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REVIEWS

Heat Conduction Beyond the Fourier Law

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Abstract—The Fourier law correctly describes heat transport in most practical macroscopic problems. However, for heat transfer in rapid processes, heat transport on micro- and nanoscales, and heat transfer in materials with an internal structure (porous media and biological tissues), other models are required that take into account nonlinear effects, as well as temporal (memory) and spatial nonlocality. Such models are considered in this review, including models with time lag, phonon and thermodynamic models, as well as models based on differential equations in fractional derivatives.

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INTRODUCTION

The Fourier model is based on defining relation

$$\mathbf{q}(\mathbf{r},t) = -\lambda\nabla T(\mathbf{r},t),$$

which, after the substitution into the energy conservation law for a stationary solid,

$$\frac{\partial(\rho CT)}{\partial t} = \nabla\mathbf{q} + Q$$

leads to the classical parabolic heat conduction equation (also known as the Fourier—Kirchhoff equation)

$$\frac{\partial(\rho cT)}{\partial t} = \nabla\cdot(\lambda\nabla T) + Q. \tag{1}$$

The Fourier law can be derived directly in the classical nonequilibrium thermodynamics based on the local equilibrium hypothesis [1].

From the standpoint of nonequilibrium thermodynamics, the Fourier law describes the linear relation between the generalized force (temperature gradient) and the generalized (heat) flux [2].

The Fourier law holds if

(i) $\frac{L}{\Lambda} \gg O(1)$,

(ii) $\frac{t}{\tau} \gg O(1)$,

(iii) $T \gg 0$ K,

where L is the characteristic size of the system, Λ is the mean free path of heat carriers, and τ is the relaxation time. Ratio $\mathrm{Kn} = \Lambda/L$ is known as the "Knudsen number," as in the dynamics of rarefied gases.

Heat waves in the form of "second sound" [3] in helium II at 1.4 K were observed by Peshkov in 1944 at a velocity of about 19 m/s, which is an order of magnitude smaller than the velocity of sound in helium II [4].

Later, second sound was observed in cryogenic conditions in other materials [5, 6], i.e., solid helium-3 [7], sodium fluoride (at 10 K [8–10]), bismuth (at 3.4 K [11]), sapphire and strontium titanate [12], and graphite at a temperature above 100 K [13].

The wave nature of heat propagation and relaxation processes become predominant, and the material "memory" effects, as well as nonlinear and nonlocal effects become significant for

(i) ultrafast heating (laser heating and melting [14],[1] for example, femtosecond heating of metal films [16–18] or thin solid argon films [19]), fast solidification of liquid metals [20], a transition to the glass-like state of supercooled liquids [21], and experiments with heat pulses at room temperature [22];

(ii) heat transport on nanoscales [23–26] (microelectronic devices [27–30], e.g., "hot" elements in nanotransistors [31–34], nanostructured thermoelectric devices [35], heterostructures [36], and laser plasma formed during irradiation of small targets [37]), and heat transfer in DNA during denaturation ("melting") and unwinding of a double spiral into two individual strips [38];[2]

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[1] A pulsed laser ensures better localization of heat as compared to a continuous-wave laser [15].

[2] The Fourier law considerably overestimates the heat dissipation rate from sources with sizes smaller than the phonon mean free path [39], which is important for analysis of thermal regimes of microelectronic devices. However, this problem becomes not so acute on account of the new "collective diffusion" effect that has been discovered recently by Hooge boom-Pot et al. [39]: when the distance between nanosize heat sources becomes smaller than the phonon mean free path, scattering of phonons by phonons "originating" from a neighboring heat source occurs, which intensifies the dissipation of heat.

(iii) heat transport in granular and porous media [40, 41] including porous silicon [42], and

(iv) heat transport in biological tissues [43–49].

It should be noted that the errors associated with the application of the Fourier law beyond the range of its application are sometimes insignificant. For example, Wilson and Cahill [50] listed the following reasons for which errors in determining the thermal conductivity of diamond are not important in analysis of thermal regimes of microelectronic devices, in which diamond heat distributers are used:

(1) ballistic–diffusion effects in polycrystalline diamond films grown by chemical deposition from the vapor phase are much weaker than in single crystals due to phonon scattering from grain boundaries;

(2) the thermal conductivity of the substrate is significant only for sizes of the active region experiencing superheating larger than 1 μm (at least, in GaN transistors with a high electron mobility);

(3) for a high density of devices, horizontal temperature gradients are small.

Pulsed lasers with a pulse duration from nanoseconds to femtoseconds are used in the wide range of medical technologies (optical tomography [51], photodynamic therapy [15, 52], and hyperthermia [53–57]). Temperature control in tissues can be enhanced by injecting nanostructures [58].

Time lag can appear in a material due to the presence of carriers with different energies [59] (the well-known example in the solid state physics is energy relaxation between the electron and phonon subsystems), e.g., energy transport from free electrons in the lattice of metals [60, 61] heated by ultrashort laser pulses [62, 63] or in materials with a heterogeneous internal structure.

Biological tissues contain cells, superstructures, as well as liquid and solid elements. The heating or cooling of biological tissues induces a series of chemical, electrical, and mechanical processes (e.g., diffusion, change in the electric potential, and osmotic flows through a cellular membrane; cellular membranes can store energy) [64]. Thus, propagation of heat in biological tissues induces energy exchange at various levels [64–66].

There are various approaches to experimental investigation of heat transfer on micro- and nanoscales:

(i) 3ω method [67] based on the measurement of the third harmonic in voltage under sample heating by a sinusoidal wave of frequency ω;

(ii) scanning thermal microscopy [68];

(iii) bimetallic devices;

(iv) optical methods;

(v) coherent X-ray methods;

(vi) analysis of thermal reflections.

In recent years, the following computational methods have been used:

(i) calculations based on “first principles” (ab initio calculations);

(ii) nonequilibrium Green’s function;

(iii) molecular dynamics method [69];

(iv) Monte Carlo method;

(v) multiscale calculations.

The relaxation time in homogeneous materials ranges from 10^{-8} to 10^{-14} s [70–72] (for example, this time is 3 ps for silicon [73], 4.0–6.4 ps for mercury, and 5.1–7.3 ps for molten gallium [74]); however, the relaxation time in granulated media and biological objects can reach 30 s. For example, Kaminski [70] reported on relaxation time of 20 s for sand and 30 s for $NaHCO_3$; Mitra et al. [75] measured relaxation time of 15 s for processed meat.

However, Grassman and Peters [40], as well as Herwig and Beckert [76], did not find any proofs of the hyperbolic type of heat propagation in materials with heterogeneous internal structure. Roetzel et al. [77] explained these discrepancies by methodical errors in early experiments, associated with independent determination of thermal and physical properties of materials and the measurement of relaxation time. Roetzel et al. determined all the parameters simultaneously from the same experiment. They confirmed the deviations from the Fourier law, but obtained smaller values for relaxation time (2.26 s for sand and 1.77 s for processed meat).

In later experiments, Antaki [78] measured a relaxation time of 2 s for processed meat.

The results of observation of thermal waves in living tissues were described in review [79]. The errors in the predicted temperature distribution in tissues in cryosurgery and cryoconservation can be manifested in the form of thermoelastic stresses [80–82] and the emergence of cracks in tissues [83, 84] because of substantial thermal expansion [85]. Mechanical waves were also observed in solid argon films under sudden heating [86].

Yu et al. [87] used the low-frequency impedance method for studying the response of biological tissues to instantaneous switching from strong cooling and heating.

For analyzing heat transfer in living biological tissues, the contribution of arterial and venous blood flows must be taken into account [49, 88, 89].

The continual models of heat transfer with account for the blood circulatory system are developed by averaging the effect of a large number of blood vessels in the part of biological system under investigation. The best-known and unconditionally most important continual model was proposed by Pennes [90] in 1948 and is known as the “Pennes equation” (sometimes, the “heatsink model” [91])

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot \lambda \nabla T + Q_p, \qquad (2)$$

$$Q_p = c_b \omega_b (T_a - T) + \dot{q}_{\text{met}} + Q_{\text{ext}}, \qquad (3)$$

where T, ρ, c, and λ are the temperature, density, specific heat, and thermal conductivity of tissues as a homogeneous medium, ω_b is the velocity of blood perfusion, c_b is the specific heat of blood, T_a is the arterial blood temperature, and $\dot{q}_{\text{met}}$ and Q_{ext} are heat sources associated with metabolic reactions (these sources can usually be ignored in the cryobiological problems) and an external energy source.

The obvious extension leading to the nonlinear (modified) Pennes equation involves the account for the temperature dependence of the blood perfusion velocity $\omega_b = \omega_b^0 + \omega_b^1 T$ [92].

Deviations from the Fourier law are also observed for low-dimensional objects (spatially bounded systems [93]) such as thin films, carbon and boron-nitride nanotubes, nanowires, graphene strips [94–96], and polymer chains [97, 98]. Low-dimensional objects demonstrate the so-called "scale effect" or "size effect": thermal conductivity decreases with the sample size. For example, the thermal conductivity of crystalline nanowires is much lower than the values for the bulk material and decreases with the wire diameter [42] and upon an increase in the surface roughness [99].

The model explaining such a behavior combines the incoherent scattering of short-wavelength phonons at the surface and almost ballistic propagation of long-wavelength phonons.

The thermal conductivity of superlattices is much lower than the thermal conductivity of materials constituting them.

