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Potential of Image Forces in a Dielectric Film

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Abstract—The potential of image forces for the charge in a three-layer medium (a dielectric film between two dielectrics) is considered in terms of electric fields (by definition) rather than potentials. It is demonstrated that the result differs from the usual result by a constant. Related issues are discussed.

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INTRODUCTION

The method of images for determining the electric field induced by a point charge near bodies of various shapes was developed by William Thomson (Lord Kelvin) [1–5]. Thomson applied it to metal bodies, but the method is easy to generalize for dielectrics. Let us consider an illustrative example of two semi-infinite media

$$\mathbf{1}: X < 0, \quad \mathbf{2}: X > 0$$

with permittivities $\varepsilon_1\varepsilon_0$ and $\varepsilon_2\varepsilon_0$ (ε_0 is the permittivity of vacuum, and ε_1 and ε_2 are the relative permittivities). If charge Q is located near the interface of two semi-infinite media at point x in medium **2**, the field in medium **2** is characterized in this method by the field of the charge itself ($Q\vec{R}/(4\pi\varepsilon_2\varepsilon_0 R^3)$ in the SI system, where $\vec{R} = (X - x, 0, 0)$ and $R = |\vec{R}|$) and the field that would be induced in medium **2** by point charge $k_{12}Q$ ($k_{12} = (\varepsilon_2 - \varepsilon_1)/(\varepsilon_2 + \varepsilon_1)$) located in medium **1** at a mirror symmetric point ($-x$, 0, 0) (i.e., this field is $k_{12}Q\vec{R}_1/(4\pi\varepsilon_2\varepsilon_0 R_1^3)$, where $\vec{R}_1 = (X + x, 0, 0)$) [1, 3]. Exactly this charge is the image of charge Q. The field in medium **1** is the same as the field that would be induced by charge k_1Q ($k_1 = 2\varepsilon_1/(\varepsilon_2 + \varepsilon_1)$) located in medium **2** at initial point (x, 0, 0) (i.e., the field is $k_1Q\vec{R}_2/(4\pi\varepsilon_1\varepsilon_0 R_2^3)$, where $\vec{R}_2 = (X - x, 0, 0)$). It can be demonstrated that the tangential components of these fields and the normal components of the electric flux density coincide (i.e., these field and flux density components are continuous and the electrostatic boundary conditions are thus satisfied) when the interface ($x = 0$) is approached from different directions. By virtue of the uniqueness theorem for electrostatic problems, this implies that these fields are the solutions of the considered problem. Note that the solution for the field is obtained this way, although it is easy to rewrite it in terms of the corresponding Coulomb potentials (which is usually performed). Note also that additional fields are actually produced by bound surface charges.

The method of images (referred to by Thomson as the "principle of images") can be generalized for the case of a point charge near a sphere (inside or outside it) and for certain more complex problems [1–5].

Charge Q is affected by image force $\vec{F}_{\text{im}}$ that may be put in correspondence with potential energy [6–8]

$$U(\vec{r}) = -\int_{\vec{a}}^{\vec{r}} \vec{F}_{\text{im}} d\vec{r}, \tag{1}$$

which is typically referred to as the image force potential (IFP). In the case of two semi-infinite media, the image force in medium **2** is directed along Ox and is written as

$$F_1 = k_{12}Q^2/(16\pi\varepsilon_2\varepsilon_0 x^2), \tag{2a}$$

while the IFP (traditionally measured relative to infinity) is

$$U_1 = k_{12}Q^2/(16\pi\varepsilon_2\varepsilon_0 x). \tag{2b}$$

Specifically, in the case of an interface with a metal ($\varepsilon_1 = \infty$, $k_{12} = -1$), the IFP is

$$U_0 = -Q^2/(16\pi\varepsilon_2\varepsilon_0 x). \tag{2c}$$

Note that the IFP is not the electrostatic potential (and does not satisfy the Laplace equation).

The IFP induces the well-known Schottky effect in thermal emission of electrons from solids (see studies [7, 8] and reviews [9, 10]).

In more complicated cases, one may use successive images as successive approximations and seek the solution in the form of a series [5]. Exactly this approach was used to calculate the image force potential in a three-layer medium (film) [11–13]; a metal–dielectric–metal (MDM) structure was considered in [12], and a more general case was discussed in [13]. The problems in these studies were analyzed in terms of potentials. The same problem is considered here in terms of fields; the results for the IFP differ by a constant.

