ for various intervals of ψ values based on physical concepts are presented in the literature, e.g., an exponential model [5, 6, 8]. The curve, which graphically represents an exponential model, is not S-shaped in the range of film and capillary moisture. For this range of the soil water content, to date, no comprehensive substantiated mathematical description of the hydrophysical properties of soil as a generalized system of functions has been proposed [10]. Therefore, regression and other empirical functions are usually applied to describe the dependences $S_e(\psi)$ and $k(S_e)$ [2, 3, 7]. In most cases, these functions are used for point approximation of direct measurement data, for example, using the Levenberg–Marquardt algorithm to minimize the squared deviations of the calculated values from the experimental data [21, 23]. These measurements are very laborious. Their complexity especially increases when measuring the scanning hysteresis branches. At the same time, to solve a number of problems in soil hydrophysics, not

only direct measurements but also a functional representation of the dependences $S_e(\psi)$ and $k(S_e)$ are required. These tasks include, for example, predicting the water supply of crops and calculating precision irrigation rates.

A parabolic partial differential equation (Richards equation) [32], where one of the coefficients is the reduced differential soil water capacity function $dS_e/d\psi$, is used to solve the first problem. The application of regression dependences (including power polynomials) and other empirical functions [2, 8], which approximate experimental data, often leads to physically absurd results (for example, when calculating the derivative $dS_e/d\psi$, since differentiation is not a stable operation with respect to approximations). This problem becomes especially acute if numerical methods with iterative procedures are used to solve the Richards equation [30].

To solve the second problem, data on the scanning wetting branches of the $S_e(\psi)$ hysteresis are required. The scanning wetting branch is known to start at the turning point from the previous drying branch. This point is described by a certain pair of S_e and ψ values corresponding to the soil water content immediately before irrigation. The scanning wetting branch ends at the turning point to the next drying branch: this point is described by another pair of S_e and ψ values corresponding to the soil water content accumulated during irrigation.

As measuring scanning branches is a highly laborious procedure, a method for calculating irrigation rates on the basis of the difference between the field water capacity and the pre-irrigation soil water content is often used in irrigation farming. Here, the field water capacity is usually determined by the main drying branch $S_e(\psi)$. On this curve, S_e reaches its maximum in comparison with any other hysteresis branch at a critical ψ value, which corresponds to the largest water supply retained by the soil. However, during soil wetting, the change in the state of soil water is described by a certain scanning wetting branch of the water-retention capacity, and not by the main drying branch [35]. Therefore, when calculating the irrigation rate using this method, the result may be overestimated. For a more accurate result, a certain (lower) S_e value on the corresponding scanning wetting branch should be used. The empirical dependence of Voronin [1] can be used to determine this (lower) S_e value.

There is a problem of unpredictability of atmospheric precipitation in the upcoming growing season (on an agricultural field) along with the laboriousness of measuring the scanning branches of $S_e(\psi)$. It is impossible to predict which scanning wetting branches will be required to calculate the precision irrigation rates. In this case, the application of a mathematical model of the hysteresis $S_e(\psi)$ has no alternative. Therefore, the development of methods to reduce the

number of direct measurements of the soil hydrophysical properties [40], as well as the construction of physically based mathematical models, which are necessary for predicting the water supply of agricultural crops and calculating precision irrigation rates are highly relevant.

Our study is aimed at a functional description of the hydrophysical properties of soil as a capillary-porous medium in the form of mathematical models, its verification, and its comparison with analogues by the example of data from literature [24]. This aim is achieved by solving the following tasks:

• A description of the dependences $S_e(\psi)$ and $k(S_e)/k_S$ as three systems of functions with the corresponding sets of common parameters (the use of common parameters makes it possible to estimate the $k(S_e)/k_s$ function values by the experimental data $S_e(\psi)$ and reduce the number of direct measurements);

• Taking into account the hysteresis and representing the $S_e(\psi)$ dependence as a function with two sets of parameters for (a) drying and (b) wetting branches. The application of such sets of parameters allows estimating the scanning branches according to the data on the main hysteresis curve $S_e(\psi)$ and reduces the number of direct measurements;

• Identification of the parameters of three systems of functions by point approximation of data from the literature on the main branches of the $S_e(\psi)$ hysteresis;

• Verification of the three systems of functions by estimating the $k(S_e)/k_s$ values and the scanning branches of $S_e(\psi)$ using the parameters identified from the data on the main branches of the hysteresis $S_e(\psi)$;

• Revealing the significant differences between the estimation errors according to the Williams−Kloot criterion [4] to select the best model.

