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ELECTRODYNAMICS AND WAVE PROPAGATION

On the Solution of Waveguide Boundary Value Problems in the Absence of the Lorentz Calibration

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Abstract—The spectral method for the solution of boundary value electrodynamic problems without the use of the Lorentz calibration is proposed. The method involves autonomous decompositions of the electromagnetic field components, and the Galerkin procedure is applied directly to the Maxwell equations. The efficiency of the method is demonstrated for the example of a rectangular waveguide that is inhomogeneously filled.

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1. FORMULATION OF THE PROBLEM

There are no analytical (closed) solutions to boundary value electrodynamic problems excluding a very bounded number of cases. As a rule, these solutions can be represented by infinite sums as decompositions in the functional space that is determined by a functional space determined by the operator equation, which specifies the zero element of the functional space. Let φ and $\vec{A}$ be the scalar and vector potentials of the electromagnetic field and ε and μ be the medium permittivity and permeability. Then, when the Lorentz calibration

$$\mathrm{div}\vec{A} = -\varepsilon\mu\frac{\partial\varphi}{\partial t}, \qquad (1)$$

is used, the boundary value problems give rise to operator equations like the Helmholtz equation.

Actually, a corollary to relationship (1) consists of the waveguide equations

$$\Delta\varphi - \varepsilon\mu\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon}, \quad \Delta\vec{A} - \varepsilon\mu\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{j}, \qquad (2)$$

where ρ and $\vec{j}$ are the densities of volume free charges and currents.

The harmonic time dependence of Eq. (2) yields the Helmholtz equations. Expressing the scalar and vector potentials in terms of Hertz vectors $\vec{\Pi}^{e,m}$

$$\varphi = -\mathrm{div}\vec{\Pi}^e; \quad \vec{A} = \varepsilon\mu\frac{\partial\vec{\Pi}^e}{\partial t}, \quad \vec{A} = \mu\mathrm{curl}\vec{\Pi}^m,$$

we obtain the wave equations

$$\Delta\vec{\Pi}^{e,m} - \varepsilon\mu\frac{\partial^2\vec{\Pi}^{e,m}}{\partial t^2} = -\frac{\vec{P}}{\varepsilon}\left(\frac{\vec{M}}{\mu}\right), \qquad (3)$$

where $\vec{P}$ and $\vec{M}$ are the polarization and magnetization vectors that determine the functions of the electromagnetic field sources.

The expressions

$$\vec{E} = -\mathrm{grad}\varphi - \frac{\partial\vec{A}}{\partial t}, \quad \vec{H} = \frac{1}{\mu}\mathrm{curl}\vec{A}$$

form the coupling between the components of the electric and magnetic fields

$$\vec{E} = \mathrm{curlcurl}\vec{\Pi}^e - \mu\mathrm{curl}\frac{\partial\vec{\Pi}^m}{\partial t},$$
$$\vec{H} = \mathrm{curlcurl}\vec{\Pi}^m + \varepsilon\mathrm{curl}\frac{\partial\vec{\Pi}^e}{\partial t}. \qquad (4)$$

Thus, Lorentz calibration (1) forms the coupling between the components of the electromagnetic field in form (4).

As a rule, the Helmholtz operator equation is the zero element of a functional space in direct variation method [1], in particular, in the Galerkin method. When condition (1) is not imposed on electromagnetic fields, there is no operator coupling (3) and (4) between the fields. As a result, the system of the Maxwell equations can directly be present as the zero element of the functional space. In this case, the components of the electric and magnetic fields, which are not analytically coupled in form (4), can autonomously be decomposed in arbitrary orthogonal bases. The coupling between the field components will later on be established with the help of the Galerkin procedure

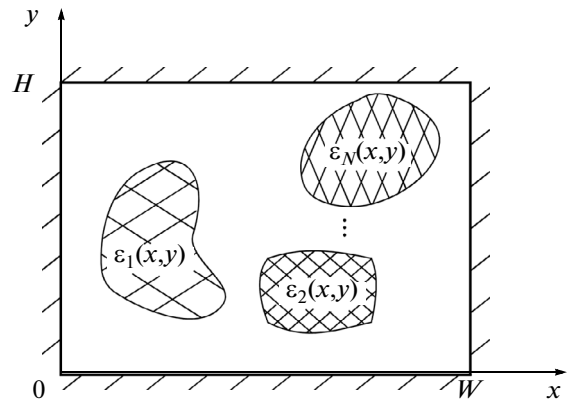

Fig. 1. General structure of the waveguide transverse section.

applied to the Maxwell equations. This approach expands the possibilities of direct variation methods. Let us formulate the spectral method for the calculation of guiding electrodynamic structures where Lorentz calibration condition (1) is not imposed on the fields.

As an example of realization of the proposed method, we consider a rectangular waveguide with a dielectric filling that is regular in longitudinal coordinate z and arbitrary in the transverse section (Fig. 1). The values of the filling permittivies can be complex as well.

