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THEORETICAL AND MATHEMATICAL PHYSICS

Shape of the Surface of a Vertical Liquid Bridge between Two Parallel Solid Planes Taking into Account the Gravity Force for Small Bond Numbers

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Abstract—A variational formulation has given the problem of the shape of the lateral surface of a small vertical liquid θ bridge between two parallel solid surfaces that take into account the gravity force in the axisymmetric case. An algorithm has been constructed for the iterative solution of the problem for small Bond numbers. The dependence of the number of solutions on the liquid bridge height has been analyzed.

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INTRODUCTION

The results of a theoretical analysis of the profile of the surface of a small liquid drop in the presence of the three-phase contact zone are essential for solving a large number of theoretical and technological problems [1, 2] (in particular to study the shape of liquid meniscuses formed during crystal growth in accordance with Stepanov's method [3]). In this case, a liquid drop of the melt is located between two solid surfaces with different properties, e.g., between a molybdenum shaping part and a crystalline seed. Different authors refer to this liquid drop in different terms; for example, in the Geguzin's monograph [4], it is referred to as a *crushed drop*; in the foreign literature, it is called a *liquid bridge* or a *capillary bridge*; in [5], it was referred to as an *oblate drop*; here, we will use the term *liquid bridge.*

In one of the first publications devoted to analyzing liquid bridges, Fortes [6] proposed that two types of objects be considered; namely, bridges with a fixed contact contour (*r* bridges) and bridges with a fixed wetting angle (θ bridges). Following [7], we can single out two trends in the study, i.e., (i) the analysis of the evolution of the shape of liquid bridges and (ii) the analysis of their stability to small perturbations. In the literature, liquid bridges between two parallel solid surfaces [6], between two spheres [8], and between two axisymmetric solids [7] have been investigated.

Liquid bridges have been investigated using both asymptotic and numerical methods for solving equations. In [9], the asymptotic form of the surface of a horizontal θ bridge was constructed for small Bond numbers. References to these investigations were cited in [7, 9].

This study is devoted to the calculation of the shape of the surface of a vertical liquid θ bridge between two solid surfaces taking into account the gravity force. We give a variational formulation of the problem and propose an algorithm for determining an approximate solution to the problem for small Bond numbers.

An analysis of the vertical liquid bridge is important for studying processes that occur when seeding crystals grown from a melt [3].

1. VERTICAL LIQUID BRIDGE: VARIATIONAL FORMULATION OF THE PROBLEM

Let us consider a liquid bridge in contact with two solid plane surfaces (Fig. 1, bottom and top). In this case, the liquid bridge is a small liquid drop between two parallel solid surfaces with preset properties (oblate drop). In view of the proposed axial symmetry, we will solve the problem of determining the profile of this drop in the cylindrical system of coordinates (r, z). The surface tensions between the media are α_{13}, α_{14}, α_{34}, α_{23}, and α_{24}, respectively. The contact region between the drop and plane $z = 0$ (bottom) is a circle with radius r_1, while the contact region between the drop and plane $z = \hat{h}$ (top) is a circle of radius r_2. We denote the sought functions that describe the profiles of the lower $(u_1(r))$ and upper $(u_2(r))$ parts of the drop as by $u_1(r)$ and $u_2(r)$. The region separating these parts

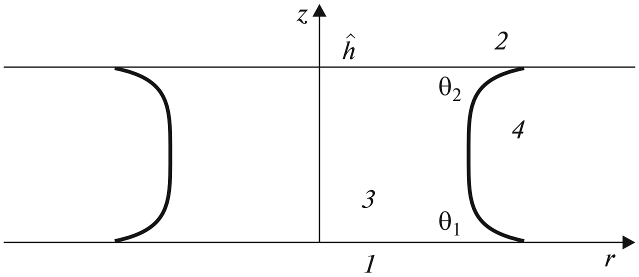

Fig. 1. Liquid bridge (in the given case, a drop between two parallel solid surfaces): (*1*, *2*) solid media, (*3*) liquid medium, and (*4*) gaseous medium; θ_1, θ_2 are wetting angles.

(neck) is the circle of radius r_* ($r_* \geq 0$). Since there are no physical reasons for the sharpening of the droplet profile, the tangent to the droplet profile at the point with abscissa $r = r_*$ must be vertical, i.e., $u_1'(r_*) = -\infty$, $u_2'(r_*) = +\infty$, $u_1(r_*) = u_2(r_*)$. The only exception is the contact between the sessile and pendent drop at the same point.