OBJECTS AND METHODS

Currently, the water-retention capacity and relative hydraulic conductivity functions proposed by Van Genuchten are widely used to describe the hydrophysical properties of soil [39]:

$$S_e(\psi) = \begin{bmatrix} (1 + (-\alpha\psi)^n)^{-(1-1/n)}, & \psi < 0; \\ 1, & \psi \geq 0, \end{bmatrix} \quad (1)$$

$$k(S_e)/k_s = \begin{bmatrix} \sqrt{S_e}(1 - (1 - S_e^{1/(1-1/n)})^{(1-1/n)})^2, & \theta < \theta_s; \\ 1, & \theta = \theta_s, \end{bmatrix} \quad (2)$$

where $n > 1$ and α (cm H_2O^{-1}) are formal parameters.

The Mualem−Van Genuchten method is the evaluation method of the relative hydraulic conductivity values from data on the water-retention capacity of the soil using relations (1) and (2) [26, 39]. The functions described by relations (1) and (2) can be designated as the WRC-VG (WRC, water-retention capacity according to Van Genuchten) and RHC-MVG (RHC, rela-

tive hydraulic conductivity according to Mualem−Van Genuchten) functions. An important advantage of these functions is that they have common parameters. However, the formality of n and α parameters is also the reason of disadvantages of these functions. The first disadvantage is the possibility of the above-mentioned differentiation of the approximation. The second disadvantage is the difficulty of the RHC-VG estimation beyond the ψ interval, for which the $\theta(\psi)$ dependence is measured. The third disadvantage is the restrictive condition $n > 1$. As this takes place at small values of the n parameter, there is a significant increase in the estimation error of the relative hydraulic conductivity values according to the water-retention capacity data on soils using formulas (1) and (2). This was noted by Van Genuchten for an alluvial clay soil (1006 Beit Netofa clay) from the Mualem catalog ($k_s = 9.5 \times 10^{-7}$ cm s^{-1}) [24, 31].

It should be emphasized that Van Genuchten did not associate the dubious result for this soil with the use of the condition $n > 1$. However, as shown in [9], this is one of the reasons for the rather high error of the Mualem−Van Genuchten method for the 1006 Beit Netofa clay.

In works [29, 36, 37] the hydrophysical functions of the soil were formulated, based on the concept of soil as a capillary-porous medium. Here, the following relationships are presented:

$$S_e(\psi) = \begin{bmatrix} \dfrac{1}{2}\mathrm{erfc}\left(\dfrac{n\sqrt{\pi}}{4}\ln(-\alpha(\psi - \psi_e)) \right), & \psi < \psi_e; \\ 1, & \psi \geq \psi_e, \end{bmatrix} \quad (3)$$

$$k(S_e)/k_s$$
$$= \begin{bmatrix} \dfrac{\sqrt{S_e}}{4}\left(\mathrm{erfc}\left(\mathrm{inverfc}(2S_e) + \dfrac{2}{n\sqrt{\pi}} \right) \right)^2, & \theta < \theta_s; \\ 1, & \theta \geq \theta_s, \end{bmatrix} \quad (4)$$

where $\mathrm{erfc}(x) = 1 - \dfrac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)dt$ is a complementary error function; $\mathrm{inverfc}(\mathrm{erfc}(x)) = x$; ψ_e (cm H$_2$O) is an additive parameter that takes into account the hysteresis (ψ_e is the capillary air-entry pressure (bubbling pressure) for the drying branches ($\psi_e \leq 0$), and ψ_e is the capillary water-entry pressure for the wetting branches ($\psi_e \geq 0$)); α (cm H$_2$O^{-1}) is a multiplicative parameter: $\alpha = -1/(\psi_0 - \psi_e)$, ψ_0 (cm H$_2$O) is the capillary water pressure, at which the probability density distribution by values of normally distributed random variable $\ln\left((\psi - \psi_e)/(\psi_0 - \psi_e)\right)$ with zero general mean and standard deviation σ reaches its maximum value, $\psi_0 < \psi_e$, and n is an exponential parameter: $n = 4/\left(\sigma\sqrt{2\pi}\right)$.