Let us represent the dielectric filling in a rectangular waveguide in the form of a piecewise continuous function

$$\varepsilon(x,y)=\begin{cases}\varepsilon_1(x,y) \text{ on } S_1,\\ \varepsilon_2(x,y) \text{ on } S_2,\\ \ldots\\ \varepsilon_N(x,y) \text{ on } S_N,\\ \varepsilon \quad\quad \text{on } S_\perp\end{cases}$$

and write the Maxwell equations for the whole region inside the considered waveguide

$$\begin{aligned}\operatorname{curl}\vec{E} &= -i\omega\mu_0\vec{H},\\ \operatorname{curl}\vec{H} &= i\omega\varepsilon(x,y)\varepsilon_0\vec{E},\end{aligned} \tag{5}$$

where ε_0 and μ_0 are the permittivity and permeability constants.

We obtain from Eqs. (5) that

$$\operatorname{curl}\operatorname{curl}\vec{E} = k_0^2\varepsilon(x,y)\vec{E}, \tag{6}$$

where $k_0^2=\varepsilon_0\mu_0\omega$.

Assuming that the field depends on longitudinal coordinate z and the time as $\exp(-i\beta z + i\omega t)$ (i.e., $\vec{E}(x,y,z) = \vec{E}_\perp(x,y)\exp(-i\beta z)$), we obtain from (6) three equations for the x, y, and z components of the electric field

$$\begin{aligned}&\frac{\partial^2 E_x}{\partial y^2}+\left(k_0^2\varepsilon(x,y)-\beta^2\right)E_x-\frac{\partial^2 E_y}{\partial x\partial y}+i\beta\frac{\partial E_z}{\partial x}=0,\\ &\frac{\partial^2 E_y}{\partial x^2}+\left(k_0^2\varepsilon(x,y)-\beta^2\right)E_y-\frac{\partial^2 E_x}{\partial x\partial y}+i\beta\frac{\partial E_z}{\partial y}=0,\\ &\frac{\partial^2 E_z}{\partial x^2}+\frac{\partial^2 E_z}{\partial y^2}+k_0^2\varepsilon(x,y)E_z+i\beta\frac{\partial E_x}{\partial x}+i\beta\frac{\partial E_y}{\partial y}=0,\end{aligned} \tag{7}$$

where β is the longitudinal wave number.

We represent components of the electric field E_x, E_y, and E_z, which are functions of transverse coordinates, in the form of autonomous decompositions

$$\begin{aligned}E_x=\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}&\left(A_p\sin(\eta_p x)+\hat{A}_p\cos(\eta_p x)\right)\\ &\times\left(a_r\sin(\vartheta_r y)+\hat{a}_r\cos(\vartheta_r y)\right),\end{aligned} \tag{8a}$$

$$\begin{aligned}E_y=\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}&\left(B_p\sin(\delta_p x)+\hat{B}_p\cos(\delta_p x)\right)\\ &\times\left(b_r\sin(\chi_r y)+\hat{b}_r\cos(\chi_r y)\right),\end{aligned} \tag{8b}$$

$$\begin{aligned}E_z=\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}&\left(C_p\sin(\kappa_p x)+\hat{C}_p\cos(\kappa_p x)\right)\\ &\times\left(c_r\sin(\sigma_r y)+\hat{c}_r\cos(\sigma_r y)\right).\end{aligned} \tag{8c}$$

Here, A_p, $\hat{A}_p$, B_p, $\hat{B}_p$,C_p, $\hat{C}_p a_r$, $\hat{a}_r$, b_r, $\hat{b}_r$, c_r, and $\hat{c}_r$ are the decomposition coefficients; $\eta_p=\dfrac{\pi p}{H}$, $\vartheta_r=\dfrac{\pi(r+1)}{H}$, $\delta_p=\dfrac{\pi(p+1)}{W}$, $\chi_r=\dfrac{\pi r}{H}$, $\kappa_p=\dfrac{\pi(p+1)}{W}$, $\sigma_r=\dfrac{\pi(r+1)}{H}$, $p=0,1,\ldots,M$, $r=0,1,\ldots,M$.

It is principal that the decomposition coefficients in (8a) and (8b) are independent when calibration condition (1) is abandoned.

Making the electric field components obey the boundary conditions

$$\begin{aligned}&E_z\big|_{x=0}=0,\quad E_z\big|_{x=W}=0,\quad E_z\big|_{y=0}=0,\\ &E_z\big|_{y=H}=0,\quad E_x\big|_{y=0}=0,\quad E_x\big|_{y=H}=0,\\ &\left.\frac{\partial E_x}{\partial x}\right|_{x=0}=0,\quad \left.\frac{\partial E_x}{\partial x}\right|_{x=W}=0,\quad E_y\big|_{x=0}=0,\\ &E_y\big|_{x=W}=0,\quad \left.\frac{\partial E_y}{\partial y}\right|_{y=0}=0,\quad \left.\frac{\partial E_y}{\partial y}\right|_{y=H}=0\end{aligned} \tag{9}$$

and replacing the infinite limit of summation by finite number M, we bring expressions (8a) and (8b) to the form

$$\begin{aligned} E_x &= \sum_{p=0}^{M}\sum_{r=0}^{M} A_{p,r}\cos(\eta_p x)\sin(\vartheta_r y),\\ E_y &= \sum_{p=0}^{M}\sum_{r=0}^{M} B_{p,r}\sin(\delta_p x)\cos(\chi_r y),\\ E_z &= \sum_{p=0}^{M}\sum_{r=0}^{M} C_{p,r}\sin(\kappa_p x)\sin(\sigma_r y). \end{aligned} \tag{10}$$

In this case, expressions (10) are the decompositions in the orthogonal bases corresponding to boundary conditions (9), because there is no operator coupling of the field components in form (3).