Upon a further decrease in its sizes, a nanowire is transformed into a molecular chain, while thin films are transformed into molecular strips. The Fermi–Pasta–Ulam (FPU) famous numerical experiment revealed that the thermal conductivity of a long chain of interacting particles can diverge upon an increase in the chain length as a positive power of the chain length in the 1D case and can demonstrate the logarithmic divergence in 2D problems [100] for so-called "integrable systems" (FPU chain, unordered harmonic chain, 1D diatomic gas, and the diatomic Toda lattice). This problem and its connection with the extremely high thermal conductivity of carbon and boron nitride nanotubes [101, 102] and graphene strips are not considered in this review. These questions were discussed in detail in the reviews by S. Lepri, R. Livi, and A. Politi, "Thermal Conduction and Classical Low-Dimensional Lattices" (*Phys. Rep.* **377** (1), 1 (2003)) and S.R. Xie, G. Zhang, B. Li Liu, and X. Xu, "Anomalous Heat Conduction and Anomalous Diffusion in Low-Dimensional Nanoscale Systems" (*Eur. Phys. J. B* **85**, 337 (2012)), and in the monograph by S. Lepri "Thermal Transport in Low Dimensions: from Statistical Physics to Nanoscale Heat Transfer," *Lecture Notes in Physics Books*, 921 (Springer, 2016).

It should be borne in mind that the temperature at a point can be determined exactly in the presence of local equilibrium; therefore, we can reliably consider the temperature difference between points separated by a distance no shorter than the mean free path of energy carriers [103, 104].

A number of models generalizing the Fourier law have been developed by modifying the defining relation between the temperature gradient and the heat flux. Most of these models take into account temporal nonlocality (material "memory"); some of them include spatial nonlocality effects for materials with an internal structure. Takahashi [105] noted that spatial nonlocality is associated with the emergence of a mesoscale intermediate between the microscopic and macroscopic scales.

1. MODELS WITH A TIME LAG

The general expression for the heat flux can be written as [106]

$$\mathbf{q} = \int_{-\infty}^{t} Q(t - t')\nabla T dt',$$

where $Q(s)$ is a positive decreasing function known as a Jeffrey relaxation kernel [107–109], which tends to zero for $s \to \infty$. For $Q(s) = \lambda\delta(s)$, where $\delta(s)$ is the Dirac delta function, we obtain Fourier law (1).

Different forms of the chosen defining relation lead to different models with time lag (see [59, 64, 110, 111] and the literature cited therein).

1.1. Maxwell–Cattaneo–Vernotte Equation

The defining relation for the Jeffrey heat flux has the form [107, 112]

$$\tau \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -\lambda \nabla T - \tau \lambda_1 \frac{\partial}{dt} \nabla T. \qquad (4)$$

For $\lambda_1 = 0$, Eq. (4) is reduced to the well-known Cattaneo equation (also known as the Maxwell–Cattaneo–Vernotte equation), which is a defining relation derived independently by Morse and Feschbach (1953), Grad (1958), and Vernotte (1958),

$$\tau \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -\lambda \nabla T, \qquad (5)$$

if, in addition, $\tau = 0$, it is reduced to the Fourier law.

The Cattaneo equation can be obtained in extended nonequilibrium thermodynamics [1, 113] dealing with dissipative flows (such as heat flows) as basic independent variables. Therefore, the entropy depends on the internal energy and heat flux, $s = s(u, q)$, and obeys the evolution equation

$$\frac{\partial s}{\partial t} + \nabla \mathbf{J}^s = \sigma^s \geq 0,$$

where $\mathbf{J}^s$ is the entropy flux and σ^s is the entropy production rate.

The definition of nonequilibrium entropy as $T^{-1} = \partial s/\partial u$ and the assumption that $\partial s/\partial q = -\alpha T$, where α is a material coefficient, leads (with account for the energy balance for a solid at rest) to equation

$$\frac{\partial s}{\partial t} = -\nabla \cdot \frac{\mathbf{q}}{T} + \mathbf{q} \cdot \left(\nabla T^{-1} - \alpha \frac{\partial}{\partial t}\right),$$

therefore,

$$\sigma^s = \mathbf{q} \cdot \left(\nabla T^{-1} - \alpha \frac{\partial \mathbf{q}}{\partial t}\right).$$

The simplest way to ensure positive entropy production rate is to presume a linear relation between the heat flux and the thermodynamic force (expression in the parentheses),

$$\nabla T^{-1} - \alpha \frac{\partial \mathbf{q}}{\partial t} = \mu \mathbf{q},$$

where μ is a positive coefficient. Introducing the notation $\alpha/\mu = \tau$ and $\mu^{-1} T^{-2} = \lambda$, we obtain the Cattaneo equation.

Jou and Casa-Vazouez [114] demonstrated that it is possible to include in this way nonlocal terms into the Cattaneo equation, assuming that the generalized entropy, the entropy flux, and the entropy production rate depend on heat flux tensor $\hat{Q}$:

$$\tau \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -\lambda \nabla T + l^2 \nabla^2 \mathbf{q}.$$

This equation differs from the Guyer–Krumhansl equation (see below) in the absence of terms of form $\nabla\nabla \cdot \mathbf{q}$.

Relaxation time τ is the time lag required for stabilization of the stationary heat flow in the volume element to which a temperature gradient is applied; time lag is a consequence of “thermal inertia.”

Thermal perturbations in this model propagate with finite velocity:

$$s = \sqrt{\frac{\lambda}{\rho C \tau}}.$$

The estimates of the relaxation time for solids and rarefied gases can be written, respectively, as [74]

$$\tau_s = \frac{3\lambda}{\bar{c}^2}$$

and

$$\tau_g = \frac{3\nu}{\bar{c}^2},$$

where $\bar{c}$ is the velocity of phonons in a solid or the average velocity of molecules in a gas,

$$\bar{c} = \sqrt{\frac{8}{\pi}\frac{k_B T}{m}},$$

where ν is the kinematic viscosity of the gas and m is the mass of a molecule.

Sometimes, the Cattaneo number is used, which is defined as

$$Ca = \frac{\kappa\tau}{L^2},$$

where $\kappa = \lambda(\rho C)$ is the thermal diffusivity.

Defining Cattaneo relation (5) can be treated as a result of the Taylor series expansion,

$$\mathbf{q}(\mathbf{r}, t + \tau) = -\lambda \nabla T(\mathbf{r}, t),$$

which is sometimes referred to as the “improved” Cattaneo relation or the single-phase lag (SPL) model [18]. Cheng et al. [115] derived the equations of the SPL model from the Boltzmann equation, using the following approximation for the time derivative:

$$\frac{\partial f}{\partial t} \approx \frac{f(t + \Delta t)}{\Delta t} = \frac{f(t + \tau)}{\tau}.$$

Recently, Li and Cao [116] have noted that the Cattaneo model should not be treated as a particular case of the SPL model because the value of discarded terms in the Taylor expansion is unknown and can be significant.

Cattaneo defining relation (5) can be written as the integral of the temperature gradient:

$$\mathbf{q} = -\frac{\lambda}{\tau} \int_{-\infty}^{t} \exp\left(-\frac{t - t'}{\tau}\right) \nabla T dt'.$$

Thus, the modified Fourier (Cattaneo) equation can be written (in the case of constant properties) as [117, 118]

$$\frac{\partial T}{\partial t} + \tau \frac{\partial^2 T}{\partial t^2} = \frac{\lambda}{\rho c} \nabla^2 T.$$

This equation can be considered as a particular case of the telegraph equation.

Frankel et al. [119] noted that an alternative formulation in terms of the heat flux (scalar equation for three components in the general case) can be useful for problems with boundary conditions including the heat flux. In this case, the temperature distribution can be found by integrating the energy conservation equation

$$T(t) = T(0) + \int_0^t \frac{1}{\rho c} [-\nabla \cdot q(t') + Q] dt'.$$

Nie and Cao [120] compared three groups of numerical methods using different representation in terms of temperature and heat flux with the hybrid representation and found that the latter approach is preferable.

Sometimes, the Cattaneo equation is called the damped version of the Fourier equation [121].

The Cattaneo model eliminates the paradox of infinite velocity of propagation of perturbation, but introduces a new one—the Cattaneo equation is not invariant to the Galileo transformations: the velocity of propagation of perturbations in a system moving with velocity U is a nonlinear function of U [122]:

$$c_{1,2} = \frac{1}{2}[U \pm \sqrt{U^2 + 4}].$$

This paradox is eliminated when instead of the partial derivative with respect to time, the material derivative is used [122]. Later, Christov [123] proposed that the Oldroyd convective derivative [124] independent of the coordinate system be used; in this case, the Cattaneo equation has form

$$\tau\left[\frac{\partial \mathbf{q}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{q} + \mathbf{q} \cdot \nabla \mathbf{v} + (\nabla \cdot \mathbf{v})\mathbf{q}\right] + \mathbf{q} = -\lambda \nabla T.$$

Joseph and Presiosi [107] proposed that the relaxation kernel be written as $R_{\mathrm{JP}} = \lambda_{1\delta}(s) + (\lambda_2/\tau)\exp(-s/\tau)$, where λ_1 is the effective thermal conductivity and λ_2 is the elastic thermal conductivity. In this case, the heat flux (in the 1D approximation) has form

$$q = -\lambda_1 \frac{\partial T}{\partial x} - \frac{\lambda_2}{\tau} \int\limits_{-\infty}^{t} \exp\left(\frac{t - s}{\tau}\right) \frac{\partial T}{\partial x} ds.$$

Barletta and Zanchini [125–127] analyzed the compatibility of the Cattaneo equation with the second law of thermodynamics and considered the Tajfel paradox (the excess of boundary values of temperature for a layer with the surfaces at different temperatures) and found that the entropy production rate can be negative in the regions in which the heat flux decreases faster than $|\partial \mathbf{q}/\partial t| > |\mathbf{q}|/\tau$. However, this result cannot be treated as a violation of the second law of classical thermodynamics based on the local equilibrium hypothesis, which does not hold [128]. The Cattaneo law is compatible with the second law in the extended nonequilibrium thermodynamics [129, 130].