For functions (3) and (4), we use the WRC-KT and RHC-MKT abbreviations, respectively. In the partic-

ular case at $\psi_e = 0$, WRC-KT and RHC-MKT functions are reduced to the models proposed by Kosugi [17−20], which are denoted as WRC-KT$_0$ and RHC-MKT$_0$, respectively.

A continuous approximation of relations (3) and (4) in the class of elementary functions is presented in [9]:

$$S_e(\psi) = \begin{bmatrix} (1 + (-\alpha(\psi - \psi_e))^n)^{-1}, & \psi < \psi_e; \\ 1, & \psi \geq \psi_e, \end{bmatrix} \quad (5)$$

$$k(S_e)/k_s = \begin{bmatrix} \sqrt{S_e}\left(1 - (1 - S_e^{-1})\exp\left(\dfrac{8}{n\pi}\right)\right)^{-2}, & \theta < \theta_s; \\ 1, & \theta \geq \theta_s, \end{bmatrix} \quad (6)$$

where ψ_e, α, n are the same parameters as in relations (3) and (4).

For functions (5) and (6), we use the WRC-HT and RHC-MT abbreviations, respectively. In the particular case at $\psi_e = 0$, WRC-HT is reduced to the model proposed by Haverkamp et al. [14], which we denote as WRC-HT$_0$. For RHC-MT at $\psi_e = 0$, we use the RHC-MT$_0$ abbreviation. It should be noted that the pair of WRC-HT$_0$ and RHC-MT$_0$ functions is a mathematically correct solution to the Van Genuchten problem in its original formulation.

As noted in [11], the problem of interpreting the multiplicative parameter α of function (1) requires further research. Indeed, there is often a very dubious interpretation of this parameter in the literature, according to which it is the reciprocal of the bubbling pressure. The bubbling pressure is understood to be the capillary pressure of water at which the entry of air begins upon drying of the initially water-saturated soil, for which the equality $S_e = 1$ is true. For function (1), this equality does not hold at $\psi = -1/\alpha$ because: $S_e = 2^{-(1-1/n)}$, and the condition $n > 1$ is applied. Therefore, the degree of the reasonableness for this interpretation seems to be more than problematic. The problem of interpreting the multiplicative parameter α in function (1) as a value inversely proportional to the bubbling pressure is also illustrated by the experimental data [11]. During the point approximation of these data using functions (3) and (5), bubbling pressure value is adequately taken into account by the additive parameter ψ_e.

Much of the research on modeling the $S_e(\psi)$ hysteresis is a development of two well-known models: (i) the Scott et al. model [33] and (ii) the Kool and Parker model [16]. The WRC-HT$_0$ function is used in the first model; the second model is based on the WRC-VG function. We use the following abbreviations, namely: Hys-SHT$_0$ for the model of Scott and coauthors and Hys-KPVG for the Kool and Parker model.

The WRC-VG function used in the Hys-KPVG model is defined only on the interval $\psi \leq 0$. However,

as the main drying and wetting branches of the hysteresis can intersect at $\psi > 0$ (in the range of displacement of air trapped in dead-end pores) [12, 13, 15], the WRC-VG function fundamentally cannot fully describe the hysteresis phenomenon.

This study considers other three mathematical models of hysteresis along with the Hys-SHT$_0$ and Hys-KPVG models. For these models, the following designations are introduced: Hys-SKT, which is based on the WRC-KT function described by relation (3); Hys-SKT$_0$, which is based on the WRC-KT$_0$ function described by relation (3) at $\psi_e = 0$; Hys-SHT, which is based on the WRC-HT function described by relation (5).