In addition, we assume that wetting angles θ_1 and θ_2 do not exceed 90°. Then, our oblate drop (liquid bridge) has a catenoidal shape $r_* < \min(r_1, r_2)$, and $u_1(r)$ and $u_2(r)$ are single-valued functions.

We assume that the droplet volume is fixed as follows:

$$I\{u_1(r), u_2(r)\} = 2\pi\left(\int_{r_*}^{r_1} u_1(r_*) r dr + \int_{r_*}^{r_2} (\hat{h} - u_2(r)) r dr\right) + \pi (r_*)^2 \hat{h} = V. \quad (1)$$

Let us introduce the functional, which includes the surface energy and the energy of the gravity force. The surface energy in turn consists of a component that corresponds to the free surface of the drop and the component that corresponds to its contact with a solid. The functional under investigation can be written in the form

$$J\{u_1(r), u_2(r)\} = 2\pi\int_{r_*}^{r_1}\left\{\alpha_{34}\sqrt{1+(u_1')^2} + \frac{1}{2}g\rho u_1^2 + \alpha_{13} - \alpha_{14}\right\} r dr$$
$$+ 2\pi\int_{r_*}^{r_2}\left\{\alpha_{34}\sqrt{1+(u_2')^2} - \frac{1}{2}g\rho(\hat{h} - u_2)^2 + \alpha_{23} - \alpha_{24}\right\} r dr \quad (2)$$
$$+ \pi\left\{\alpha_{13} - \alpha_{14} + \alpha_{23} - \alpha_{24} + \frac{1}{2}g\rho\hat{h}^2\right\}(r_*)^2,$$

where g is the acceleration due to gravity and ρ is the density of the liquid.

Therefore, we obtain the isoperimetric problem: find the minimum of functional (2) provided that functional (1) assumes a preset value V. In accordance with the Euler theorem on isoperimetric problems, we introduce the extended functional (λ is a Lagrange multiplier)

$$J\{u_1(r), u_2(r)\} + \lambda I\{u_1(r), u_2(r)\}$$
$$= 2\pi\int_{r_*}^{r_1} H_1(r, u_1(r), u_1'(r)) dr + 2\pi\int_{r_*}^{r_2} H_2(r, u_2(r), u_2'(r)) dr \quad (3)$$
$$+ \pi\left\{\alpha_{13} - \alpha_{14} + \alpha_{23} - \alpha_{24} + \frac{1}{2}g\rho\hat{h}^2 + \lambda\hat{h}\right\}(r_*)^2,$$

where

$$H_1(r, u_1(r), u_1'(r)) = \left\{\alpha_{34}\sqrt{1+(u_1')^2} + \frac{1}{2}g\rho u_1^2 + \alpha_{13} - \alpha_{14} + \lambda u_1\right\} r,$$

$$H_2(r, u_2(r), u_2'(r)) = \left\{\alpha_{34}\sqrt{1+(u_2')^2} - \frac{1}{2}g\rho(\hat{h} - u_2)^2 + \alpha_{23} - \alpha_{24} + \lambda(\hat{h} - u_2)\right\} r.$$

Performing the variations in the extended functional, we obtain two Euler equations and two transversality conditions. Let us introduce dimensionless variables $\xi = r/V^{1/3}$, $w_i = u_i/V^{1/3}$, $i = 1, 2$, and dimensionless parameters $\mu = \lambda V^{1/3}/\alpha_{34}$ (sought quantity), $h = \hat{h}/V^{1/3}$, $B = g\rho V^{2/3}/\alpha_{34}$ (preset quantities). Dimensionless constant B is the Bond number.

The problem can be formulated in dimensionless form as follows:

the Euler equations are

$$\frac{d}{d\xi}\left(\frac{\xi w_1'(\xi)}{\sqrt{1+(w_1'(\xi))^2}}\right) = B\xi w_1(\xi) + \mu\xi, \quad \xi_* < \xi < \xi_1, \quad (4)$$

$$\frac{d}{d\xi}\left(\frac{\xi w_2'(\xi)}{\sqrt{1+(w_2'(\xi))^2}}\right) = B\xi(h - w_2(\xi)) - \mu\xi, \quad \xi_* < \xi < \xi_2; \quad (5)$$

the transversality conditions are

$$\cos\theta_1 \equiv \frac{1}{\sqrt{1+(w_1'(\xi_1))^2}} = \frac{\alpha_{14} - \alpha_{13}}{\alpha_{34}} \equiv \alpha_1, \quad (6)$$