Let us represent expressions (10) in the following form:

$$\begin{aligned} E_x &= \sum_{n=0}^{N} a_n\cos\left(\alpha_n^{(x)}x\right)\sin\left(\gamma_n^{(x)}y\right),\\ E_y &= \sum_{m=0}^{N} b_m\sin\left(\alpha_m^{(y)}x\right)\cos\left(\gamma_m^{(y)}y\right),\\ E_z &= \sum_{t=0}^{N} c_t\sin\left(\alpha_t^{(z)}x\right)\sin\left(\gamma_t^{(z)}y\right), \end{aligned} \tag{11}$$

where $N = (M+1)^2 - 1$. When ordinary summation is performed in (11), the whole spectrum of functions is taken into account in (10). Transverse coefficients $\alpha_n^{(x,y,z)}$ and $\gamma_n^{(x,y,z)}$ are written as follows: $\alpha_{p+(M+1)r}^{(x)} = \eta_p$, $\gamma_{p+(M+1)r}^{(x)} = \vartheta_r$; $\alpha_{p+(M+1)r}^{(y)} = \delta_p$, $\gamma_{p+(M+1)r}^{(y)} = \chi_r$; $\alpha_{p+(M+1)r}^{(z)} = \kappa_p$, $\gamma_{p+(M+1)r}^{(z)} = \sigma_r$. Here, $p = 0, 1, \ldots, M$ and $r = 0, 1, \ldots, M$. For example, when $M = 2$, quantities $\alpha^{(x)}$ and $\gamma^{(x)}$ are written as $\alpha^{(x)} = (\eta_0, \eta_1, \eta_2, \eta_0, \eta_1, \eta_2, \eta_0, \eta_1, \eta_2)^{\mathrm{T}}$ and $\gamma^{(x)} = (\vartheta_0, \vartheta_0, \vartheta_0, \vartheta_1, \vartheta_1, \vartheta_1, \vartheta_2, \vartheta_2, \vartheta_2)^{\mathrm{T}}$. Here, sign T means the conjugation operation.

Thus, the solution of the problem of calculation of arbitrarily filled waveguides is reduced to the representation of the electric field in form (11). The coupling between the x, y, and z field components in (11) is determined by coefficients a_n, b_m, and c_t, which are found from the Maxwell equations with substituted expressions (11) upon the Galerkin procedure is carried out. This process of determination of the decomposition coefficients and couplings of the field components is called the spectral method.

The substitution of expressions (11) into (7) yields the system of functional equations

$$\begin{aligned} &-\sum_{n=0}^{N} a_n\left[\left(\gamma_n^{(x)}\right)^2 + \beta^2\right]\cos\left(\alpha_n^{(x)}x\right)\sin\left(\gamma_n^{(x)}x\right)\\ &+ k_0^2\sum_{n=0}^{N} a_n\varepsilon(x,y)\cos\left(\alpha_n^{(x)}x\right)\sin\left(\gamma_n^{(x)}y\right)\\ &+ \sum_{m=0}^{N} b_m\alpha_m^{(y)}\gamma_m^{(y)}\cos\left(\alpha_m^{(y)}x\right)\sin\left(\gamma_m^{(y)}y\right)\\ &+ i\beta\sum_{t=0}^{N} c_t\alpha_t^{(z)}\cos\left(\alpha_t^{(z)}x\right)\sin\left(\gamma_t^{(z)}y\right) = 0, \end{aligned} \tag{12a}$$

$$\begin{aligned} &-\sum_{m=0}^{N} b_m\left[\left(\alpha_m^{(y)}\right)^2 + \beta^2\right]\sin\left(\alpha_m^{(y)}x\right)\cos\left(\gamma_m^{(y)}y\right)\\ &+ k_0^2\sum_{m=0}^{N} b_m\varepsilon(x,y)\sin\left(\alpha_m^{(y)}x\right)\cos\left(\gamma_m^{(y)}y\right)\\ &+ \sum_{n=0}^{N} a_n\alpha_n^{(x)}\gamma_n^{(x)}\sin\left(\alpha_n^{(x)}x\right)\cos\left(\gamma_n^{(x)}y\right)\\ &+ i\beta\sum_{t=0}^{N} c_t\gamma_t^{(z)}\sin\left(\alpha_t^{(z)}x\right)\cos\left(\gamma_t^{(z)}y\right) = 0, \end{aligned} \tag{12b}$$

$$\begin{aligned} &-\sum_{t=0}^{N} c_t\left[\left(\alpha_t^{(z)}\right)^2 + \left(\gamma_t^{(z)}\right)^2\right]\sin\left(\alpha_t^{(z)}x\right)\sin\left(\gamma_t^{(z)}y\right)\\ &+ k_0^2\sum_{t=0}^{N} c_t\varepsilon(x,y)\sin\left(\alpha_t^{(z)}x\right)\sin\left(\gamma_t^{(z)}y\right)\\ &- i\beta\sum_{n=0}^{N} a_n\alpha_n^{(x)}\sin\left(\alpha_n^{(x)}x\right)\sin\left(\gamma_n^{(x)}y\right)\\ &- i\beta\sum_{m=0}^{N} b_m\gamma_m^{(y)}\sin\left(\alpha_m^{(y)}x\right)\sin\left(\gamma_m^{(y)}y\right) = 0. \end{aligned} \tag{12c}$$