Li and Cao [131] analyzed thermodynamic problems of the SPL model. Using the expression

$$S = -\frac{\mathbf{q} \cdot \nabla T}{T^2}$$

for the entropy production rate in the classical nonequilibrium thermodynamics, they obtained relation

$$S_{\mathrm{F}} = -\frac{q^2}{\lambda T^2} \geq 0$$

for the Fourier equation and

$$S_{\mathrm{SPL}}^{\mathrm{CIT}} = -\frac{\mathbf{q}(t) \cdot \mathbf{q}(t + \tau)}{\lambda T^2}$$

for the SPL model.

Obviously, the entropy production rate for the SPL model is not necessarily positive or zero. The second law is satisfies in the extended nonequilibrium thermodynamics [129], in which the entropy production rate is given by

$$S_{\mathrm{SPL}}^{\mathrm{EIT}} = -\frac{\mathbf{q}(t + \tau) \cdot \mathbf{q}(t + \tau)}{\lambda T^2}.$$

In biological problems, the Cattaneo equation (supplemented with the source terms from Pennes equation (2), (3)) is often referred to as the model of a thermal wave [132]; sometimes, the term "temperature wave" is also used [133].

1.2. Double Phase Lag Models

To account for relaxation effects, as well as the microscopic structure, double phase lag (DPL) models have been introduced [134–136]:

$$\mathbf{q}(\mathbf{r}, t + \tau_q) = -\lambda \nabla T(\mathbf{r}, t + \tau_T), \tag{6}$$

where τ_q and τ_T are the time lags for the heat flux and the temperature gradient appearing due to thermal inertia and microstructural interactions [137].

The DPL model is reduced to the SPL model (with a single phase lag) for $\tau_T = 0$ and to the Fourier law for $\tau_T = \tau_q = 0$.

Both relaxation times are extremely short for conventional materials. For example, τ_q and τ_T for gold are 8.5 and 90 ps, respectively [138].

The available data on relaxation time in biological tissues are quite contradictive. These time range from 14–16 to 0.043–0.056 s for processed meat [78]; experiments with cow muscles give values of 7.36–8.43 and 14.54–21.03 s [139]. Zhang [140] studied the relaxation time depending on the properties of tissues and blood and introduced the interfacial convective coefficient.

Equation (6) can be written in terms of the difference in relaxation times:

$$\mathbf{q}(\mathbf{r}, t) = -\lambda \nabla T(\mathbf{r}, t + (\tau_T - \tau_q)).$$

Thus, the DPL model is independent of relaxation times τ_T and τ_q separately and depends only on their difference [141]; therefore, the SPL and DPL models are analytically equivalent [142].

The DPL model is close [143] to hyperbolic models describing the energy exchange between electrons and phonons [144–146], which in 1D case have form

$$c_e \frac{\partial T_e}{\partial t} = -\frac{\partial q}{\partial x} - G(T_e - T_l) + Q,$$

$$c_l \frac{\partial T_l}{\partial t} = G(T_e - T_l),$$

$$\tau \frac{\partial q}{\partial t} + \lambda \frac{\partial T_e}{\partial x} + q = 0,$$

where T_e is the electron gas temperature; T_l is the lattice temperature; c_e and c_l are the specific heats of the electron gas and the lattice, respectively; and G is the electron–phonon coupling constant.

Tzou [147] estimated the relaxation times in the DPL model in terms of parameters G, c_e, and c_l and obtained the values of τ_T and τ_q for copper, gold, and lead of about 10^{-11} and 10^{-13} s, respectively.

Zhang [148] proposed the following expression for heat transfer relaxation times in tissues:

$$\tau_q = \frac{\varepsilon(1-\varepsilon)}{\dfrac{\varepsilon}{C_{tb}} + (1-\varepsilon)} \frac{\rho_b C_b}{G},$$

$$\tau_T = \frac{\varepsilon(1-\varepsilon)}{\dfrac{\varepsilon}{K_{tb}} + (1-\varepsilon)} \frac{\rho_b C_b}{G},$$

where $C_{tb} = (\rho_t C_t)/(\rho_b C_b)$ is the ratio of heat capacities of tissues and blood, $K_{tb} = \lambda_t/\lambda_b$ is the ratio of thermal conductivities, ε is the porosity of tissues, and G is the convection–perfusion parameter.

Tzou and Dai [149] considered the delays in a system with a large number of carriers. The equations for a system with N carriers can be written in the form

$$C_1 \frac{\partial T_1}{\partial t} = \lambda_1 \nabla^2 T_1 - \sum_{j=2}^{N} G_{1i}(T_1 - T_i),$$

$$C_m \frac{\partial T_m}{\partial t}$$
$$= \lambda_m \nabla^2 T_m + \sum_{j=1}^{m-1} G_{jm}(T_j - T_m) - \sum_{i=m+1}^{N} G_{mi}(T_m - T_i),$$
$$m = 2, 3, \ldots, (N-1),$$

$$C_N \frac{\partial T_N}{\partial t} = \lambda_N \nabla^2 T_N + \sum_{i=1}^{N-1} G_{iN}(T_i - T_N).$$

In deriving equations to the unified temperature in a system with three types of carriers (e.g., composite material with three components or polar semiconductors in which the energy can be transferred by electrons, holes, and phonons), Tzou and Dai detected nonlinear effects associated with τ_q^2 and τ_t^2.

Using the first terms of the expansion in τ_q and τ_T, we obtain

$$q + \tau_q + \frac{\partial q}{\partial t} = -\lambda\left[\nabla T + \tau_T \frac{\partial T}{\partial t}\right].$$

The application of this relation in the energy conservation law gives the type-I DPL model [79, 150] (also known as the linear DPL model [151]).

The equations of the model can be written in terms of the heat flux instead of temperature and even in terms of the heat flux potential defined as $\mathbf{q} = \nabla\phi$ [152].

Wang et al. [152] proved the correctness of the DPL model on a 1D interval with homogeneous Dirichlet, Neuman, or Robin boundary conditions. Later, Wang and Xu [153] generalized this result to the n-dimensional case.

The type-II DPL model is obtained using the first- and second-order Taylor expansions for q and T, respectively:

$$q + \tau_q \frac{\partial q}{\partial t} = -\lambda\left[\nabla T + \tau_T \frac{\partial \nabla T}{\partial t} + \frac{\tau_T}{2}\frac{\partial^2 \nabla T}{\partial t^2}\right],$$

while the type-III model (second-order DPL model [154]) is obtained using second-order expansions for q and T:

$$q + \tau_q \frac{\partial q}{\partial t} + \frac{\tau_q}{2}\frac{\partial^2 q}{\partial t^2}$$
$$= -\lambda\left[\nabla T + \tau_T \frac{\partial \nabla T}{\partial t} + \frac{\tau_T}{2}\frac{\partial^2 \nabla T}{\partial t^2}\right].$$

Sometimes, a different notation is used for distinguishing between DPL models with indication of orders of expansion, e.g., DPL (2, 1) [155]. Rukolaine [156] established that the solution in the DPL model is unstable. Later, he confirmed this conclusion for the type-III DPL model [157]

Quintanilla and Rake [158] (see also [159]) analyzed the stability of solutions in various versions of the DPL model. These authors introduced a parameter controlling the stability of solutions as the ratio of relaxation times of the DPL model:

$$\xi = \frac{\tau_T}{\tau_q}.$$

The authors considered the characteristic polynomial associated with the Laplace transformation in a bounded region in the case of the Dirichlet conditions. The solution is determined by the real part of the eigenvalue.

The results of investigation can be summarized as follows:

(i) when the first-order approximation in τ_q and the first- or second-order approximation in τ_T are used, the system is always stable;

(ii) when the second-order approximation in τ_q and the first-order approximation in τ_T are used, the system is stable if $\xi > 1/2$ and unstable if $\xi < 1/2$;

(iii) when the second-order approximation in τ_q and first-order approximation in τ_T are used, the system is stable if $\xi > 2 - \sqrt{3}$ and unstable if $\xi < 2 - \sqrt{3}$;

(iv) if $\xi > 1/2$, all types of the DPL models behave identically.

The conditions imposed on ratio $\xi = \frac{\tau_T}{\tau_q}$ were also introduced by Fabrizio and Lazzari from the second law of thermodynamics [160].

The compatibility of the DPL model with the second law of thermodynamics in extended irreversible thermodynamics was proved by Xu [161].

The thermal wave model, as well as the DPL model, is actively used for calculating the heat transfer in biological problems. For example, the action of laser radiation on biological tissues were investigated by Zhou et al. [162], Jaunich et al. [163], Afrin et al. [164], Ahmadikia et al. [165], Sahoo et al. [166], Liu and Wang [167], Poor et al. [168], Hooshomand et al. [169], Kumar and Srivastava [170], and Jasinsky et al. [171]. Zhou et al. [172] solved the 2D (axisymmetric) problem for two cases (surface heating and bulk heating); these authors found that multidimensional effects are significant.

Noroozi et al. used the thermal wave model for studying heat transport in a strip under heating by laser radiation and the DPL model for bulk heating of the strip [173].

Liu et al. [132] calculated the change in temperature under the ultrasonic action in the thermal wave model. Li et al. [174] used the DPL model for calculating the ex vivo temperature response to the action of focused ultrasonic heating on a homogeneous phantom of a biological tissue and on the heterogeneous liver tissue.

Kumar et al. [175] used the Galerkin finite-element wavelet method for studying hyperthermia under the assumption of the Gaussian nature of the external heat source.

Xo et al. [176] applied the Boltzmann lattice method for solving the DPL model of heat transport in a bilayer system.

Moradi and Ahmadikia [177] used the DPL model for calculating the heat transfer in ultrafast freezing of biological tissues (the cooling rate was about 1000°C/s [178]) when amorphous ice was formed in the frozen region [44].

Liu and Chen [179] investigated hyperthermia using a magnetic fluid in the framework of the DPL model.

Borjalilou et al. [180] applied the DPL model for solving the thermoelasticity problem (damped vibrations of a microbeam).