$$\cos\theta_2 \equiv \frac{1}{\sqrt{1+(w_2'(\xi_2))^2}} = \frac{\alpha_{24} - \alpha_{23}}{\alpha_{34}} \equiv \alpha_2; \quad (7)$$

the conditions of contacts of the liquid bridge with the bottom and top are

$$w_1(\xi_1) = 0, \quad (8)$$

$$w_2(\xi_2) = h; \quad (9)$$

the continuity condition for the bridge profile at the neck is

$$w_1(\xi_*) = w_2(\xi_*); \qquad (10)$$

the verticality condition for the tangent at the neck is

$$w_1'(\xi_*) = -\infty; \quad w_2'(\xi_*) = +\infty; \qquad (11)$$

and the volume conservation law is

$$2\pi\int_{\xi_*}^{\xi_1} w_1(\xi)\xi d\xi + 2\pi\int_{\xi_*}^{\xi_2} (h - w_2(\xi))\xi d\xi + \pi(\xi_*)^2 h = 1. \qquad (12)$$

2. ALGORITHM OF THE SOLUTION

To simplify the dependence on the lower limit in the integrals that appear in Eq. (12) and analogous integrals, we introduce the renormalization of the argument, sought functions, and parameters as follows:

(i) new independent variable $\eta = \xi/\xi_*$;

(ii) new sought functions $v_i(\eta) = w_i(\xi)/\xi_*$, $i = 1, 2$;

(iii) new parameters $H = h/\xi_*$, $b = B(\xi_*)^2$ (b is the modified Bond number); and

(iv) new Lagrange multiplier $M = \mu\xi_*$.

In the new dimensionless variables, Eqs. (4) and (5) assume the form

$$\frac{d}{d\eta}\left(\frac{\eta v_1'(\eta)}{\sqrt{1+(v_1'(\eta))^2}}\right) = bv_1\eta + M\eta, \quad 1 < \eta < \eta_1, \qquad (13)$$

$$\frac{d}{d\eta}\left(\frac{\eta v_2'(\eta)}{\sqrt{1+(v_2'(\eta))^2}}\right) = b(H - v_2(\eta))\eta - M\eta, \quad 1 < \eta < \eta_2. \qquad (14)$$

Integrating these equations, we obtain

$$\frac{\eta v_1'(\eta)}{\sqrt{1+(v_1'(\eta))^2}} = b\int_1^{\eta} v_1(s)sds + \frac{M(\eta^2-1)}{2} + C_1, \qquad (15)$$

$$\frac{\eta v_2'(\eta)}{\sqrt{1+(v_2'(\eta))^2}} = b\int_1^{\eta} (H - v_2(s))sds - \frac{M(\eta^2-1)}{2} + C_2. \qquad (16)$$

Passing in Eqs. (15) and (16) to the limit for $\eta \to 1$, we obtain constants C_1 and C_2: $C_1 = -1$ and $C_2 = 1$. Dividing both sides of Eqs. (15) and (16) by η, we reduce these equations to the form

$$\frac{v_1'(\eta)}{\sqrt{1+(v_1'(\eta))^2}} = \frac{1}{\eta}\left[b\int_1^{\eta} v_1(s)sds + \frac{M(\eta^2-1)}{2} - 1\right] \equiv -\Phi_1(\eta), \qquad (17)$$

$$\frac{v_2'(\eta)}{\sqrt{1+(v_2'(\eta))^2}} = \frac{1}{\eta}\left[b\int_1^{\eta} (H - v_2(s))sds - \frac{M(\eta^2-1)}{2} + 1\right] \equiv \Phi_2(\eta). \qquad (18)$$

Functions Φ_1 and Φ_2 introduced above must satisfy the inequalities $0 < \Phi_1 \le 1$ and $0 < \Phi_2 \le 1$. Solving Eqs. (17) and (18) for derivatives, we arrive at the following equations:

$$\frac{dv_1}{d\eta} = -\frac{\Phi_1(\eta)}{\sqrt{1-(\Phi_1(\eta))^2}}, \quad \frac{dv_2}{d\eta} = \frac{\Phi_2(\eta)}{\sqrt{1-(\Phi_2(\eta))^2}}. \qquad (19)$$