We project the system of equations (12a)–(12c) onto the chosen orthogonal basis, which means that we multiply (12a) by $\cos\left(\alpha_q^{(x)}x\right)\sin\left(\gamma_q^{(x)}y\right)$, $q = 0,..,N$; (12b) by $\sin\left(\alpha_q^{(y)}x\right)\cos\left(\gamma_q^{(y)}y\right)$, $q = 0,..,N$; (12c) by $\sin\left(\alpha_q^{(z)}x\right)\sin\left(\gamma_q^{(z)}y\right)$, $q = 0,\ldots,N$; and integrate over the section ($\int_0^H\int_0^W (\ldots)\,dxdy$).

Chosen basis (11) satisfies the orthogonality condition

$$\begin{aligned} N_{q,n}^{(x)} &= \int_0^H\int_0^W \cos\left(\alpha_n^{(x)}x\right)\sin\left(\gamma_n^{(x)}y\right)\cos\left(\alpha_q^{(x)}x\right)\\ &\times \sin\left(\gamma_q^{(x)}y\right)dxdy = \delta_{q,n}\frac{HW}{2}\begin{cases}1, \text{ when } \alpha_n^{(x)} = 0,\\ 0.5, \text{ when } \alpha_n^{(x)} \neq 0,\end{cases} \end{aligned} \tag{13a}$$

$$N_{q,n}^{(y)} = \int_0^H \int_0^W \sin\left(\alpha_n^{(y)} x\right) \cos\left(\gamma_n^{(y)} y\right) \sin\left(\alpha_q^{(y)} x\right) \times \cos\left(\gamma_q^{(y)} y\right) dxdy = \delta_{q,n} \frac{HW}{2} \begin{cases} 1, \text{ when } \gamma_n^{(y)} = 0, \\ 0.5, \text{ when } \gamma_n^{(y)} \neq 0, \end{cases} \tag{13b}$$

$$N_{q,n}^{(z)} = \int_0^H \int_0^W \sin\left(\alpha_n^{(z)} x\right) \sin\left(\gamma_n^{(z)} y\right) \sin\left(\alpha_q^{(z)} x\right) \times \sin\left(\gamma_q^{(z)} y\right) dxdy = \delta_{q,n} \frac{HW}{4}, \tag{13c}$$

where $\delta_{q,n}$ is the Kronecker symbol.

With allowance for (13a)–(13c), we obtain from (12a)–(12c) the system of matrix equations

$$\begin{gathered} -\left(\Gamma^{(x)} N^{(x)} + \beta^2 N^{(x)}\right) a + k_0^2 \mathbf{Q}^{(x)} a \\ + \mathbf{W}^{(x)} b + i\beta \mathbf{V}^{(x)} c = 0, \quad -\left(\Gamma^{(y)} N^{(y)} + \beta^2 N^{(y)}\right) \\ \times b + k_0^2 \mathbf{Q}^{(y)} a + \mathbf{W}^{(y)} a + i\beta \mathbf{V}^{(y)} c = 0, \\ -\Gamma^{(z)} N^{(z)} c + k_0^2 \mathbf{Q}^{(z)} c - i\beta \mathbf{W}^{(z)} a - i\beta \mathbf{V}^{(z)} b = 0, \end{gathered} \tag{14}$$

where the elements of matrices $\Gamma^{(i)}$, $\mathbf{Q}^{(i)}$, $\mathbf{W}^{(i)}$, and $\mathbf{V}^{(i)}$, $i = x, y, z$, are determined as follows:

$$\begin{gathered} \Gamma_{q,n}^{(x)} = \left(\gamma_n^{(x)}\right)^2 \delta_{q,n}, \quad \Gamma_{q,n}^{(y)} = \left(\alpha_n^{(y)}\right)^2 \delta_{q,n}, \\ \Gamma_{q,t}^{(y)} = \left[\left(\alpha_t^{(z)}\right)^2 + \left(\gamma_t^{(z)}\right)^2\right] \delta_{q,t}, \end{gathered} \tag{15a}$$

$$Q_{q,n}^{(x)} = \int_0^H \int_0^W \varepsilon(x,y) \cos\left(\alpha_n^{(x)} x\right) \sin\left(\gamma_n^{(x)} y\right) \times \cos\left(\alpha_q^{(x)} x\right) \sin\left(\gamma_q^{(x)} y\right) dxdy,$$

$$W_{q,m}^{(x)} = \alpha_m^{(y)} \gamma_m^{(y)} \int_0^H \int_0^W \cos\left(\alpha_m^{(y)} x\right) \sin\left(\gamma_m^{(y)} y\right) \times \cos\left(\alpha_q^{(x)} x\right) \sin\left(\gamma_q^{(x)} y\right) dxdy \tag{15b}$$