Chou and Yang [181] analyzed the heat transport in a multilayer structure in the 2D case using the method of spatiotemporal conservative elements [182]. This method was developed for solving the Euler and Navier–Stokes equations [183] and ensured the local and global flux conservation. The authors established different heat-transfer regimes: hyperbolic, wave, diffusion, and superdiffusion.

1.3. Triple Phase Lag Model

The triple phase lag (TPL) model is obtained from the DPL model by introducing, in addition to the heat flux relaxation times and the temperature gradient, the relaxation time for the thermal displacement gradient[3] [110, 187–189]

$$\mathbf{q}(\mathbf{r}, t + \tau_q) = -[\lambda \nabla T(\mathbf{r}, t + \tau_T) + Cc\nabla \nu(r, t + \tau_\nu)].$$

The TPL is also used in analysis of thermoelasticity problems (see, for example, [190]).

2. PHONON MODELS

Phonons are quantum oscillations of a lattice (elastic waves can exist only for certain values of energy). Phonons are energy carriers in dielectric and semiconductor crystals. Various mechanisms of phonon scattering must be considered: normal scattering (*N* process), Umklapp (*U* process), scattering at lattice defects, and scattering at boundaries.

The phonon distribution function is described by the Boltzmann equation in the form

$$\frac{\partial f}{\partial t} + \mathbf{v}\nabla f + \mathbf{F}\frac{\partial f}{\partial \mathbf{P}} = \left(\frac{\partial f}{dt}\right)_{\text{scatt}},$$

where **P** is the momentum and **F** is the external force.

For linearizing the Boltzmann equation, the following approximation of the collisional term (relaxation time approximation) is often used:

$$\left(\frac{\partial f}{dt}\right)_{\text{scatt}} = -\frac{f - f_0}{\tau},$$

where f_0 is the equilibrium distribution.

The widely used Callaway approximation [6, 96] has form

$$\left(\frac{\partial f}{dt}\right)_{\text{scatt}} = -\frac{f - f_\lambda}{\tau_N} - \frac{f - f_0}{\tau_R},$$

where f_λ is the distribution function for a uniformly drifting phonon gas, τ_N is the relaxation time of the *N* process and τ_R is the relaxation time for the *U* process.

The existence of phonons with a wide spectrum indicates the absence of a unified value of the phonon mean free path that determines the heat transport regime.

Ab initio calculations give different distributions for different materials. For example, 80% of heat in silicon are carried by phonons with a mean free path from 0.05 to 8 μm, while 80% of heat in diamond are carried by phonons with a mean free path from 0.3 to 2 μm [50]; more than 95% of heat in sapphire are car-

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[3] Thermal displacement was introduced by Helmholtz [184]. It satisfies condition $\dot{\nu} = T$. This quantity was used by Green and Naghdi [185, 186] as "scalar historical variable" $\nu = \int_0^t T(\tau)d\tau + \nu_0$.

ried by phonons with a mean free path shorter than 1 μm [39].

2.1. The Guyer–Krumhansl Equation

Guyer and Krumhansl [191, 192] solved the linearized Boltzmann equation, assuming that the normal scattering rate is much higher than the rate of *U* processes at low temperatures. They proposed a phenomenological relation between phonons and lattice vibrations associated with lattice anharmonism.

When the phonon mean free path exceeds the sample size, the behavior of the phonon gas becomes similar to the Knudsen flow or ballistic transport. These authors determined the conditions in which a Poiseuille flow makes a significant contribution to heat conduction.

The Poiseuille flow in a cylinder is escribed by equation [114]

$$\Lambda^2\nabla^2\mathbf{q} = \lambda T$$

the solution to which is $q(r) = A(R^2 - r^2)$, $A = -(\lambda\nabla T/(\Lambda^2))$.

The Guyer–Krumhansl equation is usually written in the form

$$\tau\frac{\partial\mathbf{q}}{\partial t} + \mathbf{q} = -\lambda\nabla T + \beta'\Delta\mathbf{q} + \beta''\nabla c\mathrm{dot}\nabla\mathbf{q},$$

where β' and β'' are the Guyer–Krumhansl coefficients (in the case of a rarefied gas, these coefficients are connected with the Callawey integral [22]), or

$$\tau\frac{\partial\mathbf{q}}{\partial t} + \mathbf{q} = -\lambda\nabla T + \Lambda^2(\nabla^2\mathbf{q} + 2\nabla\cdot\nabla\mathbf{q}),$$

where Λ is the phonon mean free path.

The Guyer–Krumhansl equation can be obtained in the extended nonequilibrium thermodynamics under the assumption that nonlocal terms can be included into the expression for the entropy flux as [1]

$$\mathbf{J}^s = \frac{\mathbf{q}}{T} + \gamma(\mathbf{q}\cdot\nabla\mathbf{q} + 2\mathbf{q}\nabla\cdot\mathbf{q}),$$

where γ is a positive coefficient.

Recently, Calvo-Schwartzwalder et al. [193] used the Guyer–Krumhansl equation for solving the 1D Stefan problem.

2.2. Ballistic–Diffusion Model

The ballistic–diffusion model introduced by Chen [194, 195] is based on the splitting of the distribution function (as well as internal energy and heat flux) into two components:

$$f = f_b + f_d, \quad e = e_b + e_d, \quad \mathbf{q} = \mathbf{q}_b + \mathbf{q}_d,$$

which reflects the coexistence of two types of heat carriers:

(i) ballistic phonons that are mainly scattered at the boundaries;

(ii) diffusion phonons experiencing numerous scattering events in the bulk of the system.

The relative role of these components is determined by the value of the Knudsen number and by the system geometry [196].

Yang et al. [197] used the Boltzmann equation for relative phonon intensity $I_\omega = \mathbf{v}_\omega\hbar\omega f D(\omega)/(4\pi)$ ($\mathbf{v}_\omega$ is the group velocity of carriers, ω is the phonon frequency, $D(\omega)$ is the phonon density of states per unit volume, and S_ω is the source term that can be determined, for example, by the electron–phonon scattering)

$$\frac{\partial I_\omega}{\partial t} + \mathbf{v}_\omega\nabla I_{b\omega} = -\frac{I_\omega - I_{0\omega}}{\tau_\omega} + S_\omega.$$

The equations for the ballistic and diffusion components can be written as

$$\frac{\partial I_{b\omega}}{\partial t} + \mathbf{v}_\omega\nabla I_{b\omega} = -\frac{I_{b\omega}}{\tau_\omega} + S_\omega$$

and

$$\frac{\partial I_{b\omega}}{\partial t} + \mathbf{v}_\omega\nabla I_{d\omega} = -\frac{I_{d\omega} - I_{0\omega}}{\tau_\omega} + S_\omega,$$

respectively.

Allen [198] analyzed a transition from the ballistic to diffusion regime using computer simulation and the version of the Boltzmann equation subjected to the Fourier transformation. The evolution equations for the mean population in the reciprocal space of phonon mode Q includes a number of terms (drift, scattering, and external terms)

$$\frac{\partial N_Q}{\partial t} = \left(\frac{dN_Q}{dt}\right)_{\mathrm{drift}} + \left(\frac{dN_Q}{dt}\right)_{\mathrm{scatt}} + \left(\frac{dN_Q}{dt}\right)_{\mathrm{ext}},$$

$$\left(\frac{dN_Q}{dt}\right)_{\mathrm{drift}} = -\mathbf{v}_Q\cdot\nabla N_Q = -\mathbf{v}_Q\left[\frac{dn_Q}{dT}\nabla T + \nabla\Phi_Q\right],$$

$$\left(\frac{dN_Q}{dt}\right)_{\mathrm{scatt}} = -\sum_{Q'} S_{Q,Q'}\Phi_{Q'},$$

where n_Q is the local equilibrium Bose–Einstein distribution, $\Phi_Q = N_Q - n_Q$, and $S_{Q,\,Q'}$ is the linearized scattering operator.

Vazqurz et al. [36] and Lebon et al. [1] considered the two-temperature version of the ballistic–diffusion model. Vasquez et al. used the Guyer–Krumhansl equation for describing the ballistic and diffusion heat flows. Lebon et al. used the Guyer–Krumhansl equation only for the ballistic transport; the diffusion component was described by the Cattaneo equation.

Pumarol et al. [95] reported the direct observation of the ballasting and diffusion transport in graphene.

2.3. Generalized Fourier–Hua Law

Hua et al. [199] developed the generalized Fourier law, which holds from the ballistic to the diffusion regime and is based on the Boltzmann equation for modes in the relaxation time approximation (Bhatnagar–Gross–Krook model)

$$\frac{\partial g_\mu(x,t)}{\partial t} + \mathbf{v}_\mu \cdot g_\mu(\mathbf{x},t) = -\frac{g_\mu - g_0(T,\mathbf{x},t)}{\tau_\mu} + \dot{Q}_\mu,$$

where $g_\mu(\mathbf{x}, t) = \hbar\omega_\mu(f_\mu(\mathbf{x}, t) - f_0(T))$ is the distribution function for energy difference for a phonon in state $\mu = (\mathbf{q}, s)$; $\mathbf{q}$ is the phonon wavevector; s is the phonon branch index; f_0 is the Bose–Einstein distribution function; $g_0(T) = \hbar\omega_\mu f_0(T) - f_0(T_0)) \approx C_\mu \Delta T$; C_μ is the specific heat depending on the mode; and $\dot{Q}$ is the energy supply rate per mode.

For solving this equation, the authors used the Fourier transform in time and connected the temperature gradient with mode-dependent heat capacity $\mathbf{x}$.

2.4. Phonon Hydrodynamics

Guo and Wang [200] obtained the macroscopic equation of motion of a phonon gas based on the Boltzmann equation for phonons,

$$\frac{\partial f}{\partial t} + \mathbf{v}_g = C(f),$$

where $f = f(\mathbf{x}, t, \mathbf{k})$ is the phonon distribution function and $\mathbf{v}_g = \nabla_{\mathbf{k}}\omega$ is the group velocity of phonons.