Relations (17) and (18) imply that $\Phi_1(1) = \Phi_2(1) = 1$, while Eqs. (19) lead to

$$\left.\frac{dv_1}{d\eta}\right|_{\eta=1} = -\infty, \quad \left.\frac{dv_2}{d\eta}\right|_{\eta=1} = +\infty,$$

i.e., to the fulfillment of conditions (11). In the vicinity of point $\eta = 1$, functions Φ_1 and Φ_2 can be represented in the form $\Phi_i(\eta) = 1 + \beta_i(\eta - 1) + o(\eta - 1)$, $i = 1, 2$; therefore, the singularities on the right-hand sides of Eqs. (19) are integrable,

$$v_1(\eta) = \int_{\eta}^{\eta_1} \frac{\Phi_1(s)ds}{\sqrt{1-(\Phi_1(s))^2}},$$
$$v_2(\eta) = H - \int_{\eta}^{\eta_2} \frac{\Phi_2(s)ds}{\sqrt{1-(\Phi_2(s))^2}}. \qquad (20)$$

In this case, analogs of the conditions of contact of the liquid bridge with the bottom and top are satisfied: $v_1(\eta_1) = 0$, $v_2(\eta_2) = H$. Satisfying the analog of condition (10), we arrive at the relation

$$\int_1^{\eta_1} \frac{\Phi_1(s)ds}{\sqrt{1-(\Phi_1(s))^2}} + \int_1^{\eta_2} \frac{\Phi_2(s)ds}{\sqrt{1-(\Phi_2(s))^2}} = H. \qquad (21)$$

Substituting relations (17) and (18) into the transversality conditions

$$\left.\frac{v_1'(\eta)}{\sqrt{1+(v_1'(\eta))^2}}\right|_{\eta=\eta_1} = -\Phi_1(\eta_1) = -\sqrt{1-\alpha_1^2}, \qquad (22)$$

$$\left.\frac{v_2'(\eta)}{\sqrt{1+(v_2'(\eta))^2}}\right|_{\eta=\eta_2} = \Phi_2(\eta_2) = \sqrt{1-\alpha_2^2}, \qquad (23)$$

we arrive at the relations

$$\frac{1}{2}M\eta_1^2 + \sqrt{1-\alpha_1^2}\eta_1 - \left[1 + \frac{1}{2}M - b\int_1^{\eta_1} v_1(s)sds\right] = 0, \quad (24)$$

$$\frac{1}{2}M\eta_2^2 + \sqrt{1-\alpha_2^2}\eta_2 - \left[1 + \frac{1}{2}M + b\int_1^{\eta_2} (H - v_2(s))sds\right] = 0. \quad (25)$$

Assuming temporarily that the terms containing factor b are known, we can solve Eqs. (24), (25) as quadratic equations in η_1, η_2 and obtain, respectively,

$$\eta_1^{\pm} = \frac{1}{M}\left\{-\sqrt{1-\alpha_1^2} \pm \left[1-\alpha_1^2 + 2M\left(1 + \frac{1}{2}M - b\int_1^{\eta_1} v_1(s)sds\right)\right]^{1/2}\right\}, \quad (26)$$

$$\eta_2^{\pm} = \frac{1}{M}\left\{-\sqrt{1-\alpha_2^2} \pm \left[1-\alpha_2^2 + 2M\left(1 + \frac{1}{2}M + b\int_1^{\eta_2} (H - v_2(s))sds\right)\right]^{1/2}\right\}. \quad (27)$$

Let us write the volume conservation condition (12) in new variables:

$$\xi_*^3\left(\int_1^{\eta_1} v_1(s)sds + \int_1^{\eta_2}(H - v_2(s))sds + \frac{1}{2}H\right) = \frac{1}{2\pi}. \quad (28)$$

Integrating by parts, we reduce this relation to the form

$$\xi_* = \left\{\pi\left[\int_1^{\eta_1} \frac{\Phi_1(s)s^2ds}{\sqrt{1-(\Phi_1(s))^2}} + \int_1^{\eta_2} \frac{\Phi_2(s)s^2ds}{\sqrt{1-(\Phi_2(s))^2}}\right]\right\}^{-1/3}. \quad (29)$$

As a result, we obtain nonlinear system of four equations (21), (24), (25), (29) for determining four parameters η_1, η_2, ξ_*, and M.

The last sought parameter M is the modified Lagrange multiplier. Substituting the upper integration limits $\eta_1(M)$ and $\eta_2(M)$ into formula (21), we arrive at the relation

$$H(M) = \int_1^{\eta_1(M)} \frac{\Phi_1(s)ds}{\sqrt{1-(\Phi_1(s))^2}} + \int_1^{\eta_2(M)} \frac{\Phi_2(s)ds}{\sqrt{1-(\Phi_2(s))^2}}. \quad (30)$$

Then, we can find M from the preset value of H by inverting function (30).

