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THEORY AND METHODS OF SIGNAL PROCESSING

Precise Point Positioning in Global Navigation Satellite Systems with Ambiguity Resolution of Phase Measurements[1]

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Abstract—Ambiguity resolution of phase measurements during Precise Point Positioning (PPP) in global navigation satellite systems (GNSSs) is implemented. The underlying feature of the design matrix of singular systems of GNSS linearized equations allowing one to estimate the user coordinates with ambiguity resolution of phase measurements is revealed. Details of the algebraic methods of overcoming the rank deficiency associated with the procedure are considered. Observation models at GPS and GLONASS initial frequencies are proposed and investigated. A filtering method allowing one to exclude the influence of ionospheric signal delays on estimates of user coordinates in processing of the measurement at initial GPS and GLONASS frequencies is described. Significant reduction of the PPP convergence period due to ambiguity resolution of phase measurements is shown. Comparison of the quality of user positioning with the use of decoupled satellite clocks calculated from global and local network solutions is presented.

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INTRODUCTION

Recently, the method of precise point positioning (PPP) has been actively developed in global navigation satellite systems (GNSSs) [1]. The PPP means estimation of precise user coordinates in the Earth-centered-Earth-fixed coordinate system with processing of only pseudorange (code) and phase measurements (observations) from user's navigation receiver. Currently, one can speak of the so-called global differential navigation satellite systems in which it is possible to distinguish the network solution (processing of measurements from the network of ground stations) and the user solution, i.e., the PPP.

The PPP is not an autonomous method; it requires precise ephemerides (satellite orbits and clocks) calculated in the network solution. The PPP method in GNSS is based on the use of precise ephemerides, compensation of a number of systematic biases in measurements and use of precise but ambiguous phase measurements. Today, the PPP errors in the postprocessing mode reach 1 cm or less for a static receiver and a few dm for a kinematic receiver [1].

The PPP method without taking into account the integer nature of ambiguities of phase measurements has become standard in GNSS, it is called the Float PPP method. In this method, integer ambiguities of phase measurements are estimated as float values, because they absorb unmodeled equipment biases [1]. The convergence period of the float PPP method to the 1-cm accuracy level is too long (several hours) for many applications. It is necessary to use ambiguity resolution of phase measurements during processing (Integer PPP method) to reduce it [2]. Implementation of the Integer PPP method requires separation of unmodeled equipment biases and integer ambiguities in the observation model. This procedure leads to singularity of the design matrix of the system of linearized GNSS equations, i.e., to rank deficiency.

Today, several approaches to implementation of the Integer PPP method are known [2–6]. We have chosen the method most theoretically substantiated, which is based on the decoupled clock model developed in Natural Resources Canada (NRCan) [2]. The approach assumes that the clocks for receiver and satellites in observation model are separated according to the measurement type and frequency. To remove the rank deficiency in the original systems of equations, the theory of S-transformations, which has been described in detail in the literature with application to geodetic networks but without focusing on specific GNSS features [7] is used. In this theory, all initial parameters to be estimated are grouped into linear combinations. The number of these combinations is less than the number of initial parameters; as a result, the singularity is removed. When this theory is used in the Integer PPP method, the number of estimated parameters is also reduced, but linear combinations

[1] The article was translated by the authors.

are created not for all initial parameters. Some parameters, including corrections to coarse user coordinates, are still estimated in their original form and linear combinations of phase ambiguities preserve their integer nature. Theoretic explanation of this feature and algebraic details of resolving the rank deficiency in the Integer PPP method are absent in the literature. The underlying feature of singular systems of equations in GNSS that allows implementation of the Integer PPP method will be formulated in this paper and specific details of implementation of the Integer PPP method for GPS and GLONASS will be investigated.

1. OBSERVATION MODELS WITH DECOUPLED CLOCKS FOR GPS AND GLONASS

Mathematic models of pseudorange and phase measurements of a navigation receiver with common (not decoupled) clocks are detailed in [8, 9]. Below, the observation model is meant as a set of receiver's measurements (combinations of measurements) that is used to solve some problem by means of measurement processing in GNSS. The linearized observation model with decoupled clocks at initial GPS frequencies described in [9, 10] can be written as follows:

$$\begin{aligned}
P_1^{G,j} &= R_C^{G,j} + h_X^j \Delta x + h_Y^j \Delta y + h_Z^j \Delta z \\
&+ m^j \Delta D_W + dT_{P1}^G - dt_{P1}^{G,j} + I_1^{G,j} + \varepsilon_{P1}^G, \\
P_2^{G,j} &= R_C^{G,j} + h_X^j \Delta x + h_Y^j \Delta y + h_Z^j \Delta z \\
&+ m^j \Delta D_W + dT_{P2}^G - dt_{P2}^{G,j} + k^G I_1^{G,j} + \varepsilon_{P2}^G, \\
L_1^{G,j} &= R_C^{G,j} + h_X^j \Delta x + h_Y^j \Delta y + h_Z^j \Delta z \\
&+ m^j \Delta D_W + dT_{L1}^G - dt_{L1}^{G,j} - I_1^{G,j} - \lambda_1^G N_1^{G,j} + \varepsilon_{L1}^G, \\
L_2^{G,j} &= R_C^{G,j} + h_X^j \Delta x + h_Y^j \Delta y + h_Z^j \Delta z \\
&+ m^j \Delta D_W + dT_{L2}^G - dt_{L2}^{G,j} - k^G I_1^{G,j} - \lambda_2^G N_2^{G,j} + \varepsilon_{L2}^G,
\end{aligned} \tag{1}$$

where $P_i^{G,j}$, $L_i^{G,j}$ are the pseudorange and phase measurements for GPS satellite j at frequency f_i^G ($i = 1, 2$) (m); $R_C^{G,j}$ is the geometric distance from the jth satellite to the point with coarse user coordinates (m); h_X^j, h_Y^j, and h_Z^j are the directions cosines; Δx, Δy, and Δz are the corrections to the coarse user coordinates (m); ΔD_W is the uncompensated wet component of vertical tropospheric delay (m); m^j is the mapping function for the jth satellite; dT_{P1}^G, dT_{P2}^G and dT_{L1}^G, and dT_{L2}^G are the pseudorange (code) and phase receiver clock errors relative to the GPS time scale, including receiver equipment biases in measurements $P_1^{G,j}$, $P_2^{G,j}$, $L_1^{G,j}$ and $L_2^{G,j}$ (m); $dt_{P1}^{G,j}$, $dt_{P2}^{G,j}$, $dt_{L1}^{G,j}$, and $dt_{L2}^{G,j}$ are the known from the network solution code and phase clock errors of the jth satellite relative to the GPS time scale, including satellite equipment biases in measurements $P_1^{G,j}$, $P_2^{G,j}$, $L_1^{G,j}$, and $L_2^{G,j}$ (m); $I_1^{G,j}$ is the slant ionospheric delay for the GPS satellite j at frequency f_1^G (m); $k^G = (f_1^G)^2/(f_2^G)^2$; $\lambda_1^G = c/f_1^G \approx 0.19$ (m) and $\lambda_2^G = c/f_2^G \approx 0.24$ (m) are the wavelengths of satellite carrier signals at frequency f_i^G; $N_1^{G,j}$, and $N_2^{G,j}$ are the integer ambiguities of phase measurements for the jth satellite at frequency f_i^G (cycles); and ε_{Pi}^G and ε_{Li}^G are the noise errors for corresponding measurements (m).

In (1) and further, it is supposed that measurements $P_i^{G,j}$, $L_i^{G,j}$ are corrected for systematic biases associated with relativistic and gravitational effects, phase center offsets, tidal effects, wind-up effect [1], and tropospheric signal delay. Multipath errors are considered to be insignificant and included in noise errors. Model (1) has $8 + 3M_{\text{sat}}$ (M_{sat} is the the number of processed satellites) estimated parameters: Δx, Δy, Δz, ΔD_W, dT_{P1}^G, dT_{P2}^G, dT_{L1}^G, dT_{L2}^G, $\lambda_1^G N_1^{G,1} \ldots \lambda_1^G N_1^{G,M_{\text{sat}}}$, $\lambda_2^G N_2^{G,1} \ldots \lambda_2^G N_2^{G,M_{\text{sat}}}$, and $I_1^{G,1} \ldots I_1^{G,M_{\text{sat}}}$. Due to the presence of decoupled clocks and ionospheric delays in (1), the corresponding system of equations is singular and, for $M_{\text{sat}} > 4$, the rank deficiency is three. For brevity, observation model (1) is further designated as the P1P2L1L2 model.