$$= \alpha_m^{(y)} \gamma_m^{(y)} \begin{cases} \frac{WH}{4}, \text{when } \alpha_m^{(y)} = \alpha_q^{(x)} \text{ and } \gamma_m^{(y)} = \gamma_q^{(x)}, \\ 0, \text{ when } \alpha_m^{(y)} \neq \alpha_q^{(x)} \text{ or } \gamma_m^{(y)} \neq \gamma_q^{(x)}, \end{cases}$$

$$V_{q,t}^{(x)} = \alpha_t^{(z)} \int_0^H \int_0^W \cos(\alpha_t^{(z)} x) \sin(\gamma_t^{(z)} y) \times \cos(\alpha_q^{(x)} x) \sin(\gamma_q^{(x)} y) dxdy \tag{15c}$$

$$= \alpha_t^{(z)} \begin{cases} \frac{WH}{4}, \text{ when } \alpha_t^{(z)} = \alpha_q^{(x)} \text{ and } \gamma_t^{(z)} = \gamma_q^{(x)}, \\ 0, \text{ when } \alpha_t^{(z)} \neq \alpha_q^{(x)} \text{ or } \gamma_t^{(z)} \neq \gamma_q^{(x)}, \end{cases}$$

$$Q_{q,m}^{(y)} = \int_0^H \int_0^W \varepsilon(x,y) \sin\left(\alpha_m^{(y)} x\right) \cos\left(\gamma_m^{(y)} y\right) \times \sin\left(\alpha_q^{(y)} x\right) \cos\left(\gamma_q^{(y)} y\right) dxdy, \tag{16a}$$

$$W_{q,n}^{(y)} = \alpha_n^{(x)} \gamma_n^{(x)} \int_0^H \int_0^W \sin(\alpha_n^{(x)} x) \cos(\gamma_n^{(x)} y) \times \sin(\alpha_q^{(y)} x) \cos(\gamma_q^{(y)} y) dxdy \tag{16b}$$

$$= \alpha_n^{(x)} \gamma_n^{(x)} \begin{cases} \frac{WH}{4}, \text{ when } \alpha_n^{(x)} = \alpha_q^{(y)} \text{ and } \gamma_n^{(x)} = \gamma_q^{(y)}, \\ 0, \text{ when } \alpha_n^{(x)} \neq \alpha_q^{(y)} \text{ or } \gamma_n^{(x)} \neq \gamma_q^{(y)}, \end{cases}$$

$$V_{q,t}^{(y)} = \gamma_t^{(z)} \int_0^H \int_0^W \sin\left(\alpha_t^{(z)} x\right) \cos\left(\gamma_t^{(z)} y\right) \times \sin\left(\alpha_q^{(y)} x\right) \cos\left(\gamma_q^{(y)} y\right) dxdy \tag{16c}$$

$$= \gamma_t^{(z)} \begin{cases} \frac{WH}{4}, \text{ when } \alpha_t^{(z)} = \alpha_q^{(y)} \text{ and } \gamma_t^{(z)} = \gamma_q^{(y)}, \\ 0, \text{ when } \alpha_t^{(z)} \neq \alpha_q^{(y)} \text{ or } \gamma_t^{(z)} \neq \gamma_q^{(y)}, \end{cases}$$

$$Q_{q,t}^{(z)} = \int_0^H \int_0^W \varepsilon(x,y) \sin\left(\alpha_t^{(z)} x\right) \sin\left(\gamma_t^{(z)} y\right) \times \sin\left(\alpha_q^{(z)} x\right) \sin\left(\gamma_q^{(z)} y\right) dxdy, \tag{17a}$$

$$W_{q,n}^{(z)} = \alpha_n^{(x)} \int_0^H \int_0^W \sin\left(\alpha_n^{(x)} x\right) \sin\left(\gamma_n^{(x)} y\right) \sin\left(\alpha_q^{(z)} x\right) \times \cos\left(\gamma_q^{(z)} y\right) dxdy \tag{17b}$$

$$= \alpha_n^{(x)} \begin{cases} \frac{WH}{4}, \text{ when } \alpha_n^{(x)} = \alpha_q^{(z)} \text{ and } \gamma_n^{(x)} = \gamma_q^{(z)}, \\ 0, \text{ when } \alpha_n^{(x)} \neq \alpha_q^{(z)} \text{ or } \gamma_n^{(x)} \neq \gamma_q^{(z)}, \end{cases}$$

$$V_{q,m}^{(z)} = \gamma_m^{(y)} \int_0^H \int_0^W \sin\left(\alpha_m^{(y)} x\right) \sin\left(\gamma_m^{(y)} y\right) \times \sin\left(\alpha_q^{(z)} x\right) \sin\left(\gamma_q^{(z)} y\right) dxdy \tag{17c}$$

$$= \gamma_m^{(y)} \begin{cases} \frac{WH}{4}, \text{ when } \alpha_m^{(y)} = \alpha_q^{(z)} \text{ and } \gamma_m^{(y)} = \gamma_q^{(z)}, \\ 0, \text{ when } \alpha_m^{(y)} \neq \alpha_q^{(z)} \text{ or } \gamma_m^{(y)} \neq \gamma_q^{(z)}. \end{cases}$$