We propose that the iteration method be used for finding the solution to the problem. Namely,

1. We specify the value of parameter M, determine the roots of quadratic equations (24) and (25) under the assumption that $b = 0$, and determine ξ_*, H, and functions v_1 and v_2 determining the profile of the lateral surface of the liquid bridge at the first state of iteration process.

2. We calculate the value of $b = B(\xi_*)^2$ and perform the above procedure, but now taking into account the determined functions v_1 and v_2. It is the second step of the iteration process. Further, we take the third step, and so on until the required accuracy of calculations is reached.

We cover the entire range of admissible values of parameter M (see below) and construct the dependence of the dimensionless height of the liquid bridge on this parameter. To determine the shape of the lateral surface of the liquid bridge of preset height h, we determine the corresponding values of parameter M using the plotted dependence $h(M)$ and perform the above calculations for these values.

It should be noted at the very outset that the maximal number of such values is four (four different profiles of the bridge). If the height exceeds h_{max} (h_{max} is the maximal height of the liquid bridge), there are no solutions.

3. VERTICAL LIQUID BRIDGE IN ZERO GRAVITY CONDITIONS (FIRST STEP OF ITERATION PROCESS)

Let us suppose that the Bond number is zero. We determine the range of admissible values of parameter M. If the values of parameter M are positive, we have two real-valued roots of different signs for each of Eqs. (24), (25) and choose positive roots η_1^+, η_2^+ because we cannot pass through zero point in evaluating the integrals (see the form of functions $\Phi_i(\eta)$, $i = 1, 2$).

Let us now consider the case with negative values of parameter M. An analysis of the discriminants of the quadratic equations shows that the real-valued roots exist in regions (a) $\max\{-1 + \alpha_1, -1 + \alpha_2\} \le M < 0$ and (b) $M \le \min\{-1 - \alpha_1, -1 - \alpha_2\}$. In region (b), the roots of equations (24) and (25) are smaller than unity and $\xi_{*0} < 0$ (see expression (29)). Consequently, physically meaningful solutions exist only in region (a). Therefore, for negative values of parameter M, it is sufficient to confine analysis to interval (a). In this region, each value of parameter M corresponds to four roots of Eqs. (24), (25), which lie in the region $(+1, +\infty)$, and four values of parameter h.

Let us describe the first step of the iteration process (first iteration) in greater detail. Thus,

1. We specify the value of parameter M from the range of admissible values.

2. We find positive roots of Eqs. (24), (25) for $b = 0$. If $M > 0$, there are two roots, η_1^+ and η_2^+ (see expressions (26) and (27)); however, if M belongs to region (a), there are four roots and, hence, four corresponding variants of the choice of the roots for further calculations appear, namely, version (++) with root η_1^+ of Eq. (24) and root η_2^+ of Eq. (25); version (+–) with roots η_1^+ and η_2^-, version (–+), and finally version (– –).

3. For each of the possible variants for choosing positive roots of quadratic equations (24), (25), we calculate ξ_* using expression (29) and taking into account the fact that $\Phi_1(\eta) = \Phi_2(\eta) = \Phi(\eta) = [1 - 0.5M(\eta^2 - 1)]/\eta$.

4. We determine H using formula (21) and functions $v_1(\eta)$ and $v_2(\eta)$ using formulas (20).

5. We return to the initial dimensionless variables and determine ξ_1, ξ_2, h, and then functions $w_1(\xi) = v_1(\xi/\xi_*)\xi_*$, $w_2(\xi) = v_2(\xi/\xi_*)\xi_*$ describing the profile of the liquid bridge in each variant in the first approximation.

Further, we plot the graphs of dimensionless height of the liquid bridge as a function of parameter M. For $\alpha_1 = 0.5$ and $\alpha_2 = 0.7$, this dependence is plotted in Fig. 2.