The linearized observation model with decoupled clocks at initial GLONASS frequencies can be written as

$$\begin{aligned}
P_1^{R,j} &= R_C^{R,j} + h_X^j \Delta x + h_Y^j \Delta y + h_Z^j \Delta z \\
&+ m^j \Delta D_W + dT_{P1}^R - dt_{P1}^{R,j} + I_1^{R,j} + \varepsilon_{P1}^R, \\
P_2^{R,j} &= R_C^{R,j} + h_X^j \Delta x + h_Y^j \Delta y + h_Z^j \Delta z \\
&+ m^j \Delta D_W + dT_{P2}^R - dt_{P2}^{R,j} + k^R I_1^{R,j} + \varepsilon_{P2}^R, \\
\frac{L_1^{R,j}}{\lambda_1^{R,j}} &= \frac{R_C^{R,j}}{\lambda_1^{R,j}} + \frac{h_X^j}{\lambda_1^{R,j}} \Delta x + \frac{h_Y^j}{\lambda_1^{R,j}} \Delta y \\
&+ \frac{h_Z^j}{\lambda_1^{R,j}} \Delta z + \frac{m^j}{\lambda_1^{R,j}} \Delta D_W + \frac{dT_{P1}^R}{\lambda_1^{R,j}} + dT_{L1}^R \\
&- dt_{L1}^{R,j} - \frac{1}{\lambda_1^{R,j}} I_1^{R,j} - N_1^{R,j} + \frac{\varepsilon_{L1}^R}{\lambda_1^{R,j}}, \\
\frac{L_2^{R,j}}{\lambda_2^{R,j}} &= \frac{R_C^{R,j}}{\lambda_2^{R,j}} + \frac{h_X^j}{\lambda_2^{R,j}} \Delta x + \frac{h_Y^j}{\lambda_2^{R,j}} \Delta y \\
&+ \frac{h_Z^j}{\lambda_2^{R,j}} \Delta z + \frac{m^j}{\lambda_2^{R,j}} \Delta D_W + \frac{dT_{P2}^R}{\lambda_2^{R,j}} + dT_{L2}^R \\
&- dt_{L2}^{R,j} - \frac{k^R}{\lambda_2^{R,j}} I_1^{R,j} - N_2^{R,j} + \frac{\varepsilon_{L2}^R}{\lambda_2^{R,j}},
\end{aligned} \tag{2}$$

where index R means that corresponding variables belong to GLONASS system; subscripts of corrected measurements $P_1^{R,j}$, $P_2^{R,j}$, $L_1^{R,j}$, and $L_2^{R,j}$ correspond to the number of the GLONASS frequency band (L1: 1598.0625–1605.375 MHz, L2: 1242.9375–1248.625 MHz); j is the number of the satellite frequency channel; $\lambda_1^{R,j}$ and $\lambda_2^{R,j}$ are the wavelengths of satellite carrier signals at frequencies $f_1^{R,j} = f_{L1}^{R,0} + j\Delta f_1$ and $f_2^{R,j} = f_{L2}^{R,0} + j\Delta f_2, j = -7,..., 6$ is the number of the satellite frequency channel, $f_{L1}^{R,0} = 1602$ MHz, $f_{L2}^{R,0} = 1246$ MHz, $\Delta f_1 = 0.5625$ MHz, and $\Delta f_2 = 0.4375$ MHz; $I_1^{R,j}$ is the ionospheric delay of the signal at frequency $f_1^{R,j}$ (m); dT_{P1}^R and dT_{P2}^R are the receiver code clock errors relative to the GLONASS time scale, including receiver equipment biases in measurements $P_1^{R,j}$ and $P_2^{R,j}$ (m); dT_{L1}^R and dT_{L2}^R are the receiver phase clock errors relative to GLONASS time scale, including receiver equipment biases in measurements $L_1^{R,j}/\lambda_1^{R,j}$ and $L_2^{R,j}/\lambda_2^{R,j}$ (cycles); $dt_{P1}^{R,j}$, $dt_{P2}^{R,j}$ (m) and $dt_{L1}^{R,j} = dt_{P1}^{R,j}/\lambda_1^{R,j} - \psi_{01}^j$, $dt_{L2}^{R,j} = dt_{P2}^{R,j}/\lambda_2^{R,j} - \psi_{02}^j$ (cycles) are the known from network solution code and phase clock errors of satellite j relative to the GLONASS time scale, including initial phases of satellite oscillators ψ_{01}^j, ψ_{02}^j in measurements $L_1^{R,j}/\lambda_1^{R,j}$ and $L_2^{R,j}/\lambda_2^{R,j}$; $k^R = (f_1^{R,j}/f_2^{R,j})^2$; $N_1^{R,j}$ and $N_2^{R,j}$ are the integer ambiguities of phase measurements $L_1^{R,j}/\lambda_1^{R,j}$ and $L_2^{R,j}/\lambda_2^{R,j}$ (cycles). Model (2) has $8 + 3M_{\text{sat}}$ estimated parameters: Δx, Δy, Δz, ΔD_W, dT_{P1}^R, dT_{P2}^R, dT_{L1}^R, dT_{L2}^R, $N_1^{R,1}...N_1^{R,M_{\text{sat}}}$, $N_2^{R,1}...N_2^{R,M_{\text{sat}}}$, and $I_1^{R,1}...I_1^{R,M_{\text{sat}}}$. Due to the presence of decoupled clocks and ionospheric delays in (2), the corresponding system of equations is singular; for $M_{\text{sat}} > 4$, the rank deficiency is three. Model (2) is based on the hypothesis of linearity of the phase response of navigation receiver $\psi_{\text{eq},i}^j = \psi_{\text{eq},i}^0 - \dot{\psi}_{\text{eq},i} f_i^{R,j}$, where $\psi_{\text{eq},i}^0$ is the phase bias for frequency $f_{Li}^{R,0}$, $i = \overline{1,2}$. According to the general circuit theory [11], slope $\dot{\psi}_{\text{eq},i}$ of the phase response of navigation receiver is a signal delay in the receiver equipment. The hypothesis of linearity of the receiver phase response means the its slope is a constant, i.e., the signal delays in navigation receiver equipment are constant for all GLONASS satellites radiating at different carrier frequencies. For this reason, in model (2), code clock errors dT_{P1}^R are dT_{P2}^R are present in not only code measurements but also phase measurements $L_1^{R,j}/\lambda_1^{R,j}$, $L_2^{R,j}/\lambda_2^{R,j}$. For brevity, observation model (2) is further designated as the GL_P1P2L2L2 model.