In all five above-mentioned models, the calculation of the scanning curves is carried out according to the algorithm developed by Scott et al. [33]. For these five models of hysteresis, there is a possibility of an undesirable artificial "pump effect." Briefly, when ψ oscillates in a fixed interval, the intersection of the scanning and main hysteresis branches and physically absurd values of S_e are possible. According to the opinion of the authors of this article, (i) only scanning branches can intersect; (ii) scanning (primary) branches begin from the main branches, but scanning branches cannot end on the main branches; (iii) a closed loop can form between two intersection points of two adjacent scanning branches in a sequential order; and (iv) at each point on any curve, the derivative $dS_e/d\psi$ takes only two values, which correspond to the wetting and drying branches. It can be assumed that the most preferable way to prevent the "pump effect" is to apply the condition of equality of the exponential parameter n values for drying and wetting branches: $n_d = n_w$ (hereinafter, d is used for drying branches, and w is used for wetting branches).

The above functions $S_e(\psi)$ and $k(S_e)k_s$, as well as the hysteresis models of $S_e(\psi)$, are grouped into three systems:

• system 1 (WRC-VG, RHC-MVG, Hys-KPVG);

• system 2 (WRC-KT, RHC-MKT, Hys-SKT or WRC-KT$_0$, RHC-MKT$_0$, Hys-SKT$_0$ for the case $\psi_e = 0$);

• system 3 (WRC-HT, RHC-MT, Hys-SHT or WRC-HT$_0$, RHC-MT$_0$, Hys-SHT$_0$ for the case $\psi_e = 0$).

In a previous study [9], the advantage of systems 2 and 3 over system 1 has been shown by an example of the 1006 Beit Netofa clay with respect to estimation errors of the function $k(S_e)/k_s$. Here, the exponential parameter n values for systems 2 and 3 are less than 1 (for system 1, the restrictive condition $n > 1$ is used). However, the question remains: are there any possible advantages of systems 2 and 3 over system 1 at the exponential parameter n values > 1 for systems 2 and 3?

In addition, the following questions have to answered:

• Which of the compared systems has the smallest error in the point approximation of the experimental data on the main branches of the hysteretic water-retention capacity?

• Is the artificial (methodological) "pump effect" eliminated at $n_d = n_d$? Does this condition affect the estimation errors of $k(S_e)/k_s$ function $k(S_e)/k_s$

• Which of the systems has the smallest error of $k(S_e)/k_s$ estimation?

• Which of the compared systems has the smallest error in the estimates for the scanning branches of hysteresis $S_e(\psi)$?

• Does the use of the additive ψ_e parameter affect the point approximation errors of the experimental data on the main branches of the hysteretic water-retention capacity and the estimation errors of the $k(S_e)/k_s$ function and the scanning branches of hysteresis $S_e(\psi)$?

To obtain answers to these questions, the results of a computational experiment using data from an authoritative literary source on 3501 Rubicon sandy loam from the Mualem catalog [24, 38] are presented. The particle-size distribution of studied soil: sand (0.05−2.00 mm), 65.2%; silt (0.002−0.05 mm), 25.9%; clay (<0.002 mm), 8.9%. The soil bulk density is 1.35 g cm^{-3}; $\theta_s = 0.381$ cm^3 cm^{-3}; $k_s = 3.0 \times 10^{-4}$ cm s^{-1}.

RESULTS AND DISCUSSION

Table 1 contains the parameters of the three systems of hydrophysical functions identified by the point approximation of experimental data on the main drying and wetting branches of the water-retention capacity. For $n_d \neq n_w$, these parameters were calculated using the SoilHydrophysics−v.1.0 and SoilHysteresis−v.1.0 programs developed by the authors. For $n_d = n_w$, they were taken from [16] for system 1 and from [34] for systems 2 and 3.

Using the parameters from Table 1, we studied the condition $n_d = n_w$ to prevent the "pump effect." As an example, Fig. 1 shows the results of the study using the Hys-SHT model. The arrows in Fig. 1 show the scenarios for varying the ψ values. In the saturation zone, a sequential asymptotic approximation of closed loops formed by scanning curves occurs with oscillations of ψ in a fixed range. However, in the drying zone (Fig. 1a), the scanning drying branch crosses the main wetting branch. The fulfillment of the condition $n_d = n_w$ when identifying the parameters leads to the elimination of the marked intersection, which is confirmed by Fig. 1b.