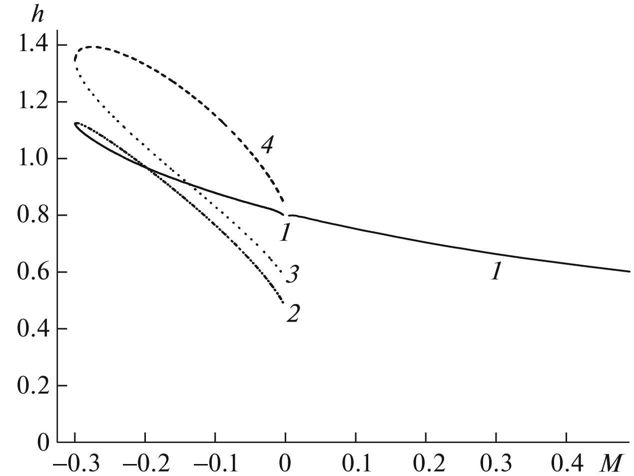

Fig. 2. Dependence of liquid bridge height h on parameter M for $\alpha_1 = 0.5$ and $\alpha_2 = 0.7$. Curve *1* corresponds to variant (++) of the choice of the roots of Eqs. (24), (25); curves *2–4* correspond to variants (+–), (–+), and (– –), respectively.

It can be seen that, in the region of positive values of parameter M increasing from zero to $+\infty$, height h of the liquid bridge decreases monotonically from its value at $M = 0$ to zero. In the range of negative values of this parameter, there exist four branches ((++), (+–), (–+), and (– –)) corresponding to different variants of the choice of the roots of the quadratic equations. In addition, it can be seen from the graph that small values of h correspond to only one value of parameter M and, hence, only one liquid bridge profile.

Thus, a fixed value of h at the first step of the iteration process can correspond to four different profiles of the drop (maximal number of branches). This situation is illustrated in Fig. 3. It shows four different profiles of the liquid bridge, which correspond to the same value of dimensionless height $h = 1$ (we consider the symmetric case when $\alpha_1 = \alpha_2 = 0.5$).

4. VERTICAL LIQUID BRIDGE (SECOND STEP OF ITERATION PROCESS)

Let us now describe the second step of the iteration process (second iteration). At this stage, we take into account the action of the force of gravity on a vertical liquid bridge.

The second step of the iteration process presumes the following operations.

1. For the same preset value of parameter M, we determine positive roots of Eqs. (24) and (25) using relations (26) and (27). Quantities ξ_*, η_1, and η_2, as well as functions $v_1(\eta)$ and $v_2(\eta)$, are taken from the results of calculations in the first approximation.

2. For each new variant of the choice of the roots, we calculate new value of ξ_* by formula (29) (functions $\Phi_1(\eta)$ and $\Phi_2(\eta)$ are defined by formulas (17) and (18)).

3. We determine H by formula (21) and functions $v_1(\eta)$ and $v_2(\eta)$ by formulas (20).

4. We return to the initial dimensionless variables and find ξ_1, ξ_2, h, and then functions $w_1(\xi) = v_1(\xi/\xi_*)\xi_*$ and $w_2(\xi) = v_2(\xi/\xi_*)\xi_*$ that describe the liquid bridge profile in each variant in the second approximation.

The next iterations can be performed in accordance with the same algorithm as the second iteration.

As an example, we calculate the liquid bridge profiles taking into account one and two iterations. The results of calculations in the variant (++) for $\alpha_1 = 0.5$, $\alpha_2 = 0.7$, and $M = -0.273$ are shown in Fig. 4. It can be seen that the inclusion of the second iteration for small values of parameter b changes the solution insignificantly.

5. ASYMPTOTIC FORM OF THE SOLUTION IN THE VICINITY OF POINT $M = 0$ AND THE SEARCH FOR THE MAXIMAL POSSIBLE LIQUID BRIDGE HEIGHT

Let us construct the asymptotic form of the solution obtained as a result on one step of the iterative process (for $b = 0$) in the vicinity of point $M = 0$. In the case of strong wetting, this point is the left boundary of

