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On the Asymptotics of Cosine Series in Several Variables with Power Coefficients[1]

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Abstract—The investigation of the asymptotic behavior of trigonometric series near the origin is a prominent topic in mathematical analysis. For trigonometric series in one variable, this problem was exhaustively studied by various authors in a series of publications dating back to the work of G.H. Hardy, 1928. Trigonometric series in several variables have got less attention. The aim of the work is to find the asymptotics of trigonometric series in several variables with the terms, having a form of "one minus the cosine" accurate to a decreasing power factor.

Keywords: trigonometric series in several variables, slowly decreasing coefficients, asymptotic behavior at zero, Hardy type asymptotics

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1. INTRODUCTION

The investigation of the asymptotic behavior of trigonometric series near the origin is a prominent topic in mathematical analysis. For trigonometric series in one variable, this problem was exhaustively studied by various authors in a series of publications dating back to the work of G.H. Hardy [4], 1928.

Trigonometric series in several variables have got less attention. The aim of the work is to partially fill this gap by finding the asymptotics of trigonometric series in several variables with the terms having a form of "one minus the cosine" accurate to a decreasing power factor.

Given a real number $\alpha > 0$, consider the function

$$F_d(\theta) = \sum_{z \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{\|z\|^{d+\alpha}} (1 - \cos\langle z, \theta\rangle), \qquad (1)$$

defined for $\theta \in \mathbb{R}^d$. Here, $\mathbb{Z}^d$ is the lattice of points from $\mathbb{R}^d$ with integer coordinates, $\langle\cdot,\cdot\rangle$ is the standard inner product and $\|\cdot\|$ is the max-norm in $\mathbb{R}^d$ which are defined by

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + \ldots + x_dy_d,$$
$$\|x\| = \max\{|x_1|, |x_2|, \ldots, |x_d|\},$$

where $x = \{x_1, x_2, \ldots, x_d\}$, $y = \{y_1, y_2, \ldots, y_d\}$.

The series in (1) is uniformly convergent for any $\alpha > 0$ and, therefore, function $F_d(\theta)$ is nonnegative and continuous, and $F_d(0) = 0$. We will be interested in the study of the asymptotic behavior of $F_d(\theta)$ as $\theta \to 0$.

For $d = 1$, function $F_d(\theta)$ can be represented in the form

$$F_d(\theta) = 2H_\alpha(\theta), \qquad (2)$$

where

$$H_\alpha(\theta) = \sum_{n=1}^{\infty} \frac{1}{n^{1+\alpha}}(1 - \cos n\theta), \quad \theta \in \mathbb{R},$$

and its asymptotics as $\theta \to 0$ can be described with the help of classical results going back to the work of Hardy [4] in which it was shown that for the functions

$$H_\alpha(\theta) = \sum_{n=1}^{\infty} a_n \cos n\theta, \quad g(\theta) = \sum_{n=1}^{\infty} a_n \sin n\theta,$$

where $0 < \alpha < 1$ and $n^\alpha a_n \to 1$, the following asymptotics as $\theta \to 0+$ are valid:

$$f(\theta) \simeq \Gamma(1-\alpha)\sin\left(\frac{\pi\alpha}{2}\right)\theta^{\alpha-1}, \qquad (3)$$

$$g(\theta) \simeq \Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\theta^{\alpha-1}. \qquad (4)$$

Here, $\Gamma(\cdot)$ is the gamma function, and, for functions $h_1(\theta)$ and $h_2(\theta)$, the notation $h_1(\theta) \simeq h_2(\theta)$ as $\theta \to \theta_0$ means that $h_1(\theta)/h_2(\theta) \to 1$ as $\theta \to \theta_0$.

[1] The article was translated by the author.

If $\alpha = 1$, that is, $na_n \to 1$, then, instead of (3) and (4), the following limit relationships [5] are valid:

$$f(\theta) \simeq |\ln\theta|, \quad g(\theta) \to \frac{\pi}{2},$$

as $\theta \to 0+$ [5]. In [1], it was shown that asymptotics (4) is also valid for $0 < \alpha < 2$, and, in [11, 12], the asymptotic behaviors of function $f(\theta)$ for all $\alpha > 1$ and of function $g(\theta)$ for all $\alpha \ge 2$ were analyzed.

So, in [1, 4, 5, 11, 12], the asymptotic behavior of functions $f(\theta)$ and $g(\theta)$ was completely investigated for all $\alpha > 0$. In particular, it follows from [11, 12] that

$$H_\alpha(\theta) \simeq H_\alpha^*(\theta), \tag{5}$$

where

$$H_\alpha^*(\theta) = \frac{1}{\alpha}\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)|\theta|^\alpha$$

for $0 < \alpha < 2$,

$$H_\alpha^*(\theta) = \frac{1}{2}\theta^2\ln\frac{1}{|\theta|}$$

for $\alpha = 2$, and

$$H_\alpha^*(\theta) = \frac{1}{2}\zeta(\alpha-1)\theta^2$$

for $\alpha > 2$, where

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

is the Riemann zeta function.