The condition $n_d = n_w$ eliminating the "pump effect" can be considered justified at the absence of sig-

**Table 1.** Parameters of three systems of hydrophysical soil properties

Function system	θ_r, cm^3 cm^{-3}	θ_s, cm^3 cm^{-3}	$\psi_{e,\,d}$, cm H$_2$O	$\psi_{0,\,d}$, cm H$_2$O	α_d, cm H$_2$O^{-1}	$\psi_{e,\,w}$, cm H$_2$O	$\psi_{0,\,w}$, cm H$_2$O	α_w, cm H$_2$O^{-1}	n_d	n_w
					$n_d \neq n_w$					
1	0.1804	0.3810	—	−86.65	0.0115	—	−25.78	0.0388	6.266	3.311
2	0.1750	0.3810	−34.38	−90.50	0.0178	26.40	−34.86	0.0163	3.676	3.105
3	0.1728	0.3810	−36.37	−90.91	0.0183	24.19	−35.05	0.0169	3.668	3.117
2, $\psi_e = 0$	0.1840	0.3810	—	−89.53	0.0112	—	−30.77	0.0325	6.566	2.711
3, $\psi_e = 0$	0.1820	0.3810	—	−89.66	0.0112	—	−30.75	0.0325	6.794	2.731
					$n_d = n_w$					
1	0.1700	0.3810	—	−73.53	0.0136	—	−26.46	0.0378	3.301	3.301
2	0.1730	0.3810	−29.17	−93.03	0.0157	29.99	−35.58	0.0153	3.421	3.421
3	0.1750	0.3810	−18.05	−89.49	0.0140	26.99	−32.39	0.0168	3.679	3.679
2, $\psi_e = 0$	0.1886	0.3810	—	−89.33	0.0119	—	−26.95	0.0371	4.733	4.733
3, $\psi_e = 0$	0.1869	0.3810	—	−89.45	0.0112	—	−27.66	0.0361	4.724	4.724

nificant differences between the estimates of $k\left(S_e\right)/k_s$, which were obtained using the parameters identified in three ways: firstly, when using data on only one main drying branch; secondly, when using data on only one main wetting branch; and thirdly, when using data on both main branches of the hysteresis $S_e\left(\psi\right)$. Figure 2 shows the results of a computational experiment on the point approximation of the main branches of the hysteresis $S_e\left(\psi\right)$ and the estimation of the $k\left(S_e\right)/k_s$ values performed in three ways for each of the compared systems of hydrophysical functions (at $\psi_e \neq 0$ for systems 2 and 3).

Table 2 contains the errors of the point approximation of the experimental data on the main branches of the hys-

teresis $S_e\left(\psi\right)$ and the estimation errors of $k\left(S_e\right)/k_s$ and the scanning branches of hysteresis $S_e\left(\psi\right)$.