Collisional term $C(f)$ includes the contributions from the two main scattering processes: normal scattering (N process) and scattering with a momentum loss (R process).

Energy is conserved in any type of scattering process, while quasi-momentum is conserved only in normal scattering.

A simplified Boltzmann equation is based on the application of the Callawey relaxation time model, which presumes that the N and R processes occur independently. At low temperatures, the N process dominates; however, at ordinary temperatures, the N process can be ignored, and the Boltzmann equation takes form

$$\frac{\partial f}{\partial t} + \mathbf{v}_g = -\frac{f - f_R^{\mathrm{eq}}}{\tau_R},$$

where f_R^{eq} is the equilibrium distribution function for the R process.

In the phonon hydrodynamics model, the following field variables are used:

phonon energy density

$$e = \int \hbar\omega f d\mathbf{k},$$

heat flux

$$\mathbf{q} = \int \hbar\omega \mathbf{v}_g f d\mathbf{k},$$

and heat flow flux

$$\hat{Q} = \int \hbar\omega \mathbf{v}_g \mathbf{v}_g f d\mathbf{k}.$$

The integration of the Boltzmann equation in the space of wavevectors gives the balance equations for the energy density,

$$\frac{\partial e}{\partial t} + \nabla \cdot \mathbf{q} = 0$$

and of heat flux

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla \cdot \hat{Q} = -\frac{\mathbf{q}}{\tau_R}.$$

These balance equations are the four-momentum field equations for the photon Boltzmann equation[4] To close the system of phonon transport equations, the heat flow flux $\hat{Q}$ must be defined in terms of four main field variables (energy density and three heat flux components).

There are several approaches to the problem of closure in the kinetic theory:

(1) the Gilbert method;
(2) the Chapman–Enskog method;
(3) the Grad method of moments;
(4) "regular method of moments" (R13 method);
(5) method of invariant manifolds.

The authors used the perturbation method relative to the four-moment phonon distribution function obtained using the entropy maximum principle.

Thus, the problem was reduced to the maximization of the following functional:

$$\Phi = -k_{\mathrm{B}} \int [f \ln(f) - (1 + f)\ln(1 + f)] d\mathbf{k} + \beta\left(e - \int \hbar\omega f d\mathbf{k}\right) + \gamma_i \left(q_i - \int v_{gi} \hbar\omega f d\mathbf{k}\right),$$

where β and γ_i are the Lagrangian multipliers.

Ultimately, the four-moment phonon distribution function took form

$$f_4 = \frac{1}{\exp\left(\beta \frac{\hbar\omega}{k_{\mathrm{B}}} + \gamma_i \frac{\hbar\omega}{k_{\mathrm{B}}}\right) - 1}.$$

Higher-order approximations for heat flow flux $\hat{Q}$ can be determined from the balance equation

$$\frac{\partial Q_{ij}}{\partial t} + \frac{\partial M_{ijk}}{\partial x_k} = \frac{1}{\partial \tau_R}\left(\frac{1}{3} v_g^2 e f_{ij} - Q_{ij}\right),$$

and third-order tensor $\hat{M}$ is defined as

$$M_{ijk} = \int v_{gi} v_{gj} v_{gk} \hbar\omega f d\mathbf{k}.$$

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[4] In [200], $\hat{Q}$ is misprinted as a vector.

The following expansion in the small parameter (Knudsen number) $\varepsilon = \mathrm{Kn}$ is used:

$$Q_{ij} = Q_{ij}^{(0)} + \varepsilon Q_{ij}^{(1)} + \dots$$

and the zeroth- and first-order terms are conserved,

$$Q_{ij} = \frac{1}{3}v_g^2\varepsilon\delta_{ij} + \frac{2}{15}\tau_R v_g^2\frac{\partial q_k}{\partial x_k}\delta_{ij} - \frac{1}{5}\tau_R v_g^2\left(\frac{\partial q_i}{\partial x_j} + \frac{q_j}{\partial x_i}\right).$$

Finally, the balance equation takes form

$$\tau_R\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -\lambda\nabla T + \frac{1}{5}\Lambda^2\left[\nabla^2\mathbf{q} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{q})\right],$$

where $\Lambda = v_g\tau_R$ is the mean free path.

This equation differs from the Guyer–Krumhansl equation only in the numerical coefficient of the nonlocal term. The authors emphasized that the analytic structure of these equation is the same, but the heat transport mechanisms underlying these equations are different. The nonlocal term in the Guyer–Krumhansl equation reflects normal scattering of phonons (this equation described heat transport at low temperatures). The nonlocal term in the phonon hydrodynamics equation is a consequence of spatial nonequilibrium effects emerging due to scattering of phonon at the boundaries or because of high temperature gradients.

Guo and Wang used the equations derived in the phonon hydrodynamics for solving a number of problems, including

(i) phonon transport in the plane of a thin film;

(ii) phonon transport in a nanowire;

(iii) nonstationary 1D phonon transport across a thin film;

(iv) high-frequency periodic heating of a semi-infinite surface, and

(v) nonstationary heat transport in a thermal grating.

2.5. Relaxon Model

Recently (2020), Simoncelli et al. [12] used the evolution of relaxons for deriving a pair of equations describing coupled collective lattice vibration in dielectric crystals.

The concept of relaxons as a collective nonequilibrium excitation of a crystal lattice, which is a linear combination of phonons, was introduced by Chepellotti and Marzari [201]. Heat conduction can be treated as the flow of a relaxon gas.

The authors proceeded from the linearized Boltzmann equation for a phonon in the form

$$\frac{\partial n_\mu}{\partial t} + \mathbf{v}_\mu\nabla n_\mu = -\frac{1}{V}\sum_{\mu'}\Omega_{\mu\mu'}\Delta n_{\mu'}.$$

The summation is carried out over all possible phonon states μ ($\mu = (\mathbf{q}, s)$, where $\mathbf{q}$ varies over the Brillouin zone and s varies over phonon branches), $\mathbf{v}_\mu$ is the phonon group velocity, V is the volume, $\Omega_{\mu\mu'}$ is the linear operator of phonon scattering, $\Delta n_\mu = n_\mu - \bar{n}_\mu$ is the deviation of the phonon distribution function from the equilibrium function, i.e., from the Bose–Einstein distribution

$$\frac{1}{\exp\left(\frac{\hbar\omega_\mu}{k_\mathrm{B}T}\right) - 1},$$

where ω_μ is the phonon frequency.

Since the Bose–Einstein distribution depends on the spatial coordinates and time only in terms of temperature, the equation can be written in the form

$$\frac{\partial\bar{n}\mu}{\partial T}\left(\frac{\partial T}{\partial t} + \mathbf{v}_\mu\cdot\nabla T\right) + \frac{\partial\Delta n_\mu}{\partial t} + \mathbf{v}_\mu\cdot\nabla\Delta n_\mu = -\frac{1}{V}\sum_{\mu'}\Omega_{\mu\mu'}\Delta n_{\mu'}.$$

The solution to this equation in closed form is possible in the unified relaxation time approximation, when the collision operator is simplified [201]:

$$\frac{1}{V}\sum_{\mu'}\Omega_{\mu\mu'}\Delta n_\mu \approx \frac{\Delta n_\mu}{\tau_\mu^{\mathrm{STA}}}.$$

The thermal conductivity in the harmonic approximation for heat flux

$$q = \sum_\mu \hbar\omega_\mu v_\mu\Delta n_\mu$$

can be written as

$$(\lambda_\mu^{ij})^{\mathrm{SMA}} = \frac{1}{V}\sum_\mu C_\mu v_\mu^i(\Lambda_\mu^j)^{\mathrm{SMA}},$$

where $(\Lambda_\mu^j)^{\mathrm{SMA}}$ is the phonon mean free path component in the j direction.

Thus, heat conduction is ensured by phonons carrying specific heat

$$\frac{C_\mu}{k_\mathrm{B}T^2}\bar{n}_\mu(\bar{n}_\mu + 1)(\hbar\omega_\mu)^2,$$

moving with velocity v_μ, and having mean free path $(\Lambda_\mu^j)^{\mathrm{SMA}}$ prior to thermalization during scattering.

Cepellotti and Marzari emphasized that the definition of the phonon lifetime or mean free path cannot be used without the assumption of the unified relaxation time because the nondiagonal terms of the scattering operator introduce the coupling between phonons, and the phonon thermalization cannot be described by the equipotential relaxation.

Operator Ω can be reduced to symmetric form using the transformation

$$\bar{\Omega}_{\mu\mu'} = \Omega_{\mu\mu'}\sqrt{\frac{\bar{n}_{\mu'}(\bar{n}_{\mu'}+1)}{\bar{n}_{\mu}(\bar{n}_{\mu}+1)}},$$

$$\Delta\bar{n}_{\mu} = \frac{\Delta n_{\mu}}{\bar{n}_{\mu}(\bar{n}_{\mu}+1)}.$$

Since $\bar{\Omega}$ is a real-valued positive matrix, it can be reduced to diagonal form, and eigenvectors θ_{μ}^{α} and eigenvalues $1/\tau_{\alpha}$ can be determined (α is the eigenvalue index):

$$\frac{1}{V}\sum_{\mu'}\Omega_{\mu\mu'} = \frac{1}{\tau_{\alpha}}\theta_{\mu}^{\alpha}.$$

Arbitrary quantity $\Delta\bar{n}_{\mu}$ can be represented as a linear combination of eigenvectors θ_{μ}^{α}:

$$\Delta\bar{n}_{\mu} = \sum_{\alpha} f_{\alpha}\theta_{\mu}^{\alpha}.$$

The Boltzmann equation can be written in the basis of eigenvectors θ^{α}:

$$\sqrt{\frac{C}{k_{\mathrm{B}}T^2}}\left(\frac{\partial T}{\partial t}\langle 0|\alpha\rangle + \Delta T + \mathbf{V}_{\alpha}\right) + \frac{\partial f_{\alpha}}{\partial t} + \sum_{\alpha'}\mathbf{V}_{\alpha\alpha'}\cdot\nabla f_{\alpha} = -\frac{f_{\alpha}}{\tau_{\alpha}}.$$

Cepelloti and Marzari derived the following equation for the thermal conductivity:

$$\lambda^{ij} = \frac{-1}{V\nabla_i T}\sum_{\mu}\hbar\omega_{\mu}\nu_{\mu}^{j}\Delta n_{\mu} = \sum_{\alpha} CV_{\alpha}^{i}\lambda_{\alpha}^{j}.$$

These authors also demonstrated that the Matthiessen rule that is widely used for estimating the relaxation time in systems with different scattering mechanisms overestimates the thermal conductivity.