The quantity $\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)$ in the previous formulas is indeterminate when $\alpha = 1$ since $\Gamma(0) = \infty$ and $\cos\left(\frac{\pi}{2}\right) = 0$. This indeterminate form can be resolved by treating $\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)$ in the case $\alpha = 1$ as

$$\lim_{\alpha\to 1-}\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right) = \frac{\pi}{2}.$$

This indeterminate form can be resolved also with the help of the identity

$$\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right) \equiv \frac{\pi}{2\Gamma(\alpha)\sin\left(\frac{\pi\alpha}{2}\right)}.$$

Note that, in works [1, 4, 5, 11, 12], as well as in subsequent publications [2, 3, 6–8], one can find quite a number of deeper results than those mentioned earlier, part of which are included in monograph [13, Ch. 5].

From (2) and (5), it follows that

$$F_d(\theta) \simeq 2H_\alpha^*(\theta)$$

for $d = 1$ and each $\alpha > 0$. For $d \ge 2$ the asymptotic behavior of function $F_d(\theta)$ at zero is much less studied. It is known [9, 10] that function $F_2(\theta)$ has a lower bound of the order $\|\theta\|^\alpha$ near the origin.

In connection with this, the aim of the work is to study the asymptotic behavior of function $F_d(\theta)$ at zero for all $d \ge 2$.

2. MAIN RESULTS

For an arbitrary set of numbers $\theta_1, \theta_2, \ldots, \theta_d$, let us define the quantity

$$\omega(\theta_1, \theta_2, \ldots, \theta_d) = \int_{-\theta_d}^{\theta_d}\ldots\int_{-\theta_2}^{\theta_2}|\theta_1 + \eta_2 + \ldots + \eta_d|^\alpha d\eta_2\ldots d\eta_d$$

and consider the symmetric function

$$A_d(\theta) = \sum_{\{i_j\}}\frac{\omega(\theta_{i_1}, \theta_{i_2}, \ldots, \theta_{i_d})}{\theta_{i_2}\cdots\theta_{i_d}}, \tag{6}$$

where the summation is taken over all permutations $\{i_j\}$ of coordinates of the vector $\theta = \{\theta_1, \theta_2, \ldots, \theta_d\}$. If some of the variables θ_i in (6) vanishes, then, the value of $A_d(\theta)$ under the corresponding values of the argument is uniquely defined by continuity.

Let $|\theta|$ denote the Euclidean norm of the vector $\theta = \{\theta_1, \theta_2, \ldots, \theta_d\}$, that is $|\theta| = \sqrt{\theta_1^2 + \theta_2^2 + \ldots + \theta_d^2}$.

Theorem 1. *Let* $d \ge 2$. *Then*,

$$F_d(\theta) \simeq \frac{2}{\alpha}\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)A_d(\theta)$$

for $0 < \alpha < 2$,

$$F_d(\theta) \simeq \frac{2^{d-1}(d+2)}{3}|\theta|^2\ln\frac{1}{|\theta|}$$

for $\alpha = 2$, and

$$F_d(\theta) \simeq \frac{1}{6}\left(\sum_{n=1}^{\infty}\frac{(2n+1)^d(n+1)}{n^{d+\alpha-1}}\right)|\theta|^2 - \frac{1}{6}\left(\sum_{n=1}^{\infty}\frac{(2n-1)^d(n-1)}{n^{d+\alpha-1}}\right)|\theta|^2$$

for $\alpha > 2$.

Function $A_d(\theta)$ is positive for $\theta \ne 0$ and homogeneous of order α, that is, $A_d(t\theta) \equiv t^\alpha A_d(\theta)$ for $t \ge 0$.

Therefore, for $0 < \alpha < 2$, the claim of Theorem 1 can be represented also in the following form:

$$F_d(\theta) \simeq \frac{2}{\alpha}\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)A_d\left(\frac{\theta}{|\theta|}\right)|\theta|^{\alpha},$$

where $0 < A_d(\theta) < \infty$ for all θ satisfying $|\theta| = 1$, and, hence,

$$c_*|\theta|^{\alpha} \le F_d(\theta) \le c^*|\theta|^{\alpha}$$

for all θ with sufficiently small values of the norm $|\theta|$, where c_* and c^* are some positive constants.