In supplementary materials, the results of a computational experiment with the Hys-SKT model on the point approximation of the main branches and the estimation of the scanning branches of hysteresis $S_e\left(\psi\right)$ are presented as an example (Fig. S1). Solid curves are calculation results, and round dots are data on the water-retention capacity of the soil. In addition, supplementary materials contain the results of revealing significant differences between the errors of the compared systems in relation to the (i) point approximation of experimental data on the main branches of the $S_e\left(\psi\right)$ hysteresis (Table 3), (ii) esti-
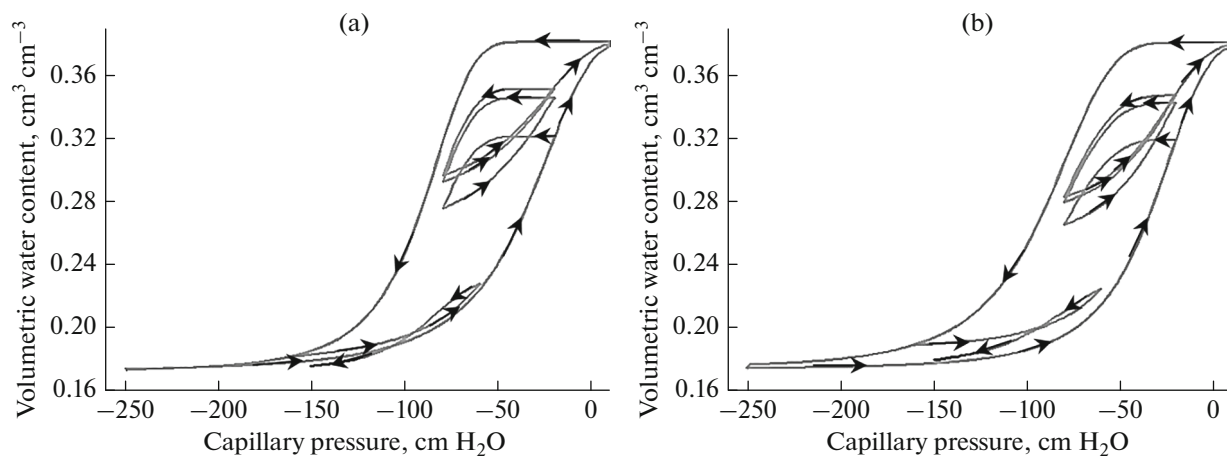
Fig. 1. Oscillation of the capillary pressure of soil moisture within fixed ranges using Hys-SHT at (a) $n_d \neq n_w$ and (b) $n_d = n_w$.
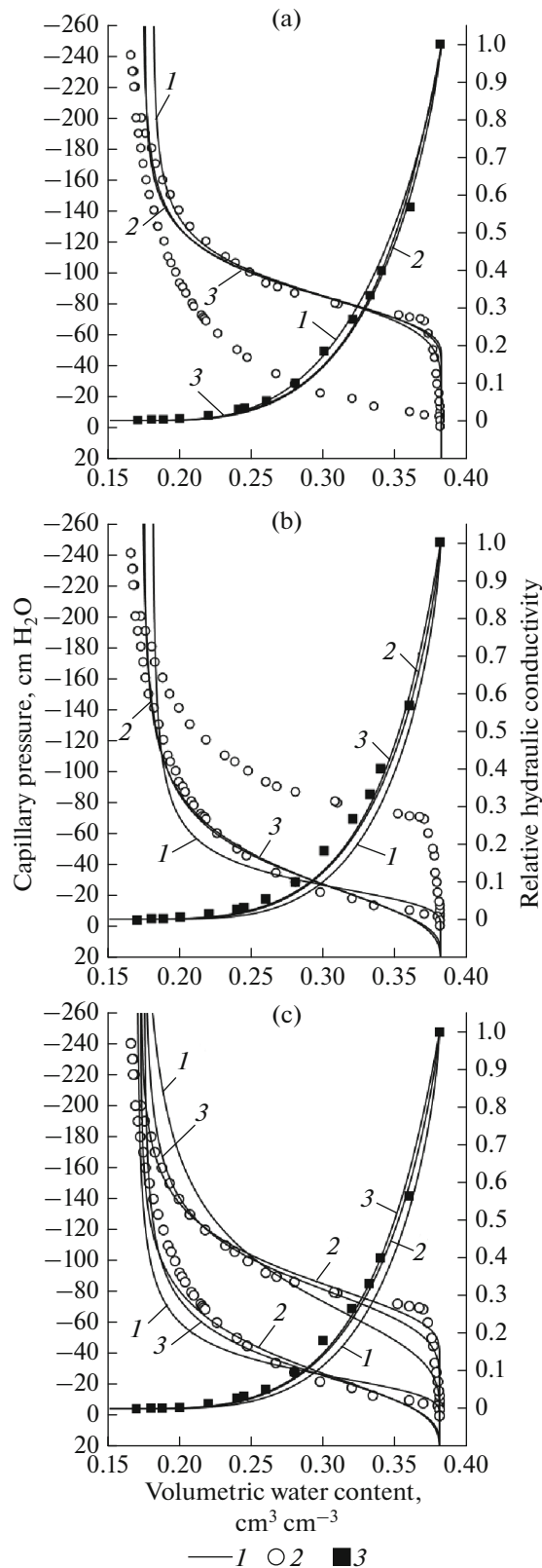
Fig. 2. Point approximation of the main branches $S_e(\psi)$ and estimation of $k(S_e)/k_s$ values using three systems of functions and parameters (a) n_d, (b) n_w, and (c) $n_d = n_w$: (*1*) calculation results, (*2*) data on water-retention capacity, and (*3*) data on the relative hydraulic conductivity of the soil.

mates of $k(S_e)/k_s$ (Table 4), and (iii) estimates of the scanning branches of hysteresis $S_e(\psi)$ (Table 5).