Relaxons possess parity, and only odd relaxons contribute to heat conduction. Even relaxons determine thermal viscosity [12]. Simoncelli et al. derived two coupled equations for the temperature and drift velocity.

3. THERMOMASS MODEL

The thermomass model is based on the old idea put forth by Tolman [202]: heat carriers exhibit the mass–energy duality and manifest energy properties in energy conversion processes and mass properties in transport processes [203].

The heat mass is defined in accordance with the Einstein principle of equivalence of mass and energy [204–206]:

$$E = Mc^2 = \frac{M_0c^2}{\sqrt{1-\frac{v^2}{c^2}}},$$

where E is the thermal energy, M_0 is the rest mass, v is the heat carrier velocity, c is the velocity of light in vacuum, and M is the relativistic mass.

If $v \ll c$, this equation is simplified:

$$E \approx (M_0 + M_k)c^2.$$

Here, M_k is an additional mass induced by the kinetic energy. Thermomass (TM) M_h is the relativistic mass of internal energy U:

$$M_h = \frac{U}{c^2}.$$

The thermomass is extremely small (10^{-16} kg for 1 J of heat) [203].

The thermomass density contained in a medium is given by [207]

$$\rho_h = \frac{\rho C_V T}{c^2},$$

where $\rho C_V T$ is the internal energy density.

A thermon is defined as a quasiparticle carrying thermal energy.

The macroscopic drift velocity of the thermon gas for a continuous medium is defined as

$$v_h = \frac{q}{\rho CT}.$$

The total energy of the thermon gas in a medium is the sum of the kinetic and potential energies [208].

$$E_T = \iint_V (\rho_T u_T du_T + dp_T)dV,$$

where V is the total volume of the medium.

3.1. Equation of State (EOS) of the Thermon Gas

3.1.1. EOS of the thermon gas in an ideal gas. The following two assumptions are made for the thermon gas in an ideal gas:

(1) thermons are coupled with gas molecules and are described by the Maxwell–Boltzmann distribution;

(2) Newtonian mechanics is applicable to the thermon gas.

The pressure in a system of n particles of mass m, which move randomly in the x direction with their velocity, u_x, is

$$P = nmu_x^2$$

with account of symmetry in the x, y, and z directions ($u_x^2 = u_y^2 = u_z^2 = \frac{1}{3}\bar{u}^2$) [206], we get

$$P = \frac{1}{3} n m_h \bar{u}^2 = \frac{1}{3} n \bar{u}^2 \left(\frac{1}{2}\frac{m\bar{u}^2}{c^2}\right) = \frac{1}{6}\frac{nm\bar{u}^4}{c^2}.$$

Using the Maxwell–Boltzmann distribution function

$$f_{\mathrm{M}}(u) = 4\pi u^2 \left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{3/2} \exp\left(\frac{mu^2}{2k_{\mathrm{B}}T}\right),$$

we obtain

$$\bar{u}^4 = \int_0^\infty u^4 f_{\mathrm{M}}(u)du = 15\left(\frac{k_{\mathrm{B}}T}{m}\right)^2$$

and, finally,

$$P = \frac{5}{3}\frac{\rho C_V R T^2}{c^2},$$

where R is the gas constant.

Thus, the pressure of the thermon gas in the ideal gas is proportional to the square of temperature.

3.1.2. EOS for a thermon gas in dielectrics. Phonons are thermons in dielectrics. The total energy of lattice vibrations is

$$E_D = E_{D0} + E_h = (M_0 + M_h)C_V T,$$

where E_h is the thermomass energy. The thermon gas pressure is then

$$P = \frac{\gamma\rho}{c^2}(C_V T)^2,$$

where γ is the Grüneisen constant.

The thermon gas pressure is proportional to the square of temperature like in an ideal gas.

The thermon gas pressure in silicon at room temperature is about 5×10^{-3} Pa [206].

3.1.3. EOS of a thermon gas in metals. In metals, thermons are attached to electrons. The thermon gas pressure is

$$P = \frac{1}{3} n m_h u_h^2,$$

where $m_h = \varepsilon/c^2$, ε being the internal energy including the contributions from electrons and from the lattice, and $u_h = \sqrt{\frac{2\varepsilon}{m}}$ is the velocity of randomly moving particles. Therefore, the pressure is given by

$$P = \frac{2}{3}\frac{n\varepsilon^2}{c^2 m}.$$

The general expression for the thermon gas pressure is

$$P = \frac{2}{3mc^2}\int_0^\infty \varepsilon^2 f(\varepsilon, T) Z(\varepsilon) d\varepsilon,$$

where

$$f(\varepsilon, T) = \frac{1}{\exp\left(\frac{\varepsilon - \varepsilon_{\mathrm{F}}}{k_{\mathrm{B}}T}\right) + 1}$$

is the Fermi–Dirac distribution function and

$$Z(\varepsilon) = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}$$

is the Sommerfeld function of the electron density of states. Wang [206] derived the following expression for the thermon gas pressure:

$$P = \frac{5}{12}\frac{\pi^2 n k_{\mathrm{B}}^2}{c^2 m} T^2.$$

3.2. Equation of Motion of the Thermon Gas

One-dimensional equations of conservation of mass and momentum have form

$$\frac{\partial\rho_h}{\partial t} + \frac{\partial\rho_h u_h}{\partial x} = \frac{S}{c^2},$$

where S is the heat source and

$$\frac{\partial}{\partial t}(\rho_h u_h) + \frac{\partial}{\partial x}(u_h \cdot \rho_h u_h) + \frac{\partial P}{\partial x} + f_h = 0;$$

where f_h is the resistance.

The continuity equation of the thermon gas is in fact the equation of energy conservation.

The flow of thermons in a solid can be treated as the motion of a compressible fluid in a porous medium; consequently, the Darcy law (K is the penetrability of the porous medium)

$$u = -K\frac{dP}{dx}$$

can be used for estimating thermomass resistance $f_h = \beta_h u_h$, β being the proportionality factor [209],

$$\beta_h = \frac{e\gamma\rho^2 C^3 T^2}{c^2\lambda}.$$

Effective resistance force f_h is introduced instead of the viscous term ($\mu_h \nabla^2 \mathbf{u}_h$) to avoid [203]

(1) the determination of viscosity μ_h of composite materials;

(2) the effects of interaction of thermons with the lattice.

The flow of a thermon gas in a solid (phonon flow) is induced by the pressure gradient (hence, by the gradient of the squared temperature [2]).

The thermomass is too small to be observed in standard conditions. However, under ultrafast heating or at extremely large value of the heat flux, the inertia of the thermomass leads to effects in heat conduction, which permit their detection.

The law of conservation of the thermon gas momentum can be written as the heat conduction equation [206]

$$\tau_h = \left(\frac{\partial q}{\partial t} + 2u_h\frac{\partial q}{\partial x} - u_h^2\rho C_V\frac{\partial T}{\partial t}\right) + \lambda\frac{\partial T}{\partial t} + q = 0.$$

Wang [206] developed a wo-step version of the thermomass theory for metals subjected to ultrafast laser heating under the following assumptions:

(1) electrons absorb laser energy and transfer it to the lattice;

(2) scattering at defects and grain boundaries is disregarded;

(3) the electron–phonon interaction is described by the coupling coefficient.

Analogously to the flow of a compressible fluid in a porous medium, Brinkman's correction $\mu\nabla^2\mathbf{q}$, which takes into account the additional resistance due to the walls, can be introduced into the equation of motion of the thermon gas. This correction is significant only for large values of the Knudsen number [207].

The entropy production during the motion of the thermon gas is ensured by dissipation of mechanical energy,

$$\frac{dE_h}{\partial t} + \nabla\mathbf{J}_{E_h} = \mathbf{f}_h\cdot\mathbf{u}_h,$$

where E_h is the mechanical energy of the thermon gas and $\mathbf{J}_{E_h}$ is the flux of E_h.

The total derivative of the entropy density can be written as [205, 210]

$$\frac{dS}{dt} = -\nabla\mathbf{J}_s + \sigma_{TM} = \frac{\mathbf{q}}{\lambda T^2}\cdot(\mathbf{q} + \lambda\nabla T) - \frac{\nabla\cdot\mathbf{q}}{T},$$

where $\mathbf{J}_s$ is the entropy flux.

3.3. Heat Flux Trapping

The thermon gas is a compressible fluid, which demonstrates effects typical of a compressible fluid (e.g., air).

One of such effects is the behavior of a gas in a convergent nozzle for the Mach number equal to unity.

The thermal Mach number is defined as

$$\mathrm{Ma}_h = \frac{u_h}{C_h},$$

where the velocity of sound (e.g., in dielectrics) is $C_h = \sqrt{2\gamma C_V T}$.

During the compressible air flow induced by a pressure gradient, the velocity in a convergent channel increases, while the pressure decreases in the direction of the flow. The flux trapping occurs when the Mach number attains unity; in this case, a jump appears in the pressure values.

The drift velocity of the phonon gas induced by the temperature gradient increases in the direction opposite to the temperature gradient. The heat flux trapping occurs when the thermal Mach number attains unity; in this case, the temperature experiences a jump.