To exclude the influence of ionospheric delays, a GPS ionosphere-free observation model with decoupled clocks was developed at NRCan [2]:

$$\begin{aligned} P_3^{G,j} &= R_C^{G,j} + h_X^j\Delta x + h_Y^j\Delta y + h_Z^j\Delta z \\ &+ m^j\Delta D_W + dT_{P3}^G - dt_{P3}^{G,j} + \varepsilon_{P3}^G, \\ L_3^{G,j} &= R_C^{G,j} + h_X^j\Delta x + h_Y^j\Delta y + h_Z^j\Delta z \\ &+ m^j\Delta D_W + dT_{L3}^G - dt_{L3}^{G,j} - \lambda_3^G N_3^{G,j} + \varepsilon_{L3}^G, \\ A_4^{G,j} &= b_{r,A4}^G - b_{A4}^{j,G} - \lambda_4^G N_4^{G,j} + \varepsilon_{A4}^G, \end{aligned} \tag{3}$$

where $A_4^{G,j}$ is the Melbourne-Wubbena measurements combination [2], $\lambda_4^G N_4^{G,j} = \lambda_1^G N_1^{G,j} - \lambda_2^G N_2^{G,j}$ is the integer widelane phase ambiguity at the difference frequency $f_1^{R,j} - f_2^{R,j}$ (m); dT_{P3}^G, dT_{L3}^G and $dt_{P3}^{G,j}$, $dt_{L3}^{G,j}$ are the code and phase receiver and satellite j clock errors relative to the GPS time scale, including receiver and satellite equipment biases in ionosphere-free combinations of measurements $P_3^{G,j}$ and $L_3^{G,j}$ (m); $b_{r,A4}^G$ and $b_{A4}^{j,G}$ are the equipment biases for the receiver and the jth satellite in measurement $A_4^{G,j}$, ε_{P3}^G, ε_{L3}^G, and ε_{A4}^G are the noise errors for measurements $P_3^{G,j}$, $L_3^{G,j}$, and $A_4^{G,j}$. For the PPP mode (user positioning), satellite corrections $dt_{P3}^{G,j}$, $dt_{L3}^{G,j}$, and $b_{A4}^{j,G}$ are known from the network solution. For model (3), there are $7 + 2M_{\text{sat}}$ estimated parameters: Δx, Δy, Δz, ΔD_W, dT_{P3}^G, dT_{L3}^G, $b_{r,A4}^G$, $\lambda_3^G N_3^{G,1}...\lambda_3^G N_3^{G,M_{\text{sat}}}$, and $\lambda_4^G N_4^{G,1}...\lambda_4^G N_4^{G,M_{\text{sat}}}$ (in practice, transition from ambiguities $N_3^{G,j}$ and $N_4^{G,j}$ to ambiguities $N_1^{G,j}$ and $N_4^{G,j}$ or $N_1^{G,j}$ and $N_2^{G,j}$ is used [12]). Due to the presence of decoupled clocks in (3), the corresponding system of equations is singular; for $M_{\text{sat}} > 4$, the rank deficiency is two. For brevity, observation model (3) is further designated as the P3L3A4 model.

In all considered observation models with decoupled clocks, integer ambiguities of phase measurements are separated from unmodeled equipment biases, i.e., these observation models can be used for implementation of the Integer PPP method.

2. UNDERLYING FEATURE OF SINGULAR SYSTEMS OF GNSS EQUATIONS

Systems of linearized equations corresponding to the above observation models with decoupled clocks (1)–(3) can be written in the generalized matrix view as

$$\mathbf{y}_{m\times 1} = \mathbf{A}_{m\times n}\mathbf{x}_{n\times 1}, \tag{4}$$

where $\mathbf{x}_{n\times 1}$ is the vector of estimated parameters, $\mathbf{A}_{m\times n}$ is the design matrix, and $\mathbf{y}_{m\times 1}$ is the vector of measurements with known covariance matrix $\mathbf{W}^{-1}$. System (4) is inconsistent and underdetermined, i.e., $\mathbf{y} \notin R(\mathbf{A})$ and $R(\mathbf{A}) \subset \mathbf{R}^m$ and, for rank r of matrix **A**, condition $r < \min(m,n)$ is fulfilled ($R(\mathbf{A})$ is the linear vector column space of matrix **A**, i.e., the linear space spanned by the columns of matrix **A** and $\mathbf{R}^m$ is the vector space of all real-valued m-dimensional vectors). This means that system (4) has an infinite set of least-squares solutions (LS-solutions), i.e. system (4) is singular (contains rank deficiency).

Multiple computing experiments for observation models (1)–(3) with decoupled clocks performed for real GNSS pseudorange and phase measurements of both GLONASS and GPS systems have shown the following: in all cases, in the space of estimated parameters, the coordinate axes corresponding to geometric parameters (Δx, Δy, Δz and ΔD_W) are orthogonal to null space $N(\mathbf{A})$ of matrix **A** in system of linear equations (4). An alternative and more mathematically abstract form of this feature can be formulated as the following statement: all vectors $\mathbf{x}$ such that $\mathbf{x} \in N(\mathbf{A})$ have zero components corresponding to geometric parameters. If follows from this feature that geometric parameters can be estimated unambiguously irrespective of singularity of system (4). For observation model (3), an additional unambiguously estimated parameter is receiver code clock error dT^G_{P3}. In the space of estimated parameters, coordinate axes corresponding to all other estimated parameters (not geometric) of system (4) are not orthogonal to null space $N(\mathbf{A})$ of matrix **A**; therefore, such parameters can not be estimated unambiguously.

The approach that uses the above feature is used in [2, 3, 12, 13] and implicitly in [4]. However, these publications do not provide theoretical details of the approach and explanations of the possibility of direct estimation of some parameters, whereas the remaining parameters are estimated as linear combinations. In [14], the independency of the ranks of corresponding blocks of the design matrix is pointed out as the reason of such a division of estimated parameters into two groups. From our point of view, this rank independency is just a consequence, whereas the reason is the described above feature of the orthogonality of the null space to some coordinate axes.

The revealed above feature of the systems of linearized GNSS equations allows us to formulate the general rule for coefficients of non-geometric parameters in estimated linear combinations. The coefficients should be chosen in such a way that new estimated parameters (formed as linear combinations of initial non-geometric parameters) in the space of initial estimated parameters should correspond to directions orthogonal to null space $N(\mathbf{A})$ of matrix **A**. For observation model P1P2L1L2 (1), the following combinations of ambiguously estimated (non-geometric) parameters correspond to directions orthogonal to null space $N(\mathbf{A})$ of matrix **A** (4):

$$\begin{aligned}
&\underline{a}\left(\frac{k^G}{k^G-1}\right)dT^G_{P1} - \underline{a}\left(\frac{1}{k^G-1}\right)dT^G_{P2},\\
&\underline{a}dT^G_{P1} + \underline{a}I^{G,j}_1,\\
&\underline{a}dT^G_{P2} + \underline{a}k^G I^{G,j}_1,\\
&\underline{a}dT^G_{L1} - \underline{a}\lambda^G_1 N^{G,i}_1 - \underline{a}I^{G,j}_1,\ i = 1,\dots,M_{\text{sat}},\ j = 1,\dots,M_{\text{sat}}\\
&\underline{a}dT^G_{L2} - \underline{a}\lambda^G_2 N^{G,i}_2 - \underline{a}k^G I^{G,j}_1,\ i = 1\dots M_{\text{sat}},\ \ j = 1\dots M_{\text{sat}} \quad \underline{a} \in R.\\
&\underline{a}\lambda^G_1 N^{G,i}_1 - \underline{a}\lambda^G_1 N^{G,j}_1,\ i = 1,\dots,M_{\text{sat}},\ j = 1,\dots,M_{\text{sat}},\ i \neq j\\
&\underline{a}\lambda^G_2 N^{G,i}_2 - \underline{a}\lambda^G_2 N^{G,j}_2, i = 1,\dots,M_{\text{sat}},\ j = 1,\dots,M_{\text{sat}},\ i \neq j\\
&\underline{a}I^{G,i}_1 - \underline{a}I^{G,j}_1, i = 1,\dots,M_{\text{sat}},\ \ j = 1,\dots,M_{\text{sat}},\ i \neq j.
\end{aligned} \tag{5}$$