CONCLUSIONS

The description of the $S_e(\psi)$ and $k(S_e)/k_s$ dependences in the form of three systems of functions is presented in this study. Each system contains a corresponding set of common parameters. The parameters of these systems have been identified from the point approximation of published data on the main wetting and drying branches of the hysteretic water-retention capacity of the 3501 Rubicon sandy loam [24, 38]. Hysteresis loops for ψ oscillations in a given range of values for $n_d \neq n_w$ and $n_d = n_w$ have been constructed. The $k(S_e)/k_s$ values and values of the scanning branches of hysteresis $S_e(\psi)$ have been estimated using three systems of functions. The Williams−Kloot criterion has been applied to identify the differences between the estimation errors.

The results of this study allow us to make the following conclusions.

(1) With regard to the errors of the point approximation of data on the main branches of the hysteretic water-retention capacity:

• At $n_d \neq n_w$, system 1 is reliably inferior to systems 2 and 3 in the case of using the ψ_e parameter, while there are no significant differences between systems 2 and 3. System 3 is reliably superior to systems 1 and 2, if the ψ_e parameter is not used (at $\psi_e = 0$), while there are no significant differences between systems 1 and 2. The application of the ψ_e parameter reliably reduces the error of estimates;

• At $n_d = n_w$, system 1 is reliably inferior to systems 2 and 3 in the case of using the ψ_e parameter, while system 2 is superior to system 3. The compared systems are indistinguishable if the ψ_e parameter is not used (at $\psi_e = 0$). The application of the ψ_e parameter significantly reduces the error of estimates.

(2) With regard to the estimation errors of the relative hydraulic conductivity of the soil:

• For each of the compared systems, there are no significant differences between the errors of estimates obtained using the parameters identified from data on the main drying branch and from data on the main wetting branch;

• For each of the compared systems, there are no significant differences between the errors of estimates obtained using the parameters identified from data on the main drying branch only and from data on both main wetting and drying branches;

• For systems 1 and 2, the estimates obtained using the parameters identified from data on both main wetting and drying branches are more accurate than the estimates obtained using the parameters identified

Table 2. Root mean square error (RMSE) of deviations of the modeling results from the experimental data

Computational experiment	$n_d \neq n_w$					$n_d = n_w$				
	Systems of hydrophysical soil functions									
	1	2	3	$\psi_e = 0$		1	2	3	$\psi_e = 0$	$\psi_e = 0$
				2	3				2	3
Parameter identification by 74 points (dots) on the main branches	0.0117	0.0080	0.0081	0.118	0.0111	0.0204	0.0090	0.0113	0.0185	0.0182
Estimation of relative hydraulic conductivity for 15 points (n_d)	0.0262	0.0167	0.0203	0.0347	0.0420	0.0396	0.0184	0.0205	0.0200	0.0242
(n_w)	0.0484	0.0273	0.0245	0.0496	0.0436					
Estimation of scanning wetting branches for 33 points	0.0154	0.0094	0.0101	0.0164	0.0166	0.0148	0.0057	0.0100	0.0212	0.0194
Estimation of scanning drying branches for 26 points	0.0255	0.0182	0.0183	0.0267	0.0250	0.0177	0.0181	0.0135	0.0301	0.0284
Estimation of all scanning branches for 129 points	0.0205	0.0140	0.0143	0.0216	0.0208	0.0161	0.0127	0.0117	0.0255	0.0238

from data on the main wetting branch only, if ψ_e is used; However, there are no significant differences between the estimation errors obtained in this way, if ψ_e is not used (at $\psi_e = 0$);