We denote $c' = i\beta c$ and rewrite system of equations (14) in the form

$$\begin{gathered} \left(k_0^2 \mathbf{Q}^{(x)} - \Gamma^{(x)} N^{(x)} - \beta^2 N^{(x)}\right) a + \mathbf{W}^{(x)} b + \mathbf{V}^{(x)} c' = 0, \\ \mathbf{W}^{(y)} a + \left(k_0^2 \mathbf{Q}^{(y)} - \Gamma^{(y)} N^{(y)} - \beta^2 N^{(y)}\right) b + \mathbf{V}^{(y)} c' = 0, \\ \beta^2 \mathbf{W}^{(z)} a + \beta^2 \mathbf{V}^{(z)} b + \left(k_0^2 \mathbf{Q}^{(z)} - \Gamma^{(z)} N^{(z)}\right) c' = 0. \end{gathered} \tag{18}$$

System (18) can be represented in the form of the single matrix equation

$$\Delta\begin{bmatrix} a \\ b \\ c' \end{bmatrix} = 0,$$

where

$$\Delta = \begin{bmatrix} k_0^2\mathbf{Q}^{(x)} - \Gamma^{(x)}N^{(x)} - \beta^2 N^{(x)} & \mathbf{W}^{(x)} & \mathbf{V}^{(x)} \\ \mathbf{W}^{(y)} & k_0^2\mathbf{Q}^{(y)} - \Gamma^{(y)}N^{(y)} - \beta^2 N^{(y)} & \mathbf{V}^{(y)} \\ \beta^2\mathbf{W}^{(z)} & \beta^2\mathbf{V}^{(z)} & k_0^2\mathbf{Q}^{(z)} - \Gamma^{(z)}N^{(z)} \end{bmatrix}. \quad (19)$$

Equating the determinant of matrix Δ to zero, we obtain the dispersion equation of the waves of a rectangle waveguide with an arbitrary dielectric filling.

We note the particular properties of the proposed method. It is seen from formulas (15a)–(15c), (16a)–(16c), and (17a)–(17c) that dielectric filling function $\varepsilon(x, y)$ affect only matrices $\mathbf{Q}^{(x)}$, $\mathbf{Q}^{(y)}$, and $\mathbf{Q}^{(z)}$, which depend on neither frequency nor the value of longitudinal wave number β. Thus, for an arbitrarily complex structure, all matrices entering (19) are calculated only once, after which they are simply multiplied by k_0^2 and β^2 as it is seen from (19). This circumstance makes it possible to reduce substantially the time necessary for the solution of the dispersion equation in the plane (ω, β).

2. NUMERICAL REALIZATION OF THE METHOD

2.1. Calculation of a Rectangular Waveguide with an Angular Coordinate Dielectric Filling

To demonstrate the efficiency of the proposed method, we consider a test problem that makes it possible to compare it with the partial domain method (PDM). Let us apply the proposed method to the calculation of the guiding structure shown in Fig. 2 and compare the results with those shown in [2].

The parameters of the structure are as follows: $W = 8$ mm, $w = 5$ mm, $H = 10$ mm, $h = 6$ mm, and $\varepsilon = 6$. The calculation is performed by two methods: the proposed spectral method and PDM. In the case of the PDM, the analyzed structure is split in regions *1* and *2* (see Fig. 2), where the LM- and LE-wave apparatus is applied to represent the fields.

The solutions to the dispersion equation that are obtained by the two methods in various approximations are presented in the table and in Fig. 3.

In the table, we show the results of the solution of the dispersion problem obtained by the PDM and spectral method. In the first column, approximation order M is shown. Using this order, the problem is solved. The data of the table is graphically shown in Fig. 3.

It is seen from the table and Fig. 3 that the results converge to one and the same value, and they are close already for small approximations. When $M = 5$, the normalized propagation constants obtained by the two methods differ by only 0.2%.

The PDM and spectral method are applied to calculate the dispersion characteristics of the considered waveguide in the frequency range from 2 GHz to 16 GHz (Fig. 4). Curves *1*, *2*, *3*, and *4* correspond to $\mathrm{Re}(HE_1)$, $\mathrm{Re}(HE_2)$, $\mathrm{Re}(HE_3)$, and $\mathrm{Re}(HE^k)$, and curves *1'*, *2'*, *3'*, *4'*, and *5'* correspond to $\mathrm{Im}(HE_1)$, $\mathrm{Im}(HE_2)$, $\mathrm{Im}(HE_3)$, $\mathrm{Im}(HE^k)$, and $\mathrm{Im}(HE_4)$. The cal-

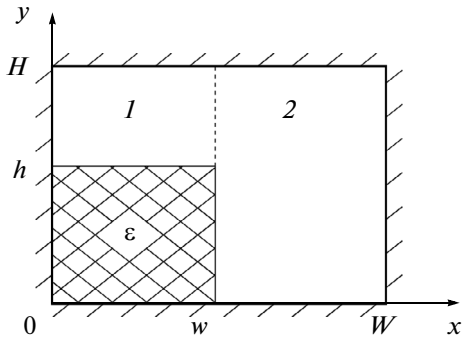

Fig. 2. Rectangular waveguide with the angular coordinate dielectric filling.