It is necessary to use an assumption for matrix **A** in observation model (2) that wavelengths $\lambda^{R,j}_1$ and $\lambda^{R,j}_2$, $j = \overline{1\dots M_{\text{sat}}}$ are equal for different satellite frequency channels j in frequency bands L1 and L2. In this case, combinations of integer ambiguities in observation model (2) correspond to directions that are strictly orthogonal to null space $N(\mathbf{A})$ of matrix **A** (4). We recommend to use $\lambda^{R,j}_1 = f^{R,0}_{L1}$ and $\lambda^{R,j}_2 = f^{R,0}_{L2}$, $j = \overline{1\dots M_{\text{sat}}}$. The error of such an assumption does not exceed 0.46 mm. Using this assumption for observation model GL_P1P2L1L2 (2), we find that the following combinations of non-geometric parameters correspond to directions orthogonal to the null space $N(\mathbf{A})$ of matrix **A** (4):

$$
\begin{aligned}
&\underline{a}\left(\frac{k^R}{k^R-1}\right)dT_{P1}^R-\underline{a}\left(\frac{1}{k^R-1}\right)dT_{P2}^R,\\
&\underline{a}dT_{P1}^R+\underline{a}I_1^{R,i},\\
&\underline{a}dT_{P2}^R+\underline{a}k^R I_1^{R,i},\\
&\underline{a}\frac{1}{\lambda_1^{R,i}}dT_{P1}^R+\underline{a}dT_{L1}^R-\underline{a}N_1^{R,i}-\underline{a}\frac{1}{\lambda_1^{R,i}}I_1^{R,i}, i=1,\dots,M_{\text{sat}},\\
&\underline{a}\frac{1}{\lambda_2^{R,i}}dT_{P2}^R+\underline{a}dT_{L2}^R-\underline{a}N_2^{R,i}-\underline{a}k^R\frac{1}{\lambda_2^{R,i}}I_1^{R,i}, 1,\dots,M_{\text{sat}},\\
&\underline{a}N_1^{R,i}-\underline{a}N_1^{R,j}, i=1,\dots,M_{\text{sat}},\ j=1,\dots,M_{\text{sat}},\ i\neq j \qquad \underline{a}\in R.\\
&\underline{a}N_2^{R,i}-\underline{a}N_2^{R,j}, i=1,\dots,M_{\text{sat}},\ j=1,\dots,M_{\text{sat}},\ i\neq j\\
&\underline{a}\frac{1}{\lambda_1^{R,i}}I_1^{R,i}-\underline{a}\frac{1}{\lambda_1^{R,j}}I_1^{R,j}, i=1,\dots,M_{\text{sat}},\ j=1,\dots,M_{\text{sat}},\ i\neq j,\\
&\underline{a}k^R\frac{1}{\lambda_2^{R,i}}I_1^{R,i}-\underline{a}k^R\frac{1}{\lambda_2^{R,j}}I_1^{R,j}, i=1,\dots,M_{\text{sat}},\ j=1,\dots,M_{\text{sat}},\ i\neq j.
\end{aligned} \tag{6}
$$

For observation model P3L3A4 (3), the following combinations of ambiguously estimated (non-geometric) parameters correspond to directions orthogonal to null space $N(\mathbf{A})$ of matrix $\mathbf{A}$ (4):

$$
\begin{cases}
\underline{a}dT_{L3}^G-\underline{a}\lambda_3^G N_3^{G,i},\ i=1,\dots,M_{\text{sat}},\\
\underline{a}b_{r,A4}^G-\underline{a}\lambda_4^G N_4^{G,i},\ i=1,\dots,M_{\text{sat}},\\
\underline{a}\lambda_3^G N_3^{G,i}-\underline{a}\lambda_3^G N_3^{G,j}, i=1,\dots,M_{\text{sat}},\ j=1,\dots,M_{\text{sat}},\ i\neq j,\\
\underline{a}\lambda_4^G N_4^{G,i}-\underline{a}\lambda_4^G N_4^{G,j}, i=1,\dots,M_{\text{sat}},\ j=1,\dots,M_{\text{sat}},\ i\neq j.
\end{cases} \qquad \underline{a}\in R. \tag{7}
$$

Directions that are opposite to the directions corresponding to combinations of parameters (5)–(7) (i.e. combinations (5)–(7) taken with opposite signs) are also orthogonal to null space $N(\mathbf{A})$ of matrix $\mathbf{A}$ (4).

3. ALGEBRAIC METHODS FOR REMOVAL OF THE RANK DEFICIENCY IN THE INTEGER PPP METHOD

The revealed feature of systems of linearized GNSS equations (4) allows us to use the following block-wise division of matrix $\mathbf{A}$ in the PPP (by the example of observation model (1)):

$$
\mathbf{y}=\mathbf{A}\mathbf{x}\Leftrightarrow\begin{bmatrix}\mathbf{y}_{\text{I}}\\ \mathbf{y}_{\text{II}}\end{bmatrix}=\left[\mathbf{A}_{\text{I}}\,\vdots\,\mathbf{A}_{\text{II}}\right]\begin{bmatrix}\mathbf{x}_{\text{I}}\\ \hline \mathbf{x}_{\text{II}}\end{bmatrix}
\Leftrightarrow \mathbf{y}=\underbrace{\begin{bmatrix}
\mathbf{A}_r & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}\\
\mathbf{A}_r & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & k^G\mathbf{I}\\
\mathbf{A}_r & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & -\mathbf{I}\\
\mathbf{A}_r & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & -\mathbf{I} & -k^G\mathbf{I}
\end{bmatrix}}_{4M_{\text{sat}}\times(8+3M_{\text{sat}})}
\times\underbrace{\left[\mathbf{r}\,\vdots\,dT_{P1}^G\ dT_{P2}^G\ dT_{L1}^G\ dT_{L2}^G\ \boldsymbol{\lambda}_1^{\mathbf{G}}\mathbf{N}_1^{\mathbf{G}}\ \boldsymbol{\lambda}_2^{\mathbf{G}}\mathbf{N}_2^{\mathbf{G}}\ \mathbf{I}_1^{\mathbf{G}}\right]^T}_{4M_{\text{sat}}\times(8+3M_{\text{sat}})}, \tag{8}
$$

where vector $\mathbf{x}_{\text{I}}$ contains geometric parameters $\mathbf{r}=[\Delta x\ \Delta y\ \Delta z\ \Delta D_W]^T$, $\mathbf{A}_r$ is the corresponding block of matrix $\mathbf{A}$, $\mathbf{x}_{\text{II}}$ is the vector of ambiguously estimated parameters corresponding to block $\mathbf{A}_{\text{II}}$ of matrix $\mathbf{A}$, $\mathbf{1}$ is the unit vector of appropriate length, and $\mathbf{I}$ is the identity matrix with appropriate dimensions. In block-wise division (8), the following condition holds true: $\mathbf{R}^{3M_{\text{sat}}}=R(\mathbf{A}_{\text{I}})\oplus R(\mathbf{A}_{\text{II}})$.

To obtain a unique solution of system (4), orthogonal bases $\mathbf{S}$ and $\mathbf{S}^{\perp}$ of linear subspaces $R(\mathbf{S})$ and $R(\mathbf{S}^{\perp})$ are considered:

$$
\begin{cases}
\mathbf{R}^n=N(\mathbf{A}_{\text{II}})\oplus R(\mathbf{S}),\\
\mathbf{R}^n=R(\mathbf{S})\oplus R(\mathbf{S}^{\perp}).
\end{cases} \tag{9}
$$

For the sake of clarity, Fig. 1 shows the set of solutions to singular system (4), the null space of design matrix $\mathbf{A}$, and bases $\mathbf{S}$ and $\mathbf{S}^{\perp}$ for the simplest case of space of three estimated parameters. In coordinate plane x_2x_3 shown in Fig. 1 for fixed basis $\mathbf{S}$, the entire set of vectors $\mathbf{x}_{\text{II}}$ corresponding to system $\mathbf{y}_{\text{II}}=\mathbf{A}_{\text{II}}\mathbf{x}_{\text{II}}$ can be projected onto subspace $R(\mathbf{S})$ and along null space $N(\mathbf{A})$ [7]:

$$
\mathbf{x}^{\mathbf{S}}=\mathbf{P}_{R(\mathbf{S}),N(\mathbf{A})}\mathbf{x}_{\text{II}}, \tag{10}
$$

where $\mathbf{P}_{R(\mathbf{S}),N(\mathbf{A})}$ is the projector (projection matrix) onto $R(\mathbf{S})$ and along $N(\mathbf{A})$. Functional transformation (10) produces combinations of variables of system $\mathbf{y}_{\text{II}} = \mathbf{A}_{\text{II}}\mathbf{x}_{\text{II}}$ that are estimated unambiguously. Expression (10) can be written also as $\left(\mathbf{S}^{\perp}\right)^{\mathbf{T}} \mathbf{x}_{\text{II}} = \mathbf{0}$, then system $\mathbf{y}_{\text{II}} = \mathbf{A}_{\text{II}}\mathbf{x}_{\text{II}}$ is extended to the following [14]:

$$\begin{bmatrix} \mathbf{y}_{\text{II}} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{\text{II}} \\ \left(\mathbf{S}^{\perp}\right)^{T} \end{bmatrix} \mathbf{x}_{\text{II}}, \tag{11}$$

where matrix $\mathbf{A}_{\text{II}}^{*} = \begin{bmatrix} \mathbf{A}_{\text{II}} \\ \left(\mathbf{S}^{\perp}\right)^{T} \end{bmatrix}$ has full rank, i.e., system (11) is inconsistent and overdetermined with unique LS-solution $\hat{\mathbf{x}}_{\text{II}}^{*}$.