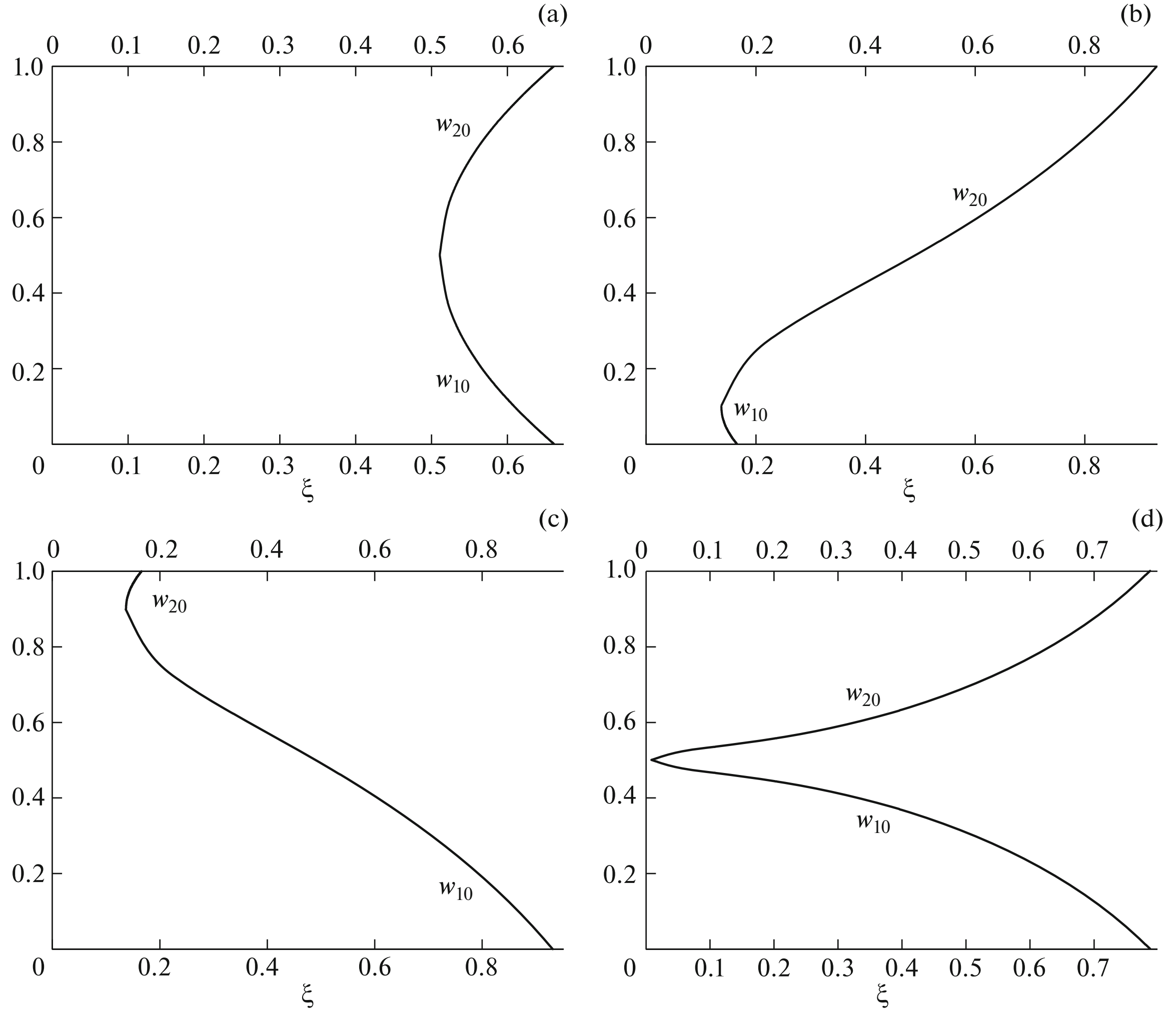

Fig. 3. Four different profiles of the liquid bridge, corresponding to the same height $h = 1.0$ for $\alpha_1 = \alpha_2 = 0.5$. (a) Variant (++), $M = -0.36$; (b) (+–), $M = -0.216$; (c) (–+), $M = -0.216$; and (d) (– –), $M = -0.019$.

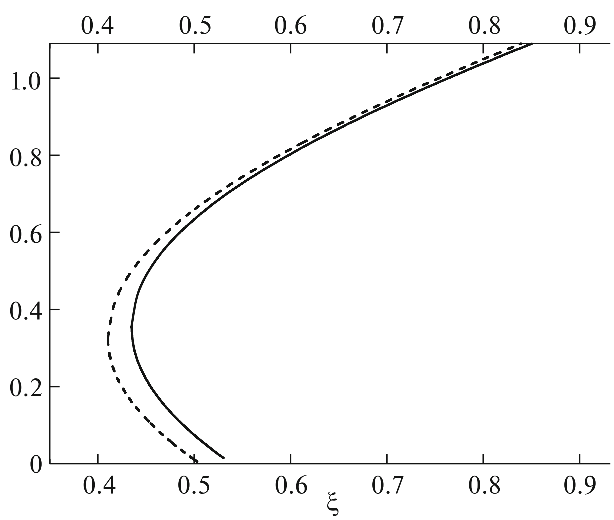

Fig. 4. Liquid bridge profiles after the first iteration step (solid curve) and after the second step (dashed curve); $\alpha_1 = 0.5$, $\alpha_2 = 0.7$, $M = -0.273$, $B = 0.224$, and $b = 0.042$. Variant (++).

the range of admissible values of parameter M. The asymptotic forms are different in different variants of the choice of roots for quadratic equations (24) and (25).