• At $n_d = n_w$, system 1 is reliably inferior to system 2 at using the ψ_e parameter, while there are no significant differences between systems 1 and 3, as well as between systems 2 and 3. System 2 is reliably superior to system 3, if ψ_e is not used (at $\psi_e = 0$), while there are no significant differences between systems 1 and 2 and between systems 1 and 3. The use of the ψ_e parameter significantly reduces the errors of system 3;

• The absence of significant differences between the estimation errors obtained using the parameters identified in three ways indicates that $k(S_e)/k_s$ function is not hysteretic, in contrast to the $k(S_e(\psi))/k_s$ complex function, in which $S_e(\psi)$ is hysteretic;

• Estimation errors of systems 2 and 3 at $n > 1$, along with errors of previously obtained estimates of these systems at $n < 1$ [9] attest to a reliable advantage of systems 2 and 3 over system 1.

(3) With regard to the estimation errors of the scanning branches of $S_e(\psi)$ hysteresis:

• At $n_d \neq n_w$, system 1 is reliably inferior to systems 2 and 3 in the case of using the ψ_e parameter, while system 2 is superior to system 3. There are no significant differences between systems 1 and 3, if the ψ_e parameter is not used (at $\psi_e = 0$), while systems 1 and 3 are more accurate than system 2. The use of the ψ_e parameter reliably reduces errors;

• At $n_d = n_w$, system 1 is reliably inferior to systems 2 and 3 in the case of using the ψ_e parameter, while system 3 is superior to system 2. System 1 is more accurate than systems 2 and 3, if ψ_e is not used (at $\psi_e = 0$), while system 3 is superior to system 2. The use of the ψ_e parameter reliably reduces errors;

• The artificial (methodological) "pump effect" is eliminated at $n_d = n_w$, while there is no increase in the estimation errors of the $k(S_e)/k_s$ values.

(4) The multiplicative parameter α of system 1 is not a value inversely proportional to the bubbling pressure. This pressure is described by the additive ψ_e parameter of systems 2 and 3. Firstly, using ψ_e reliably reduces the errors of the point approximation of experimental data on the main branches of the $S_e(\psi)$ hysteresis and the estimation errors of the $k(S_e)/k_s$ values and the scanning branches of $S_e(\psi)$ hysteresis. Secondly, this allows describing the hysteresis phenomenon in the whole range of ψ values, including the positive region, where, as a rule, air trapped in dead-end pores is displaced at the final stage of soil saturation with water, and the main branches of the $S_e(\psi)$ hysteresis connect.

(5) Estimating the $k(S_e)/k_s$ values of system 3 at $\psi_e = 0$ using the parameters of the model proposed in [14] and identified by point approximation of the (ψ) data is a mathematically correct solution to the Van Genuchten problem in its original form [39]. The advantages of systems 2 and 3 make it possible to recommend these systems for modeling the hydrophysi-

cal properties of soil and solving problems of irrigation farming. In system 3, $S_e(\psi)$ and $k(S_e)/k_s$ relations are formulated in a rather simple form using elementary mathematical functions. Moreover, in many cases, the estimation errors obtained using systems 2 and 3 are indistinguishable. Therefore, the authors of this article give preference to system 3 (WRC-HT, RHC-MT, Hys-SHT at $n_d = n_w$) with a physically interpreted additive ψ_e parameter.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

SUPPLEMENTARY INFORMATION

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Fig. S1. Point approximation of data on the main branches and estimation of scanning wetting branches using the Hys-SKT model: (a) five wetting branches at $n_d \neq n_w$; (b) four drying branches at $n_d \neq n_w$; (c) five wetting branches at $n_d = n_w$; and (d) four drying branches at $n_d = n_w$.

Table S1. Comparison of the errors of the point approximation of the experimental data on the main branches of the hysteretic soil water-retention capacity according to the Williams−Kloot criterion for confidence probabilities of 0.95 and 0.975.

Table S2. Comparison of the estimation errors of the relative hydraulic conductivity of the soil according to the Williams−Kloot criterion for confidence probabilities of 0.95 and 0.975.

Table S3. Comparison of the estimation errors of the scanning branches of the hysteretic soil water-retention capacity according to the Williams−Kloot criterion for confidence probabilities of 0.95 and 0.975.

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