Application of the methods of studying the asymptotic behavior of trigonometric series, developed in [1–8, 11, 12] for analyzing the 1D case, causes a certain difficulty in passing to dimensions $d \ge 2$. Therefore, to prove Theorem 1 we used a method of "reduction to dimension one", which allows to express function $F_d(\theta)$ as some explicit combination of 1D functions $F_1(\cdot)$, or rather functions $H_\alpha(\cdot)$, and thereby, to reduce analysis of the case we are interested in to the 1D case. A similar idea was used in [9, 10].

Definition (6) of the function $A_d(\theta)$ can be found inconvenient in practical applications, since it eventually requires the calculation of integrals determining the function $\omega(\theta_1, \theta_2, \ldots, \theta_d)$. Let us simplify definition (6) by taking advantage of the fact that the multiple integrals determining $\omega(\theta_1, \theta_2, \ldots, \theta_d)$ can be computed explicitly and expressed via the functions

$$x^{(m,\alpha)} = x^m|x|^{\alpha}.$$

From here, the next theorem about an alternative representation of function $A_d(\theta)$ follows.

Theorem 2. *For function $A_d(\theta)$, the following alternative representation holds*:

$$A_d(\theta) = \sum_{\{i_j\}}\frac{\nu(\theta_{i_1}, \theta_{i_2}, \ldots, \theta_{i_d})}{\theta_{i_2}\cdots\theta_{i_d}}, \qquad (7)$$

where summation is taken over all permutations $\{i_j\}$ of coordinates of the vector $\theta = \{\theta_1, \theta_2, \ldots, \theta_d\}$, and function $\nu(\cdot)$ is determined by the equality

$$\nu(\theta_1, \theta_2, \ldots, \theta_d) = \frac{1}{(\alpha+1)\ldots(\alpha+d-1)}$$
$$\times \sum_{s_2,\ldots,s_d=\pm1} s_2\cdots s_d(\theta_1 + s_2\theta_2 + \cdots + s_d\theta_d)^{(d-1,\alpha)}.$$

The resulting expression for function $A_d(\theta)$ is albeit more cumbersome than (6) but has the advantage that it does not require to calculate any integrals.

Let us present an example of calculating the asymptotic behavior of function $F_d(\theta)$ for $d = 2$ and $0 < \alpha < 2$ with the help of Theorems 1 and 2.

Example 1. Let $0 < \alpha < 2$, then,

$$F_2(\theta) \simeq \frac{2}{\alpha(\alpha+1)}\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\tilde{F}_2(\theta),$$

where

$$\tilde{F}_2(\theta) = \frac{1}{\theta_2}(\theta_1+\theta_2)|\theta_1+\theta_2|^{\alpha} - \frac{1}{\theta_2}(\theta_1-\theta_2)|\theta_1-\theta_2|^{\alpha}$$
$$+ \frac{1}{\theta_1}(\theta_2+\theta_1)|\theta_2+\theta_1|^{\alpha} - \frac{1}{\theta_1}(\theta_2-\theta_1)|\theta_2-\theta_1|^{\alpha}.$$

Now consider the case when function $F(\theta)$ is defined by a series of more general than (1) form.

Theorem 3. *Let $0 < \alpha \le 2$ and let*

$$F(\theta) = \sum_{z \in \mathbb{Z}^d\setminus\{0\}} a_z(1 - \cos\langle z, \theta\rangle), \quad \theta \in \mathbb{R}^d,$$

where

$$a_z\|z\|^{d+\alpha} \to 1 \text{ as } \|z\| \to \infty. \qquad (8)$$

Then, $F(\theta) \simeq F_d(\theta)$ *as* $\theta \to 0$.

Thus, for $0 < \alpha \le 2$ the asymptotics of $F(\theta)$ can be calculated explicitly by using Theorem 1 for $F_d(\theta)$. In the case $\alpha > 2$, one can only say that $F(\theta)$ has the same order of decrease at zero as function $F_d(\theta)$, that is $F(\theta) \sim F_d(\theta)$ as $\theta \to 0$. Here, for functions $h_1(x)$ and $h_2(x)$, we write $h_1(x) \sim h_2(x)$ as $x \to x_0$ if there exist c, $C \in (0, \infty)$ such that $c \le h_1(x)/h_2(x) \le C$ as $x \to x_0$.

If, in Theorem 3, the condition (8) is replaced by a less restrictive condition

$$0 < a_* \le a_z\|z\|^{d+\alpha} \le a^* < \infty, \quad z \in \mathbb{Z}^d\setminus\{0\},$$

then,

$$a_*F_d(\theta) \le F(\theta) \le a^*F_d(\theta),$$

and one can only say that the order of decrease of function $F(\theta)$ for each $\alpha > 0$ is the same as the order of decrease of function $F_d(\theta)$, that is, in this case, $F(\theta) \sim F_d(\theta)$ as $\theta \to 0$. However, it is difficult to obtain exact asymptotics for function $F(\theta)$ in this case.

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