In view of (11), system (8) is transformed to the following system:

$$\begin{bmatrix} \mathbf{y}_{\text{I}} \\ \mathbf{y}_{\text{II}} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{\text{I}} & \mathbf{A}_{\text{II}} \\ \mathbf{0} & \left(\mathbf{S}^{\perp}\right)^{T} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{\text{I}} \\ \mathbf{x}_{\text{II}} \end{bmatrix} \Leftrightarrow \mathbf{y}^{*} = \mathbf{A}^{*}\mathbf{x}. \tag{12}$$

Generally, basis $\mathbf{S}$ can be chosen arbitrarily according to (9) (Fig. 1). However, in practice, it is more convenient to choose basis $\mathbf{S}$ so that subspace $R(\mathbf{S})$ is orthogonal to such number of coordinate axes that is equal to the rank deficiency in system of equations (4). In this case, components of vectors $\mathbf{x}^{S}$ corresponding to these orthogonal axes are zeros and the remaining components of vectors $\mathbf{x}^{S}$ are biased by the values of combinations of variables corresponding to orthogonal axes. Then, instead of system (12), the following system can be used:

$$\mathbf{y} = \underline{\mathbf{A}}^{*}\underline{\mathbf{x}}^{*} \Leftrightarrow \begin{bmatrix} \mathbf{y}_{\text{I}} \\ \mathbf{y}_{\text{II}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{\text{I}} & \underline{\mathbf{A}}_{\text{II}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{\text{I}} \\ \underline{\mathbf{x}}_{\text{II}} \end{bmatrix}, \tag{13}$$

where $\underline{\mathbf{x}}_{\text{II}}$ contains only nonzero components of vector $\mathbf{x}^{S}$, and, in matrix $\underline{\mathbf{A}}_{\text{II}}$, there are no columns corresponding to the coordinate axes orthogonal to $R(\mathbf{S})$. Here, $\underline{\mathbf{A}}_{\text{II}}$ is a full rank matrix and the length of vector $\underline{\mathbf{x}}^{*}$ equals the rank of matrix $\underline{\mathbf{A}}^{*}$. Thus, transition from system (4) to system (13) reduces the number of estimated linear combinations in vector $\underline{\mathbf{x}}_{\text{II}}$ to the value of the rank of matrix $\underline{\mathbf{A}}_{\text{II}}$.

Functional components of subvector $\mathbf{x}_{\text{II}}$ in (8) whose coordinate axes are orthogonal to the chosen subspace $R(\mathbf{S})$ form linear combinations with other functional components of subvector $\mathbf{x}_{\text{II}}$ in (8). For this reason, subspace $R(\mathbf{S})$ should be chosen orthogonal to some coordinate axes so as to preserve the integer nature of linear combinations of integer functional components of subvector $\mathbf{x}_{\text{II}}$ in (8). For this purpose, base $\mathbf{S}$ in observation model (1) should be chosen so

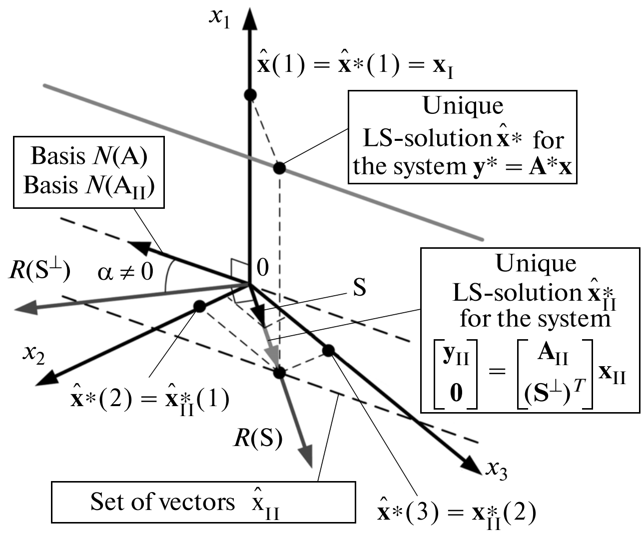

Fig. 1. Simplified geometrical example illustrating the general case of selection of the **S** basis.

that subspace $R(\mathbf{S})$ is orthogonal to three coordinate axes corresponding to two ambiguities $\lambda_1^G N_1^{G,j}$ and $\lambda_2^G N_2^{G,j}$ and ionospheric delay $I_1^{G,j}$ of the signal of the *j*th satellite. These three parameters will form linear estimated combinations with other parameters and keep integrality of the combinations of ambiguities. In observation models GL_P1P2L1L2 (2) and P3L3A4 (3), such basis $\mathbf{S}$ should be chosen that subspace $R(\mathbf{S})$ is orthogonal to three coordinate axes corresponding to two ambiguities $N_1^{R,j}$ and $N_2^{R,j}$ and ionospheric delay $I_1^{R,j}$, and two coordinate axes corresponding to two ambiguities $\lambda_3^G N_3^{G,j}$ and $\lambda_4^G N_4^{G,j}$ for the chosen *j*th satellite.

4. FILTER EXCLUDING THE INFLUENCE OF IONOSPHERIC DELAYS ON ESTIMATES OF THE USER COORDINATES

The presence of ionospheric delays as nuisance parameters in observation models P1P2L1L2 (1) and GL_P1P2L1L2 (2) hampers application of these models in practice. These models can be used with direct estimation of combinations of ionospheric delays in the user solution, whereas, in the network solution the estimated satellite decoupled clocks become biased by values of ionospheric delays of corresponding satellites. Thus, the PPP network solution cannot be solved with in the presence of ionospheric delays as nuisance parameters. In [9, 10, 15], a filtering method allowing one to avoid estimation of nuisance parameters (a filter excluding the influence of ionospheric delays on estimated user coordinates) is

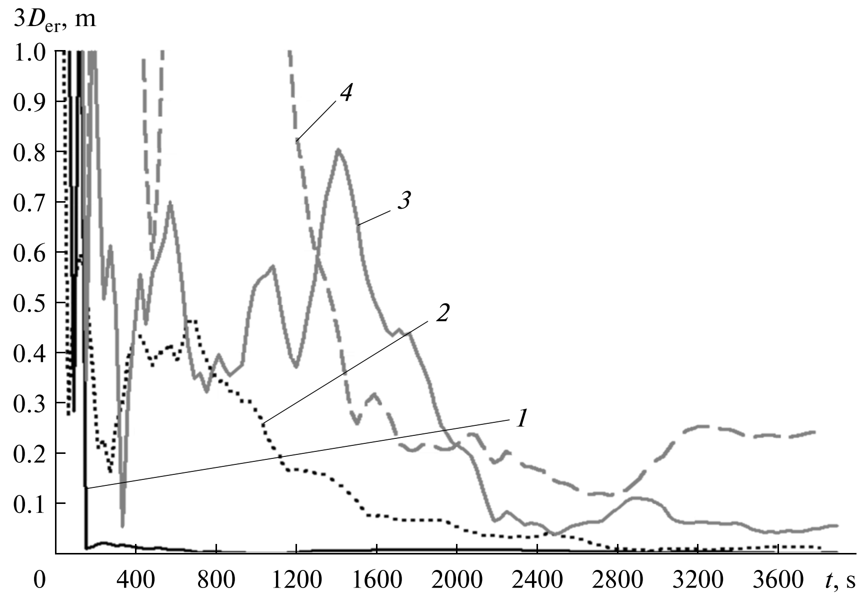

Fig. 2. Comparative plots of 3D positioning error $3D_{er}$ as functions of measurement time t for the Integer PPP method (curve *1* is the P3L3A4 fixed solution and curve *2* is the P3L3A4 float solution) and the Float PPP method (curve *3* is the P3L3 Final and curve *4* is the P3L3).