1. IMAGE FORCE POTENTIAL FIELDS IN A FILM

Let us consider three media located successively in regions $x < 0$ (medium **1** with relative permittivity ε_1), $0 < x < d$ (medium **2** with ε_2), and $d < x$ (medium **3** with ε_3). Charge Q is located at point x in medium **2**. Let us first characterize the boundary conditions at interface **1/2**. We introduce (as was made above) the field that would be induced in medium **2** by charge $k_{12}Q$ located at point $-x$ in medium **1**. In order to satisfy the boundary condition for the field and the flux density of this charge at interface **2/3**, the field that would be induced in medium **2** by charge $k_{12}k_{32}Q$ at point $2d + x$ in medium **3** (here, $k_{32} = (\varepsilon_2 - \varepsilon_3)/(\varepsilon_2 + \varepsilon_1)$) should be considered. The boundary condition at interface **1/2** for this charge can be satisfied using charge $k_{12}(k_{12}k_{32})Q$ placed at point $-2d - x$ in medium **1**. Charge $(k_{12}k_{32})^2Q$ placed at point $4d + x$ in medium **3** is then needed; afterwards, charge $k_{12}(k_{12}k_{32})^2Q$ at point $-4d - x$ in medium **1**, and so on.

Thus, two sequences of images are needed in order to satisfy the boundary condition at interface **1/2**: these images are "charges" $k_{12}(k_{12}k_{32})^nQ$ at points $-2nd - x$ in medium **1**, which are located at distances $2nd + 2x$ from the initial charge ($n = 0, 1, 2, \ldots$), and "charges" $(k_{12}k_{32})^mQ$ at points $2md + x$ in medium **3** located at distances $2md$ from the initial charge ($m = 1, 2, \ldots$). These "charges" describe the field only in medium **2**. The fields at point x (field components along axis Ox) from the first and the second sequences are

$$\frac{Q}{4\pi\varepsilon_2\varepsilon_0}k_{12}\sum_{n=0}^{\infty}\frac{(k_{12}k_{32})^n}{(2nd+2x)^2} \tag{3a}$$

and

$$-\frac{Q}{4\pi\varepsilon_2\varepsilon_0}\sum_{m=1}^{\infty}\frac{(k_{12}k_{32})^m}{(2md)^2} \tag{3b}$$

(the series converge absolutely, and their terms can be rearranged). The minus sign in (3b) is an indication of the fact that two sequences of images are located on opposite sides of the initial charge.

Likewise, the boundary condition at interface **2/3** leads to "charges" $k_{32}(k_{32}k_{12})^rQ$ at points $2rd + (2d - x)$ in medium **3**, which are located at distances $2rd - 2(d - x)$ from the initial charge ($r = 0, 1, 2, \ldots$), and "charges" $(k_{32}k_{12})^sQ$ at points $-2sd + x$ in medium **1** located at distances $2sd$ from the initial charge ($s = 1, 2, \ldots$). These "charges" also describe the field only in medium **2**. The fields of these charges at point x are

$$-\frac{Q}{4\pi\varepsilon_2\varepsilon_0}k_{32}\sum_{r=0}^{\infty}\frac{(k_{32}k_{12})^r}{[2rd+2(d-x)]^2} \tag{3c}$$

and

$$\frac{Q}{4\pi\varepsilon_2\varepsilon_0}\sum_{s=1}^{\infty}\frac{(k_{32}k_{12})^s}{(2sd)^2}. \tag{3d}$$

Formulas (3a) and (3b) can be transformed into (3c) and (3d) by swapping indices 1 and 3 substituting x with $d - x$.

Note that individual terms of sums (3b) and (3d) and the sums themselves are equal in magnitude and have opposite signs; i.e., their overall contribution to the total field at point x is zero. The image force acting on the initial charge is then written as

$$F_2 = \frac{Q^2}{16\pi\varepsilon_2\varepsilon_0}\sum_{n=0}^{\infty}(k_{12}k_{32})^n \times\left[\frac{k_{12}}{(nd+x)^2}-\frac{k_{32}}{(nd+d-x)^2}\right]. \tag{4}$$

At $|k_{12}k_{32}| < 1$, the series converge absolutely, and the IFP can be defined as

$$U_2 = \frac{Q^2}{16\pi\varepsilon_2\varepsilon_0}\sum_{n=0}^{\infty}(k_{12}k_{32})^n \times\left(\frac{k_{12}}{nd+x}+\frac{k_{32}}{nd+d-x}\right)+A, \tag{5}$$

where constant A can be chosen by setting the reference point for the IFP. Note that infinities (in x) are inaccessible; therefore, we may choose, e.g., the center of the film as the reference point.