The effect of the heat flux trapping was confirmed in experiments on heat conduction in single-wall carbon tubes suspended between metal electrodes.

4. MESOSCOPIC EQUATIONS FOR MOMENTS

In the framework of the kinetic theory, Bergamasco et al. [121] developed a number of moment (two-moment and three-moment) equations by introducing the Knudsen number as the ratio of the mean free path of heat carriers to the characteristic size of the system. These authors also introduced the concept of a "ghost" moment.

5. THERMODYNAMIC MODELS

Thermodynamic models of heat conduction are derived from thermodynamic limitations following from the second law of thermodynamics [211–218].

For example, Jou and Cimmelli [216] introduced an additional new variable, i.e., second-order tensor $\hat{Q}$, and wrote the balance equation in the form

$$\tau_l\dot{\mathbf{q}} + \mathbf{q} = -\lambda\nabla T + \nabla\cdot\hat{Q},$$

where τ_l is the relaxation time. Tensor $\hat{Q}$ is assumed to be symmetric and can be split as $\hat{Q} = Q\hat{I} + \hat{Q}_s$, where scalar Q is one-third of the trace of $\hat{Q}$ and $\hat{Q}_s$ is the deviator part of $\hat{Q}$.

These authors derived the general equation including the Cattaneo and Guyer–Krumhansl equations as particular cases:

$$\tau_R\mathbf{q} + \mathbf{q} + (\mu\nabla\mathbf{q} + \mu'\nabla^t\mathbf{q})$$
$$-\lambda(1 + \xi\mathbf{q}\cdot\mathbf{q})\nabla T + l_p^2(\nabla^2\mathbf{q} + 2\nabla\nabla\cdot\mathbf{q}),$$

where μ, μ', and ξ are material coefficients and $\nabla^t\mathbf{q}$ indicates the transposition of $\nabla\mathbf{q}$.

Kovacs and Van [213] also introduced a second-order tensor as an intrinsic variable and proposed that the entropy flux can be written as

$$\mathbf{J} = \hat{b}\cdot\mathbf{q} + \hat{B} : \hat{Q},$$

where $\hat{b}$ is a second-order tensor and $\hat{B}$ is a third-order tensor, which are known as current factors (Niiri factors).

Using the limitations following from the second law of thermodynamics (nonnegative entropy production) and excluding the intrinsic variable, the authors derived the general defining relation for the heat flux:

$$
\begin{aligned}
&m_1 m_2 \partial_{tt} q + (m_1 l_1 + m_1 k_1)\partial_t q \\
&- (m_1 n + m_2 k_2)\partial_{xxt} q + n k_2 \partial_x^4 q - (l_1 + K)\partial_{xx} q \\
&+ k_1 l_1 q = m_2 \partial_{xt}\frac{1}{T} + k_1 \partial_x \frac{1}{T} + \partial_x^3 \frac{1}{T}.
\end{aligned}
$$

Choosing the material coefficients that can be set at zero, we can obtain a number of familiar heat conduction models, such as

(i) Fourier model;

(ii) Cattaneo model;

(iii) ballistic–diffusion model; and

(iv) Guyer–Krumhansl model.

Rogolino et al. [218] chose as main variables the specific internal energy (per unit volume), internal energy, heat flux, and the flux of the heat flow. Presuming the form of the corresponding balance equation, using entropy limitations and the Lagrange–Farkas multipliers, the authors obtained two versions of the generalized heat conduction law:

(1) second-order equation in space and first-order equation in time, disregarding nonlocal effects;

(2) fourth-order equation in space and second-order equation in time, including nonlocal effects.

6. NONLOCAL MODELS WITH FRACTIONAL DERIVATIVES

6.1. Fractional Derivatives

There is no generally accepted definition of a fractional derivative [219–222]. The most popular are the Riemann–Liouville and Caputo definitions. Both derivatives are based on the Riemann–Liouville fractional integral that is defined for any $\lambda > 0$ as [223, 224]

$$J_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^1 (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

Here, $\Gamma(\alpha) = \int_0^\infty \exp(-\alpha) u^{\alpha - 1} du$ is the Euler gamma function.

This integral exists if $f(t)$ is locally integrable and behaves for $t \to 0$ as $O(t^{-\nu})$, where $\nu < \alpha$.

(i) The Riemann—Liouville fractional derivative is defined as [225]

$$
\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}
= \begin{cases}
\dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^1 \frac{\partial u(x,s)}{\partial t} (t-s)^{-\alpha} dt, & \alpha \in (0,1) \\
\dfrac{\partial u(x,t)}{\partial t}, & \alpha = 1.
\end{cases}
$$

(ii) The Caputo fractional derivative is defined as [225]

$$
\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}
= \begin{cases}
\dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t \frac{\partial u(x,s)}{\partial t} (t-s)^{-\alpha} dt, & \alpha \in (0,1) \\
\dfrac{\partial u(x,t)}{\partial t}, & \alpha = 1.
\end{cases}
$$

There exist a number of other definitions of fraction derivatives apart from the Riemann—Liouville and Caputo definitions (by Grünfeld–Letnikov, Riesz, Weyl, Marchaud, Caputo–Fabrizio, Yang, Chen, He, Jao, Qi–Jiang, and others), which are not equivalent [226].

6.2. Differential Equations with Fractional Derivatives

Usually, a fractal medium cannot be treated as a continuous medium. The noninteger-dimension spaces [227] are required for describing a fractal medium with the help of continual models [228]. Equations with fractional derivatives [229] (Abel and Liouville were the first to use fractional calculus) are nonlocal (i.e., can include memory effects and spatial correlations) and can describe anomalous diffusion (both subdiffusion and superdiffusion), as well as anomalous heat conduction [230] (e.g., heat conduction in a porous medium is described by the superdiffusion model [231]).

The initial conditions for the Caputo fractional derivative can be formulated in terms of initial conditions in the integer derivatives. Zero initial conditions for the Riemann–Liouville, Caputo, and Grünfeld–Letnikov derivatives are the same [232].

6.3. Fourier Fractional Model

Deng and Ge [233] studied heat transport in a fractal medium using the Helmholtz fractional equation

$$\frac{\partial^{\alpha} T(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\beta} T(x,y)}{\partial y^{2\beta}} + k^2 T(x,y) = f(x,y),$$

where $0 < \alpha \le 1$ and $0 < \beta \le 1$.

He and Liu [234] used the fractional version of the Fourier law

$$\lambda^{2\alpha} \frac{\partial^{\alpha} T}{\partial x^{\alpha}} = q$$

for studying heat transfer in the system of silk cocoons.

Beibalaev et al. [235, 236] used an analogous approach for analyzing heat transport in a fractal medium and for studying soil freezing.

He at al. [237] used the time-independent equation

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\lambda \frac{\partial^{\alpha} T}{\partial x^{\alpha}}\right) = 0,$$

while Wang et al. [238] used the time-dependence equation

$$\frac{\partial T}{\partial t}+\frac{\partial^\alpha}{\partial x^\alpha}\left(\lambda\frac{\partial^\alpha T}{\partial x^\alpha}\right)=0$$

for analyzing heat transport in the fractal model of polar bear fur.

Meylanov and Shabanova [239] solved 1D problems with the fractional equation

$$\frac{\partial^\alpha T}{\partial t^\alpha}=\lambda\frac{\partial^\beta T}{\partial t^\beta}.$$

Voller et al. [240] used the time-fractional model, while Meylanov et al. [241] employed the model fractional in time and space for solving the Stefan problem. Voller et al. considered both sharp and diffuse boundaries between the liquid and solid phases.

Sirotsyuk et al. [242] used the time-fractional Fourier equation for calculating heat transport in a heterogeneous semi-infinite beam.

6.4. Pennes Fractional Model

The generalization of Pennes equation (2) with the help of the Caputo fractional derivative

$$\rho c\frac{\partial^\alpha T}{\partial t^\alpha}=\nabla\cdot\lambda\nabla T+c_b\omega_b(T_a-T)+\dot{g}_{\text{met}}+Q^{\text{ext}},\quad (7)$$

was used by Darmor et al. [243] for analyzing hyperthermia and anomalous heat conduction in dermal tissues under a constant and sinusoidal heating of the surface [244, 245] and by Ezzat et al. [246] for calculating the nonstationary heat transport in skin under instantaneous heating of the surface.

It was noted by Ferras et al. [247] that Eq. (7) is incorrect as regards the dimension, and it is necessary to either redefine the coefficients in the classical equation or introduce factor $\tau^{1-\alpha}$ for obtained the "new" thermal conductivity.

Singh et al. [248] employed the equation fractional in time and space,

$$\rho C\frac{\partial^\beta}{\partial t^\beta}T=\lambda\frac{\partial^\alpha}{\partial x^\alpha}+Q_p,\quad 0<\beta\le 1<\alpha\le 2.$$

to analyze the temperature field in tissues during hyperthermia.

6.5. Zingales Fractional Model

Zingales [249] (see also [250]) considered two components of heat transfer in solids at rest:

(1) "close" heat transport described by the conventional Fourier law;

(2) heat transfer between remote elementary volumes located at points $\mathbf{x}$ and $\mathbf{y}$, which is proportional to

(i) the product of interacting masses;

(ii) temperature difference $T(\mathbf{x})-T(\mathbf{y})$;

(iii) decreasing function of distance $g(\|\mathbf{x}-\mathbf{y}\|)$.

Thus, this author proposed that function g decreases as a power-law function of distance:

$$g(\|\mathbf{x}-\mathbf{y}\|)=\frac{1}{d_n(\overline{\alpha})}\cdot\frac{1}{\|\mathbf{x}-\mathbf{y}\|^{n+\alpha}},$$

where $d_n(\overline{\alpha})$ is the normalizing factor connected with decreasing exponent α and with dimension n of the topological space of the body.