described. For GNSS, systems of equations of type (4) can be written as follows:

$$\mathbf{y} = [\mathbf{A}_A \;\; \mathbf{A}_B]\begin{bmatrix}\mathbf{x}_A \\ \mathbf{x}_B\end{bmatrix} = \mathbf{A}_A\mathbf{x}_A + \mathbf{A}_B\mathbf{x}_B, \quad (14)$$

where $\mathbf{x}_A$ is the vector of useful estimated parameters; $\mathbf{x}_B$ is the vector of nuisance estimated parameters, which are the ionospheric delays in the network solution with observation models (1) and (2); and $\mathbf{A}_A$ and $\mathbf{A}_B$ are the corresponding blocks of the design matrix. It is shown in [8, 16] that the maximum-likelihood estimate $\hat{\mathbf{x}}_A$ obtained from solution of system (14) can also be obtained from solution to the following reduced system:

$$\mathbf{y} = \mathbf{A}_A\mathbf{x}_A, \quad (15)$$

under the condition of replacement of weight matrix $\mathbf{W}$ with modified weight matrix $\mathbf{W}_a = \mathbf{W} - \mathbf{W}\mathbf{A}_b\left(\mathbf{A}_b^T\mathbf{W}\mathbf{A}_b\right)^{-1}\mathbf{A}_b^T\mathbf{W}$ during the maximum-likelihood estimation.

Based on reduced model (15) and modified weight matrix $\mathbf{W}_a$, it is possible to form the Kalman filter filtering only useful parameter $\mathbf{x}_a$ (filter excluding the influence of ionospheric delays on estimated user coordinates). The estimate of estimated parameters $\mathbf{x}_a$ at the νth filtering epoch in this filter is calculated as

$$\hat{\mathbf{x}}_{a,\nu} = \mathbf{C}_{\nu-1}\hat{\mathbf{x}}_{a,\nu-1} + \mathbf{K}_\nu\left(\mathbf{y}_\nu - \mathbf{A}_\nu\mathbf{C}_{\nu-1}\hat{\mathbf{x}}_{a,\nu-1}\right), \quad (16)$$

where $\mathbf{C}_{\nu-1}$ is the prediction matrix for vector $\mathbf{x}_a$ from the (ν-1)th to the νth filtering epoch, $\mathbf{K}_\nu = \mathbf{G}_\nu\mathbf{A}_\nu^T\mathbf{W}_{a\nu}$, $\mathbf{G}_\nu^{-1} = (\mathbf{C}_{\nu-1}^T\mathbf{G}_{\nu-1}\mathbf{C}_{\nu-1} + \mathbf{F}_\nu)^{-1} + \mathbf{A}_\nu^T\mathbf{W}_{a\nu}\mathbf{A}_\nu$, $\mathbf{F}_\nu$ is the covariance matrix of the process noise, $\mathbf{G}_\nu$ is the covariance matrix of estimation errors of vector $\hat{\mathbf{a}}_\nu$, and $\mathbf{W}_{a\nu}$ is the modified weight matrix of vector of measurements $\mathbf{y}_\nu$ at the νth filtering epoch. In (14), vector of measurements $\mathbf{y}$ depends on not only vector of useful estimated parameters $\mathbf{x}_A$ but also vector of nuisance parameters $\mathbf{x}_B$. The filter excluding the influence of ionospheric delays on estimated user coordinates is quasi-optimal, because, in this filtering method, there is no estimation and prediction of nuisance parameters $\mathbf{x}_B$. Expression (16) is a kind of the covariance form of the Kalman filter, which allows one to work with singular weight matrix $\mathbf{W}_{a\nu}$.

5. USER SOLUTION IN THE INTEGER PPP METHOD

In this study, a comparative analysis of the Float PPP and Integer PPP methods was done with the use of different GPS observation models, different accuracies of ephemerides and measurement data from receivers located in different geographical regions. To implement the Integer PPP method, decoupled satellite clocks, which were calculated by NRCan with the use of the global network of ground stations and provided for test purposes, were used.

Figure 2 shows comparative results for positioning in the Integer PPP (observation model P3L3A4 (3)) and Float PPP (observation model P3L3—ionosphere-free combinations of code and phase measurements) methods. Shown results are for measurements obtained from the IGS (International GNSS Service) station BRUS in 2008 year (the sampling interval is 30 seconds). Rapid IGS orbits were used for Integer PPP processing; Rapid and Final orbits and clocks were used for Float PPP processing. The convergence period of the Integer PPP fixed solution (curve *1*) to an accuracy of 1 cm is about 600 sec, whereas the convergence period of the Float PPP solution (curve *4*) is 18 hours (after 1 hour of processing, the Float PPP 3D position error is 0.23 m, Fig. 2). Figure 2 also shows that the maximum accuracy and the best convergence period of the Float PPP method (curve *3*, Final IGS products) are worse than those of the Integer PPP float solution calculated for observation model (3) (curve *2*).

6. NETWORK SOLUTION IN THE INTEGER PPP METHOD

The integer PPP method requires as network products not only satellite orbits (high-accuracy values of satellite coordinates) but also application of decoupled (according to the measurement type and frequency) biases to the satellite clocks (observation models (1)–(3)).

In this study, we calculated a local network solution, i.e., decoupled satellite clocks for GPS observation model P3L3A4 (3) were calculated based on the local network of European stations shown in Fig. 3 with large circles ($N_{st} = 5$). To simplify the calculations, it was assumed that all $N_{st} = 5$ stations collected measurements from the same set of $M_{sat} = 6$ satellites. The time interval of measurements that satisfied this restriction was 2 hours and 10 min.

For observation model (3), the system of linear equations corresponding to the network solution with N_{st} stations and M_{sat} satellites can be written in following matrix form:

$$\mathbf{y}^N_{N_{meas}} = \mathbf{A}^N_{N_{meas}\times N_{est}}\mathbf{X}^N_{N_{est}\times 1} + \boldsymbol{\varepsilon}^N_{N_{meas}\times 1}$$
$$\Leftrightarrow \mathbf{y}^N = \left[\mathbf{A}^N_I \,\middle|\, \mathbf{A}^N_{II}\right]\begin{bmatrix}\mathbf{X}^N_I \\ \hline \mathbf{X}^N_{II}\end{bmatrix} + \boldsymbol{\varepsilon}^N \quad (17)$$
$$\Leftrightarrow \begin{bmatrix}\mathbf{y}^N_{\mathbf{P3}} \\ \mathbf{y}^N_{\mathbf{L3}} \\ \mathbf{y}^N_{\mathbf{A4}}\end{bmatrix} = \left[\begin{array}{c|cccccccc}\mathbf{A}^N_r & \mathbf{A}_{P3} & \mathbf{0} & \mathbf{0} & \mathbf{A}^{P3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}^N_r & \mathbf{0} & \mathbf{A}_{L3} & \mathbf{0} & \mathbf{0} & \mathbf{A}^{L3} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{A4} & \mathbf{0} & \mathbf{0} & \mathbf{A}^{A4} & \mathbf{0} & -\mathbf{I}\end{array}\right]\begin{bmatrix}\mathbf{r}^N \\ \hline \mathbf{dT}^{G,N}_{P3} \\ \mathbf{dT}^{G,N}_{L3} \\ \mathbf{b}^{G,N}_{r,A4} \\ \mathbf{dt}^{G,N}_{P3} \\ \mathbf{dt}^{G,N}_{L3} \\ \mathbf{b}^{G,N}_{A4} \\ \boldsymbol{\lambda}^{\mathbf{G}}_{\mathbf{3}}\mathbf{N}^{\mathbf{G},N}_{\mathbf{3}} \\ \boldsymbol{\lambda}^{\mathbf{G}}_{\mathbf{4}}\mathbf{N}^{\mathbf{G},N}_{\mathbf{4}}\end{bmatrix} + \begin{bmatrix}\boldsymbol{\varepsilon}^N_{P3} \\ \boldsymbol{\varepsilon}^N_{L3} \\ \boldsymbol{\varepsilon}^N_{A4}\end{bmatrix},$$