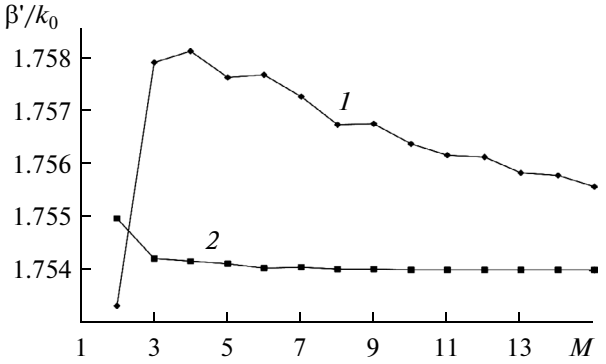

Fig. 3. Convergence of the (curve *1*) spectral method and (curve *2*) PDM at the frequency $f = 14.0$ GHz.

Results of calculation of the structure by the PDM and spectral method

M	Partial domain method β/k_0	Spectral method β/k_0
2	1.755013	1.753340
3	1.754223	1.757953
4	1.754177	1.758168
5	1.754127	1.757674
6	1.754045	1.757714
7	1.754059	1.757299
8	1.754024	1.756771
9	1.754026	1.756791
10	1.754022	1.756399
11	1.754013	1.756185
12	1.754017	1.756157
13	1.754009	1.755862
14	1.754010	1.755804
15	1.754009	1.755589

culation results obtained by the two methods practically coincide (the dispersion curves are graphically indistinguishable).

It is seen from the figure that the spectrum of the waveguide eigenwaves contain the complex waves (CWs) [3–5] whose dispersion characteristics start in the Jordan multiplicity points of wave numbers [5].

Thus, the effectiveness of the proposed method is demonstrated for the example of a rectangular waveguide partly filled with a dielectric.

2.2. Calculation of a Rectangular Waveguide with a Diagonal Filling

The proposed method is applied to calculate a rectangular waveguide diagonally filled by a dielectric. The parameters and dispersion characteristics of the calculated waveguide are given in Fig. 5. The notations of the curves in Fig. 5 correspond to those in Fig. 4. Note that it is rather problematic to calculate this guiding structure by any other method (in particular, the PDM).

As it is seen from Fig. 5, there are CWs in the spectrum of the eigenwaves of the considered waveguide.

It is known [5, 6] that CWs do not carry energy : in the transverse section, the positive power flux compensates the negative one. To check the compensation of the direct and reverse CW power fluxes, we numerically integrate the Umov–Poynting vector over two inner regions of the waveguide: over region S_1 filled by dielectric and region S_2 free from dielectric (Fig. 6). As a result, we obtain

$$\vec{\Pi}_1 = \oint_{S_1} \left(\vec{E} \times \vec{H}^*\right) d\vec{S} = \int_0^W \int_0^{y(x)} \left(\vec{E} \times \vec{H}^*\right) dy dx$$
$$= 6.27215453947621 - 1.52814418520953i,$$

$$\vec{\Pi}_2 = \oint_{S_2} \left(\vec{E} \times \vec{H}^*\right) d\vec{S} = \int_0^W \int_{y(x)}^H \left(\vec{E} \times \vec{H}^*\right) dy dx$$
$$= -6.27215453947629 + 1.52814418520947i,$$

where $y(x) = H - \dfrac{H}{W}x$ is the formula describing the interface between the dielectrics and the sign * denotes the complex conjugation operation.

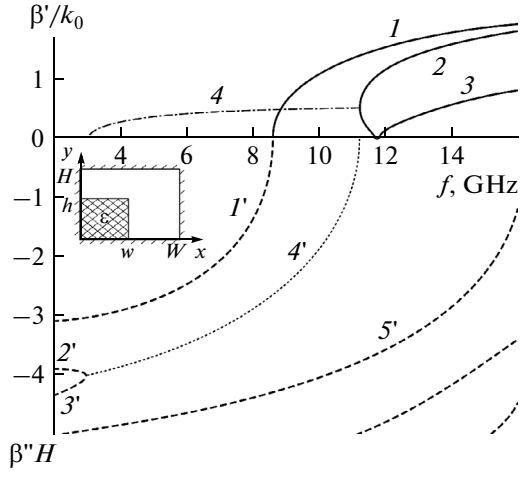

Fig. 4. Dispersion characteristics of the rectangular waveguide with the angular coordinate dielectric filling. The waveguide is shown in Fig. 2; $W = 8$ mm, $w = 5$ mm, $H = 10$ mm, $h = 6$ mm, and $\varepsilon = 6$.