Finally, the energy conservation equation can be written in the form

$$\rho C\frac{\partial T}{\partial t}=-\nabla\mathbf{q}+\rho^2\lambda_\alpha D_x^\alpha T,$$

where D_x^α is the Marchaud fractional derivative of order α, which is defined as

$$D_x^\alpha T=\frac{1}{d_n(\overline{\alpha})}\int_{V_y}^{\square}\frac{T(\mathbf{x})-T(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|^{n+\alpha}}dV_y.$$

6.6. Cattaneo and SPL Fractional Models

The fractional version of the Cattaneo equation is sometimes called the Cattaneo–Vernotte "nonlocal" equation, which reflects the properties of fractional derivatives [251].

Liu et al. [252] used the Christov modification of the Cattaneo model for developing the fractional equation using the spatially fractional Riesz derivative.

The time-fractional SPL model of the biological heat transfer was formulated in [253]:

$$\rho c\left(\frac{\partial^\alpha T}{\partial t^\alpha}+\tau\frac{\partial^{1+\alpha}T}{\partial t^{1+\alpha}}\right)=\nabla\cdot\lambda\nabla T+c_b\omega_b(T_a-T)+\dot{g}_{\text{met}}+Q^{\text{ext}}.\quad (8)$$

Calculations show that the fractional SPL model gives the same temperature distribution as the DPL model [253].

The spatially fractional SPL model was formulated by Kumar et al. [254] for the 1D case:

$$\rho c\frac{\partial T}{\partial t}=-\frac{\partial^\alpha q}{\partial x^\alpha}+c_b\omega_b(T_b-T)+Q^{\text{ext}},\quad (9)$$

where

$$q(x,t)+\tau\frac{\partial q(x,t)}{\partial x}=-\lambda\nabla T,$$

and for $m-1<\alpha<m$,

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}=\frac{1}{\Gamma(m-\alpha)}\int\frac{\partial^m u(x,s)}{\partial s^m}(x-s)^{m-\alpha-1}ds,$$

where $\Gamma(z)$ is the gamma function.

Jiang and Qi [255] derived the fractional model of the heat wave, by modifying the Cattaneo relation

$$\frac{\tau^{\alpha}}{\alpha!}D_t^{\alpha}\mathbf{q} + \mathbf{q} = -\lambda\nabla T,$$

where D_t^{α} is the modified Riemann–Liouville derivative of order α.

Qi et al. [256] studied laser heating by generalizing the Cattaneo relation as

$$\tau^{p}D_t^{p}\mathbf{q} + \mathbf{q} = -\lambda\nabla T,$$

where D_t^{p} is the Caputo derivative of order p; factor τ^p was introduced for correcting dimension.

Xu et al. proposed the fractional Cattaneo equation using two Caputo derivatives of different orders:

$$\frac{\partial^{\beta-1}\mathbf{q}}{\partial t^{\beta-1}} + \tau\frac{\partial^{\alpha-1}\mathbf{q}}{\partial t^{\alpha-1}} + \mathbf{q} = -\lambda\nabla T, \quad 0 < \beta \le \alpha \le 2,$$

$$\frac{\tau^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial^{\alpha}\mathbf{q}}{\partial t^{\alpha}} + \mathbf{q} = -\lambda\nabla T, \quad 0 < \alpha \le 1.$$

Mishra and Rai [257] used the fractional SPL model for analyzing heat transport in thin films.

Moroz and Maslovskaya [58] employed the spatially fractional SPL model for simulating the heat conduction in ferroelectrics (triglycine sulfate).

Christov [259] developed the spatially fractional equation of nonstationary heat transport with a damping term described using the Caputo–Fabrizio derivative [260], which modifies the Caputo derivative

$$D_t^{\alpha}f(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right]\frac{df(t)}{dt}ds,$$

where $M(\alpha)$ is a normalizing function such that $M(0) = M(1) = 1$.

Alkahtani and Atangana [261] considered the numerical solution of this problem.

Tang et al. [262] (see also [263]) introduced for the fractional heat transport the fractional derivative of the singular kernel as a modification of the Riemann–Liouville derivative

$$D_{\alpha^+}^{(\nu)}T = \frac{\mathcal{R}(\nu)}{(1-\nu)}\frac{d}{dx}\int_a^x \exp\left[-\frac{\nu}{1-\nu}(x-x')\right]Tdx',$$

where $\mathcal{R}(\nu)$ is the normalizing function.

6.7. Fractional DPL Model

Ji [264, 265] used the following expression for the time-fractional DPL model (${}_0^C D_t^{\alpha}$ is the Caputo fractional derivative):

$$\rho C\left(\frac{\partial T}{\partial t} + \frac{\tau_q^{\alpha}}{\Gamma(1-\alpha)}\,{}_0^C D_t^{1+\alpha}T\right) = \lambda\left(T + \frac{\tau_T^{\alpha}}{\Gamma(1-\alpha)}\,{}_0^C D_t^{\alpha}T\right)$$

for studying heat transfer in thin films.

Xu et al. [266] studied biological heat transfer using the equation with the Caputo derivatives of orders α and β:

$$\mathbf{q} + \tau_q^{\alpha}\frac{\partial^{\alpha}\mathbf{q}}{\partial t^{\alpha}} = -\lambda\left(\nabla T + \tau_T^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\nabla T\right).$$

These authors replaced relaxation times τ_q and τ_T of the DPL model by τ_q^{α} and τ_T^{β} to preserve the dimension.

Liu et al. [267] used the convective derivative introduced by Christov [123] and the Caputo derivative of order α to formulate model

$$\tau_q\left[\frac{\partial^{\alpha}\mathbf{q}}{\partial t^{\alpha}} + \mathbf{v}\cdot\nabla\mathbf{q} + \mathbf{q}\cdot\nabla\mathbf{v} + (\nabla\cdot\mathbf{v})\mathbf{q}\right] + \mathbf{q} = -\lambda\nabla\left(1 + \tau_T\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)T.$$

6.8. Fractional TPL Model

Ahbarzadeh et al. generalized the defining relation of the TPL model and retained terms up to the $2\alpha_F$th order for τ_q and up to the α_F order for τ_T and τ_ν and obtained

$$\left(1 + \frac{\tau_q^{\alpha_F}}{\alpha_F!}\frac{\partial^{\alpha_F}}{\partial t^{\alpha_F}} + \frac{\tau_q^{2\alpha_F}}{(2\alpha_F)!}\frac{\partial^{2\alpha_F}}{\partial t^{2\alpha_F}}\right)\mathbf{q} = \left[\left(\lambda + \frac{\lambda\tau_q^{\alpha_F}}{(\alpha_F)!}\frac{\partial^{\alpha_F-1}}{\partial t^{\alpha_F-1}} + \frac{\lambda\tau_q^{\alpha_F}}{(\alpha_F)!}\frac{\partial^{\alpha_F}}{\partial t^{\alpha_F}}\right)\nabla T + \lambda\nabla\nu\right].$$

The fractional TPL model, like the ordinary TPL model, is used for solving problems of thermoelasticity [268, 269].

CONCLUSIONS

In this review, an attempt has been made to include the entire variety of models intended for analysis of heat transport beyond the Fourier model.

The most promising are apparently the thermodynamic and fractional models; for studying heat transport in dielectric, the relaxon model is most attractive.

Analysis of the literature shows that the DPL model is more suitable for studying heat transport in biological tissues.

APPENDIX

SOME EXACT SOLUTIONS

Exact analytic solutions facilitate analysis of heat transfer processes and serve as standards in developing numerical methods.

Fan and Wang [270] published a comprehensive review of exact solutions to biological heat transfer problems including the Pennes equation, the heat wave model, and the DPL model [270].

Sarkar et al. [271] considered the steady-state heat transport in a multilayer dermic tissue using the Pennes equation and the phase-lag models in the cases with a preset temperature or heat flux.

The Barletta and Zanchini [272] used the phase-lag models to analyze heat transport in an infinitely wide strip with a preset heat flux with the help of the Cattaneo equation. Such a problem was solved in cylindrical coordinates by Saerdodin et al. [273].

Ahmadikia et al. [274, 275] and Kundu and Dewanjee [276] used the heat wave model and the Pennes equation for simulating the action of constant and pulsed heating on a dermal tissue.

Al-Khairy et al. [277] analyzed laser (constant, instantaneous, and exponential) action on a moving semi-infinite medium using the Cattaneo model.

The DPL model was employed by Askarizadeh and Ahmadikia [278, 279] and Lin [280] for studying nonstationary heating of dermal issues.

Al-Khairy [281] solved the equations of the DPL model for a homogeneous material using the Green function; Dai and Nassar [282] used the initial form of the DPL model without the Taylor expansion.

Kulish and Novozhilov [141] derived the integral equation connecting temperature with its gradient using the Laplace transform.

Zhang [148] used the method of "prepared solutions" based on guessing of a possible solution and determining the boundary and initial conditions.

Zhukovski [283] (see also [284]) obtained the exact solution to the Guyer–Krumhansl equation using the operator method and found that the solution may violate the maximum principle in some regimes. However, this conclusion was not confirmed in the recent article by Kovacs [285]. The initial and boundary conditions corresponded to a laser experiment. The solution was sought in two intervals: $0 < t < \tau_\Delta$ and $t > \tau_\Delta$ (τ_Δ is the laser pulse duration).

The fractional models of Getinkaya and Kiymaz [286] were based on the method of generalized differential transformation for solving diffusion equations with the Caputo derivative.

Povstenko obtained exact solution for thermal conductivity in two conjugate half-lines [287], a semi-infinite composite material [288], and in a medium with spherical inclusions [289].

Junyi and Mingyu [290] obtained the solution to the Stefan problem; Yang et al. [291] solved 1D nonstationary problem for equations with a local fractional derivative.

Kazemi and Erjaee [292] analyzed the fractional diffusion equation.

Ghosh et al. [293] obtained the solution to the linear fractional equation with the Jumarie derivative in terms of the Mittag–Leffler functions.

CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

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