where $N_{meas} = 3N_{st}M_{sat}$ is the number of measurements in $\mathbf{y}^N$ (index N means Network); $N_{est} = 3M_{sat} + 4N_{st} + 2M_{sat}N_{st}$ is the number of estimated parameters; $\mathbf{A}^N_{r,(N_{st}\cdot M_{sat})\times N_{st}}$ is the block of design matrix $\mathbf{A}^N$ of the network solution corresponding to geometric parameters in the network solution $\mathbf{r}^N_{N_{st}\times 1}$ (N_{st} troposphere delays $\Delta D_{W,i}(i = \overline{1, N_{st}})$ for network stations); $\mathbf{A}_{P3,(N_{st}\cdot M_{sat})\times N_{st}} = \mathbf{A}_{L3} = \mathbf{A}_{A4}$ are the blocks of design matrix $\mathbf{A}^N$ of the network solution corresponding to vectors $\mathbf{dT}^{G,N}_{P3,N_{st}\times 1}$, $\mathbf{dT}^{G,N}_{L3,N_{st}\times 1}$; $\mathbf{A}^{P3}_{(N_{st}\cdot M_{sat})\times M_{sat}} = \mathbf{A}^{L3} = \mathbf{A}^{A4}$ are the blocks of design matrix $\mathbf{A}^N$ of the network solution corresponding to vectors $\mathbf{dt}^{G,N}_{P3}$, $\mathbf{dt}^{G,N}_{L3}$ and $\mathbf{b}^{G,N}_{A4}$ with decoupled satellite clocks of M_{sat} satellites for measurements $\mathbf{dt}^{G,N}_{P3}$, $L^{G,j}_3$, and $A^{G,j}_4$, respectively; $\boldsymbol{\lambda}^{\mathbf{G}}_{\mathbf{3}}\mathbf{N}^{\mathbf{G},N}_{\mathbf{3},(N_{st}\cdot M_{sat})\times M_{sat}}$ and $\boldsymbol{\lambda}^{\mathbf{G}}_{\mathbf{4}}\mathbf{N}^{\mathbf{G},N}_{\mathbf{4},(N_{st}\cdot M_{sat})\times M_{sat}}$ are the vectors of integer ambiguities $\lambda^G_3 N^{G,j}_3$ and $\lambda^G_4 N^{G,j}_4$ corresponding to measurements $L^{G,j}_3$ and $A^{G,j}_4$ (m).

Decoupled satellite clocks $\mathbf{dt}^{G,N}_{P3}$, $\mathbf{dt}^{G,N}_{L3}$, and $\mathbf{b}^{G,N}_{A4}$ in (17) are estimated ambiguously as combinations $\overline{\mathbf{dt}^{G,N}_{P3}}$, $\overline{\mathbf{dt}^{G,N}_{L3}}$ and $\overline{\mathbf{b}^{G,N}_{A4}}$ with integer components of vector $\mathbf{X}^N_{\mathrm{II}}$, i.e., they are biased by integer values. It is not an obstacle to apply them for the Integer PPP method, because integer values of biases in these combinations are taken into account by user during ambiguity resolution.

The rank deficiency of matrix $\mathbf{A}^N$ equals the sum of rank deficiencies of its separated blocks $\mathbf{A}^N_{P3}$, $\mathbf{A}^N_{L3}$, and $\mathbf{A}^N_{A4}$, which correspond to measurement vectors $\mathbf{y}^N_{P3}$, $\mathbf{y}^N_{L3}$, and $\mathbf{y}^N_{A4}$. To reduce the rank deficiency in each of the blocks by one, it is sufficient to choose a reference

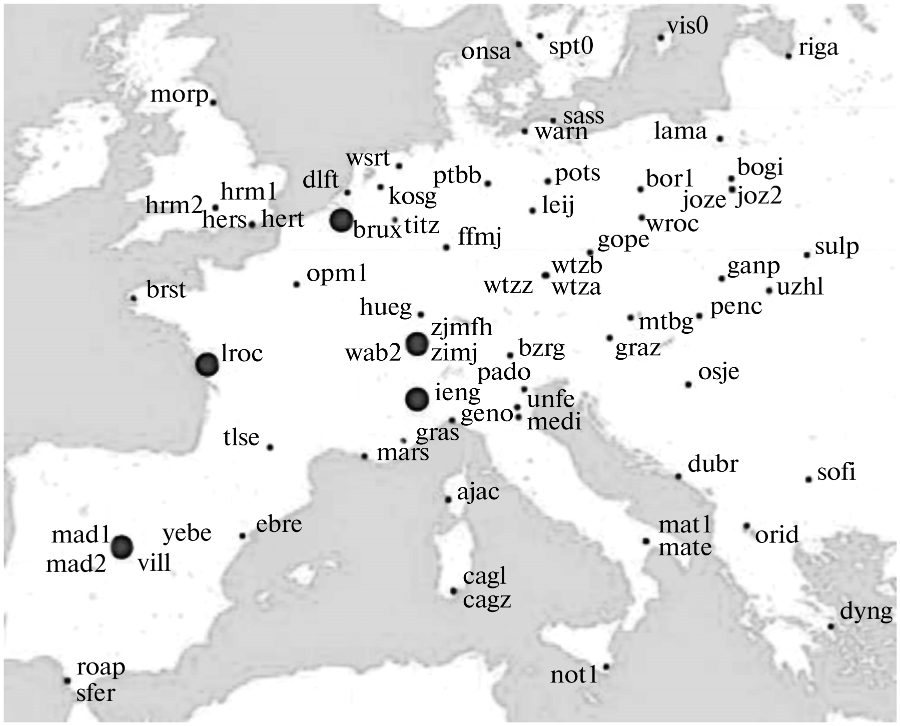

Fig. 3. Local network of European stations used in the study.

station (reference, REF) whose decoupled clocks $dT^{G,N}_{P3,\mathrm{REF}}$, $dT^{G,N}_{L3,\mathrm{REF}}$ and $b^{G,N}_{r,A4,\mathrm{REF}}$ are not estimated in the network solution. After that, the value of the rank deficiency in (17) becomes $(N_{\mathrm{st}} + M_{\mathrm{sat}} - 1)$ for blocks $\mathbf{A}^{N}_{L3}$ and $\mathbf{A}^{N}_{A4}$.

Basis $\mathbf{S}$ supplementing $N(\mathbf{A}^{N}_{\mathrm{II}})$ to $\mathbf{R}^{N_{\mathrm{est}}}$ ($N(\mathbf{A}^{N}_{\mathrm{II}}) \oplus R(\mathbf{S}) = \mathbf{R}^{N_{\mathrm{est}}}$) is chosen so that $R(\mathbf{S})$ is orthogonal to such a number of coordinate axes corresponding to parameters $\mathbf{X}^{N}_{\mathrm{II}}$ (17) that equals the rank deficiency of matrix $\mathbf{A}^{N}$. In this case, all components of vector $\mathbf{X}^{N}_{\mathrm{I}}$ (i.e., geometric parameters $\mathbf{r}^{N}_{N_{\mathrm{st}}\times 1}$) are estimated unambiguously and all nonzero components of vector $\mathbf{X}^{N}_{\mathrm{II}}$ are estimated ambiguously as combinations (inconsistent underdetermined system (17) is transformed to an inconsistent overdetermined system with a unique least-squares solution. To preserve the integrality of estimated linear combinations of ambiguities in vector $\mathbf{X}^{N}_{\mathrm{II}}$, it is necessary to choose $R(\mathbf{S})$ so as to make it orthogonal to the coordinate axes of ambiguities $\lambda^{G}_{3} N^{G,j}_{3}$ and $\lambda^{G}_{4} N^{G,j}_{4}$ ($j = \overline{1, N_{\mathrm{st}} + M_{\mathrm{sat}} - 1}$), which correspond to $(N_{\mathrm{st}} + M_{\mathrm{sat}} - 1)$ satellite–station pairs in the network. As has been shown in [17], the choice of such a set of ambiguities can be formulated in terms of the graph theory. A network with N_{st} stations and M_{sat} satellites can be described with a bipartite graph of measurements (Fig. 4). The edges in such a graph correspond to integer phase ambiguities $\lambda^{G}_{3} N^{G,j}_{3}$ or $\lambda^{G}_{4} N^{G,j}_{4}$ ($j = \overline{1, N_{\mathrm{st}} M_{\mathrm{sat}}}$). To remove the rank deficiency in blocks $\mathbf{A}^{N}_{L3}$ and $\mathbf{A}^{N}_{A4}$, one graph can be used. Subspace $R(\mathbf{S})$ is chosen to be orthogonal to the coordinate axes of such ambiguities, which correspond to the edges of the minimum spanning tree [17] (MST, Fig. 4, edges of the MST are shown with bold lines).