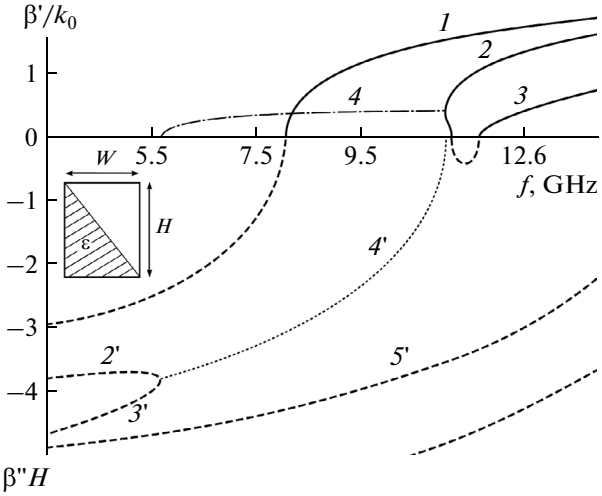

Fig. 5. Dispersion characteristics of the waveguide diagonally filled with a dielectric; $W = 8$ mm, $H = 10$ mm, and $\varepsilon = 6$.

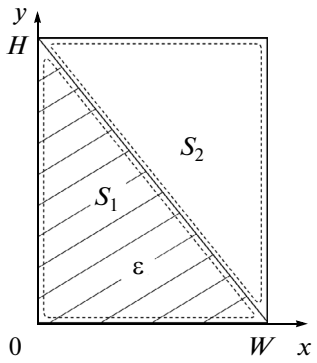

Fig. 6. Regions of integration of the power flux.

It is seen from the obtained results that the power fluxes in the two partial regions have opposite signs and modules equal to the 12th sign after the comma. The imperfection of the equality is due to the finite approximation applied to the solution of the dispersion problem and to the error of the numerical calculation of integrals. It is shown in [7] that the correctness of formulation and solution of problems in a nonclosed form can be checked by the convergence of the CW power flux to zero that is observed as the approximation number grows. In addition, the orthogonality of the waveguide eigenwaves is checked to confirm the correctness of the obtained results.

The proposed method is applied to calculate the longitudinal wave numbers of the five eigenwaves of the waveguide (in the sequence order of critical frequencies) at the frequency of 12.6 GHz:

$$\beta = \begin{pmatrix} 462.721360416437 \\ 357.384066437223 \\ 113.826952982819 \\ -i299.601303189174 \\ -i426.274899445668 \end{pmatrix} \mathrm{m}^{-1}.$$

The orthogonality of the waveguide eigenwaves is checked using the calculation of the determinant of the **Ort** matrix whose elements are determined as follows:

$$\mathrm{Ort}_{k,j} = \oint_S \left(\vec{E}_k \times \vec{H}_j\right) d\vec{S}$$

$$= \int_0^H \int_0^W \left(E_k^{(x)} H_j^{(y)} - E_k^{(y)} H_j^{(x)}\right) dxdy,$$

$$k = 0,\ldots,4 \quad j = 0,\ldots,4.$$

As a result, we obtain

Ort

$$= \begin{pmatrix} 1.00 & 1.34\times10^{-4} & -7.41\times10^{-5} & -9.46\times10^{-5}-9.46\mathrm{i}\times10^{-5} & -5.13\times10^{-5}-5.13\mathrm{i}\times10^{-5} \\ 7.52\times10^{-5} & 1.00 & 6.06\times10^{-5} & 7.74\times10^{-5}+7.74\mathrm{i}\times10^{-5} & -7.13\times10^{-5}-7.13\mathrm{i}\times10^{-5} \\ 4.03\times10^{-5} & 4.22\times10^{-6} & 1.00 & 2.52\times10^{-5}+2.52\mathrm{i}\times10^{-5} & -1.28\times10^{-4}-1.28\mathrm{i}\times10^{-4} \\ -6.48\times10^{-5}+6.48\mathrm{i}\times10^{-5} & 3.64\times10^{-5}-3.64\mathrm{i}\times10^{-5} & 2.64\times10^{-5}-2.64\mathrm{i}\times10^{-5} & 1.00 & 8.64\times10^{-6} \\ -5.01\times10^{-5}+5.01\mathrm{i}\times10^{-5} & -2.85\times10^{-5}+2.85\mathrm{i}\times10^{-5} & 2.73\times10^{-5}-2.73\mathrm{i}\times10^{-5} & -1.89\times10^{-6} & 1.00 \end{pmatrix}$$

It is seen that the matrix approaches the unit one and the maximum relative error does not exceed 10^{-4} (i.e., 0.01%).

CONCLUSIONS

The proposed method for calculation of waveguides inhomogeneously filled by a dielectric is universal and has the only limitation, which is in the fact that the outer metallized boundary must be a coordinate one. Since the Lorentz calibration is abandoned, the orthogonal basis of field representation is easily formed.

The method is preferred over the PDM in the solution of similar problems, because it does not need the continuous spectrum of eigenfunctions to be introduced in partial domains where the Sturm–Liouville boundary problem cannot be formulated [8].

The method is simple to algebraize dispersion problems. It is mathematically substantiated among a lot of computer methods and does not depend on the noncoordinate character of boundaries between the regions of the transverse section.

In the presented method, the number of functions of the electromagnetic field decomposition is limited instead of reducing the matrix equations.

The results of investigations of guiding structures that are based on the ideology described in this paper are published in [9–14].

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