Similar calculations in terms of the potential result in the following expression [13]:

$$U_2' = \frac{Q^2}{16\pi\varepsilon_2\varepsilon_0}\sum_{n=0}^{\infty}(k_{12}k_{32})^n \times\left[\frac{k_{12}}{nd+x}+\frac{k_{32}}{nd+d-x}+\frac{2k_{12}k_{32}}{(n+1)d}\right]. \tag{6}$$

The third terms in (6) are associated with those images at points $-2md + x$ and $2md + x$ ($m = 1, 2, \ldots$) that produce opposite fields at the considered point x. They are taken into account in (6) as equal (in magnitude and sign) Coulomb potentials measured from infinity. Note in this respect that the terms for forces in (3b) and (3d) corresponding to the discussed images are independent of coordinate x. Therefore, they are unable to produce Coulomb potentials even at an intermediate stage in the process of transition from forces to the potential energy (during integration over the coordinate). Naturally, the *differences in the poten-*

tial energy between neighboring media are of greater interest. However, in order to determine these quantities, we should consider the continuous (without a stepwise change) dependence of the permittivity on coordinates at the interfaces [14].

Let us then consider the case of $|k_{12}\,k_{32}| = 1$, when the series in (5) do not converge absolutely and diverge at $k_{12}\,k_{32} = 1$. The latter case corresponds to an MDM structure with $k_{12} = k_{32} = -1$. If the IFP is measured from the film center ($x = d/2$), successive approximations yield the following expression for the IFP:

$$U_3 = -\frac{Q^2}{16\pi\varepsilon_2\varepsilon_0}\sum_{n=0}^{\infty}\left\{\frac{1}{nd+x} + \frac{1}{nd+d-x} - \frac{2}{d[n+(1/2)]}\right\}. \tag{7}$$

Naturally, in principle, the bracketed constants in (7) could be chosen differently (e.g., $-2/[(n+1)d]$, as in (6)), but the association with the reference point could be then lost. In the case of an MDM structure ($k_{12} = k_{32} = -1$), formula (6) takes the form

$$U_2'' = -\frac{Q^2}{16\pi\varepsilon_2\varepsilon_0}\sum_{n=0}^{\infty}\left[\frac{1}{nd+x} + \frac{1}{nd+d-x} - \frac{2}{d(n+1)}\right]. \tag{8}$$

This expression differs from (7) by the constant $\Delta = U_2'' - U_3$:

$$\Delta = -\frac{Q^2}{8\pi\varepsilon_2\varepsilon_0 d}\sum_{n=0}^{\infty}\left[\frac{1}{n+(1/2)} - \frac{1}{n+1}\right]. \tag{9}$$

It is evident that $\Delta = U_2''(x = d/2)$.

Note also that the sums in the considered formulas for the IFP have a characteristic form and can be expressed in terms of function $\psi(z) = d\ln\Gamma(z)/dz$, which appears in the theory of gamma-function $\Gamma(z)$ (see, e.g., [15]). Here,

$$\Gamma(z) = \int_0^{\infty}\exp(-t)t^{z-1}dt, \quad \mathrm{Re}\,z > 0.$$

This function was used to describe the electrostatic potential [16] and the IFP [11] in the case of a charge in a vacuum gap between two semi-infinite metals (in a "vacuum film"). Let us recall the following useful formulas (see, e.g., [17]):

$$\psi(z) = -\gamma - \sum_{n=0}^{\infty}\left(\frac{1}{n+z} - \frac{1}{n+1}\right), \tag{10a}$$

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2\ , \tag{10b}$$

where $\gamma = 0.577...$ is the Euler constant [15]. It follows from (10a) and (10b) that

$$\sum_{n=0}^{\infty}\left[\frac{1}{n+(1/2)} - \frac{1}{n+1}\right] = 2\ln 2. \tag{11}$$

Then (see [11]),

$$\Delta = -\frac{Q^2}{4\pi\varepsilon_2\varepsilon_0 d}\ln 2. \tag{12}$$

The following formulas are valid in the case when the charge is located in medium **1** at point x ($x < 0$):

$$F_4 = \frac{Q^2}{16\pi\varepsilon_1\varepsilon_0} \times \left\{\frac{k_{12}}{x^2} - k_1k_2k_{32}\sum_{n=0}^{\infty}\frac{(k_{12}k_{32})^n}{[d(n+1)-x]^2}\right\}, \tag{13}$$

$$U_4 = \frac{Q^2}{16\pi\varepsilon_1\varepsilon_0} \times \left[-\frac{k_{12}}{|x|} + k_1k_2k_{32}\sum_{n=0}^{\infty}\frac{(k_{12}k_{32})^n}{d(n+1)+|x|}\right] + B, \tag{14}$$

where $k_2 = 2\varepsilon_2/(\varepsilon_2 + \varepsilon_1)$. If the IFP is measured from $x = -\infty$, constant B is zero; however, the reference point should be the same for all three media. In limiting cases, when $\varepsilon_1 = \varepsilon_2$ or $\varepsilon_2 = \varepsilon_3$, formulas (13) and (14) are reduced to standard expressions for two media.