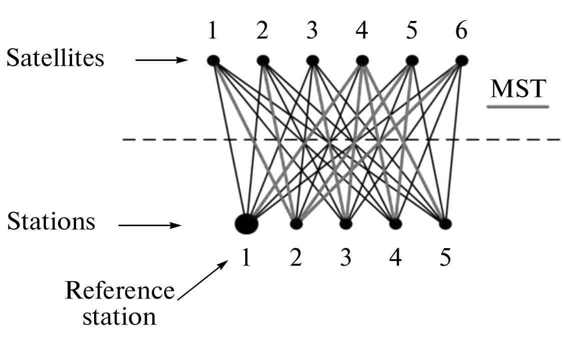

Fig. 4. Bipartite graph of measurements.

The efficiency of application of decoupled clocks calculated inside the local network in the Integer PPP method was compared with application of decoupled clocks calculated by NRCan with the use of the global network of stations. Figure 5 shows 3D user position-

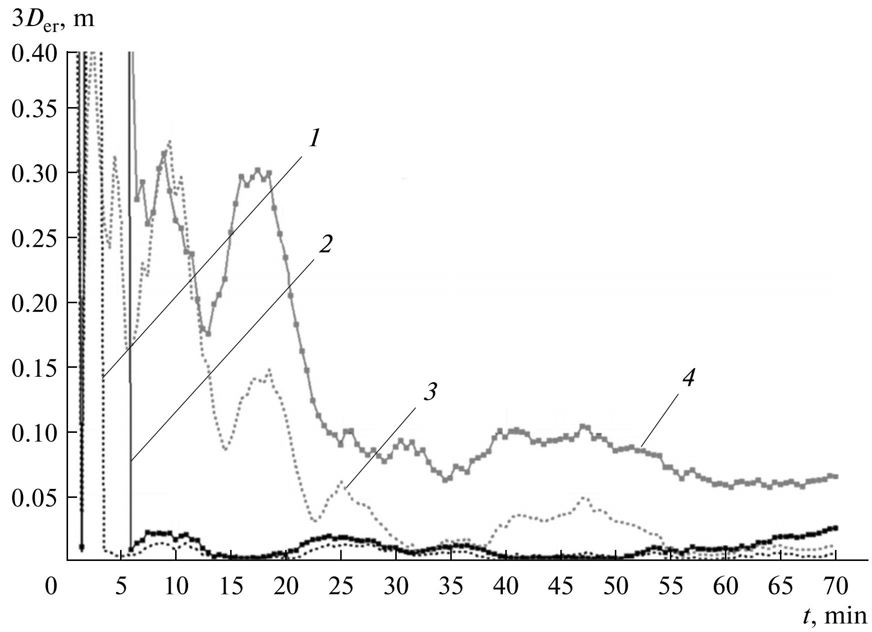

Fig. 5. Values of 3D positioning error $3D_{er}$ as a function of measurement time t for the Integer PPP method using decoupled satellite clocks from global and local networks, station BRUS: integer solution using decoupled satellite clocks from (*1*) local and (*2*) global networks; real-valued solution using decoupled satellite clocks from (*3*) local and (*4*) global networks.

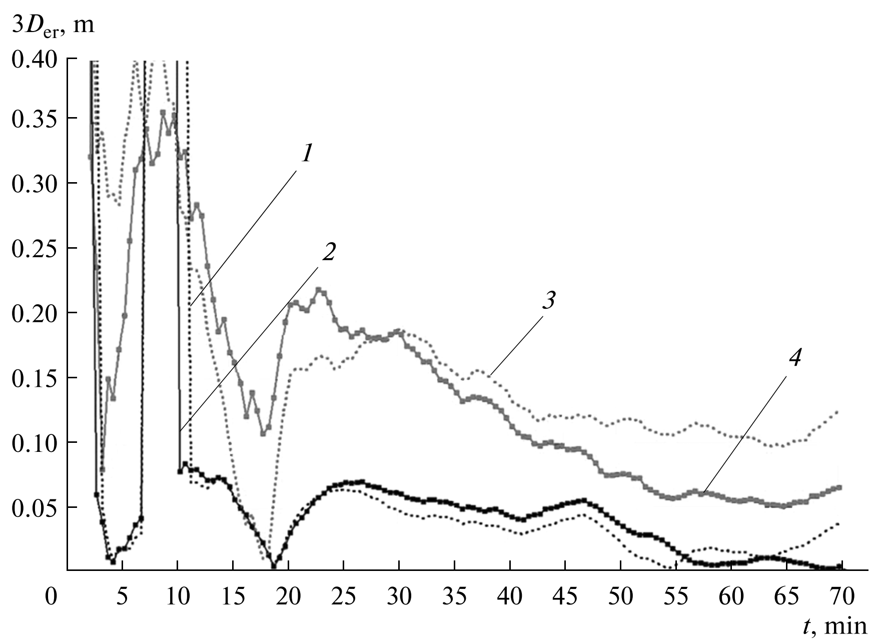

Fig. 6. Values of 3D positioning error $3D_{er}$ as a function of measurement time t for the Integer PPP method using decoupled satellite clocks from global and local networks, station OPMT: integer solution using decoupled satellite clocks from (*1*) local and (*2*) global networks; real-valued solution using decoupled satellite clocks from (*3*) local and (*4*) global networks.

ing errors (station BRUS) for the Integer PPP method with the use of two aforementioned sets of satellite decoupled clocks. One can see that satellite decoupled clocks from local and global networks provide comparable convergence period for the Integer PPP solution. Figure 6 shows the same comparative results for another station in the considered local network (station OPMT). One can see that, for the OPMT station, there is a step in the estimated coordinates in the case of application of satellite decoupled clocks from both global and local networks. Hypothetically, this step and general reduction of the user positioning quality for the OPMT station is associated with worse environmental conditions around the receiver at the OPMT station as compared to the BRUS station.

Currently, satellite decoupled clocks for observation models with initial frequencies P1P2L1L2 (1) and GL_P1P2L1L2 (2) are not calculated. For this reason, the local network solution was calculated for observation model P3L3A4 (3), which corresponds to decoupled satellite clocks calculated by NRCan.

CONCLUSIONS

The PPP with ambiguity resolution of phase measurements (Integer PPP) in GNSS has been considered. The basic underlying feature of the singular system of linearized GNSS equations has been formulated. In the Integer PPP method, this feature allows one to divide all estimated parameters into two groups and, in one of them, the estimated parameters are estimated as linear combinations. Using this feature and the theory of S-Transformations, algebraic details of resolving of the rank deficiency have been described for user and network solutions. Observation models with decoupled clocks at initial frequencies for GPS and GLONASS have been developed. For these models, directions orthogonal to the null space of the design matrix have been described. Significant reduction of the convergence period (from 18 hours to 600 seconds) for the Integer PPP user solution due to resolution of the ambiguity of phase measurements has been demonstrated. Qualities of the satellite decoupled clocks calculated in local and global networks have been compared. It has been concluded that quality of satellite decoupled clocks inside the local network is comparable for local and global networks. A filtering method that allows one to avoid estimation of nuisance parameters when working with the presented observation models at initial GPS and GLONASS frequencies has been described. Thus, new algebraic details of the Integer PPP method in GNSS have been described and experimental results which correspond to similar experimental results published before have been presented.

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