2. ENERGY OF THE SYSTEM

Let us turn to an instructive example of the explicit relation between the IFP and the electrostatic energy of charges for the case of charge Q located in vacuum at point $(x,0,0)$ near a metal. Surface charge density $\sigma(\vec{\rho})$ at point $\vec{\rho} = (0, y, z)$ is written as

$$\sigma(\vec{\rho}) = -\frac{Qx}{2\pi(x^2+y^2+z^2)^{3/2}}.$$

Evidently, the field produced by surface charges at point $(0, \rho_0, 0)$ has only the y-component, which can be described by the improper integral

$$\mathscr{E}_1 = -\frac{1}{4\pi\varepsilon_0}\iint\frac{\sigma(\vec{\rho})\cos\theta}{\rho_1^2}dydz = -\frac{Q}{4\pi\varepsilon_0}\frac{\rho_0}{(\rho_0^2+x^2)^{3/2}},$$

where

$$\rho_1^2 = (y-\rho_0)^2 + z^2,$$
$$\cos\theta = (y-\rho_0)/\rho_1.$$

This field is equal in magnitude and opposite in sign to the surface component of the field of charge Q at this point. Thus, the total field in the considered plane is zero (as it should be).

Note that since the electrostatic potential has a break at charged surfaces, the Maxwell boundary condition for the discontinuity of normal components of the electric flux density is described, as it is demonstrated in the courses, in equations of mathematical physics (see, e.g., [18]) with the use of corresponding limits of normal derivatives of the potential (or, equivalently, the field limits). In this approach, the field at the interface itself (on the surface) remains undefined. In the present case, this field is equal to the normal component of the field of charge Q (i.e., it presses the surface charges against the metal surface).

Thus, if we mentally assume that a force opposite to the IFP force is applied to charge Q, the system of charges becomes in equilibrium (naturally, by virtue of the Earnshaw's theorem, a purely electrostatic equilibrium is impossible [19]). In the case of quasi-equilibrium motion of charge Q and surface charges, one can demonstrate using the condition of the force balance in equilibrium that the IFP (U_0) is equal to the electrostatic (Coulomb) potential energy $E = E_1 + E_2$ of the system. Here, E_1 is the energy of interaction between charge Q and the surface charges:

$$E_1 = \frac{1}{4\pi\varepsilon_0}\iint \frac{\sigma(\vec{\rho})Q}{\sqrt{x^2+y^2+z^2}}dydz = -\frac{Q^2}{8\pi\varepsilon_0 x}, \quad (15)$$

and E_2 is the energy of interaction of the surface charges with each other:

$$E_2 = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\iint\iint \frac{\sigma(\vec{\rho}_1)\sigma(\vec{\rho}_2)}{|\vec{\rho}_1 - \vec{\rho}_2|}d\vec{\rho}_1 d\vec{\rho}_2 = \frac{Q^2}{16\pi\varepsilon_0 x} = -E_1/2, \quad (16)$$

where $\vec{\rho}_i = (0, y_i, z_i)$ $(i = 1, 2)$. Thus, total electrostatic energy E is two times lower than E_1 $(E = E_1/2)$:

$$E = -\frac{Q^2}{16\pi\varepsilon_0 x} = U_0. \quad (17)$$

Naturally, it coincides with the IFP.

Note that E_1 is equal to the energy of interaction between charges Q and $-Q$ spaced by $2x$. The "extra" coefficient 2 in the denominator of the expression for the IFP is associated with the interaction of the surface charges with each other.

CONCLUSIONS

The potential of image forces in a three-layer system has been considered in terms of fields (by definition) rather than potentials. The result (see Eqs. (4), (5), (13), and (14)) differs from the standard result by a constant (see formula (12) for the vacuum "film").

The potential of image forces affects electron and ion processes near the interfaces of solids (specifically, in nanodimensional films). These processes are the processes of segregation, thermal emission, tunneling, etc. The common effect is barrier lowering under the influence of the IFP. However, the IFP raises the barrier near the interface in a medium with a higher permittivity value. This may affect, for example, the emission of electrons from systems with negative electron affinity.

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