Let us consider variant (++) and assume that parameter $M \to 0$. In this case, auxiliary function Φ can be approximated as follows: $\Phi(\eta) = 1/\eta$. Integrating in expressions (29) and (21) and assuming that

$$\eta_1 \sim (1-\alpha_1^2)^{-1/2}, \quad \eta_2 \sim (1-\alpha_2^2)^{-1/2},$$

we obtain approximate relations

$$\xi_* \sim \left(\frac{2}{\pi}\right)^{1/3}\left\{\frac{\alpha_1}{1-\alpha_1^2}+\frac{\alpha_2}{1-\alpha_2^2} + \frac{1}{2}\ln\left(\frac{(1+\alpha_1)(1+\alpha_2)}{(1-\alpha_1)(1-\alpha_2)}\right)\right\}^{-1/3}, \tag{31}$$

$$H \sim \frac{1}{2}\ln\left(\frac{(1+\alpha_1)(1+\alpha_2)}{(1-\alpha_1)(1-\alpha_2)}\right).$$

In addition, we can obtain approximate expressions from (20) that describe the shape of the lateral surface of a liquid bridge as follows:

$$v_1(\eta) \sim \ln((1+\alpha_1)^{1/2}(1-\alpha_1)^{-1/2}[\eta+(\eta^2-1)^{1/2}]^{-1}),$$
$$v_2(\eta) \sim \ln((1+\alpha_1)^{1/2}(1-\alpha_1)^{-1/2}[\eta+(\eta^2-1)^{1/2}]). \quad (32)$$

Variants (+–), (–+), and (– –) are only realized for negative values of parameter M; we will consider the case when $M \to -0$. In this case, a more accurate approximation of the integrands in expressions (21) and (29) is required, namely,

$$\frac{\Phi(\eta)}{\sqrt{1-(\Phi(\eta))^2}} \sim \frac{1-0.5M(\eta^2-1)}{\sqrt{\eta^2-1}}.$$

Then, for variant (+–), we obtain

$$\eta_1 \sim (1-\alpha_1^2)^{-1/2}, \quad \eta_2 \sim \frac{-2}{M}(1-\alpha_2^2)^{1/2},$$
$$\xi_* \sim -M(2\pi)^{-1/3}(1-\alpha_2^2)^{-2/3},$$
$$\xi_1 \sim -M(2\pi)^{-1/3}(1-\alpha_1^2)^{-1/2}(1-\alpha_2^2)^{-2/3}, \quad (33)$$
$$\xi_2 \sim (2)^{2/3}(\pi)^{-1/3}(1-\alpha_2^2)^{-1/6}, \quad H \sim \frac{-(1-\alpha_2^2)}{M},$$
$$h \sim (2\pi)^{-1/3}(1-\alpha_2^2)^{1/3}.$$

For variant (–+) ($M \to -0$), we have

$$\eta_1 \sim \frac{-2}{M}(1-\alpha_1^2)^{1/2}, \quad \eta_2 \sim (1-\alpha_2^2)^{-1/2},$$
$$\xi_* \sim -M(2\pi)^{-1/3}(1-\alpha_1^2)^{-2/3},$$
$$\xi_1 \sim (2)^{2/3}(\pi)^{-1/3}(1-\alpha_1^2)^{-1/6},$$
$$\xi_2 \sim -M(2\pi)^{-1/3}(1-\alpha_1^2)^{-2/3}(1-\alpha_2^2)^{-1/2}, \quad (34)$$
$$H \sim \frac{-(1-\alpha_1^2)}{M},$$
$$h \sim (2\pi)^{-1/3}(1-\alpha_1^2)^{1/3}.$$

For variants (– –) ($M \to -0$), we can write

$$\eta_1 \sim \frac{-2}{M}(1-\alpha_1^2)^{1/2}, \quad \eta_2 \sim \frac{-2}{M}(1-\alpha_2^2)^{1/2},$$
$$\xi_* \sim -M(2\pi[(1-\alpha_1^2)^2+(1-\alpha_2^2)^2])^{-1/3},$$
$$\xi_1 \sim (2)^{2/3}(1-\alpha_1^2)^{1/2}(\pi[(1-\alpha_1^2)^2+(1-\alpha_2^2)^2])^{-1/3},$$
$$\xi_2 \sim (2)^{2/3}(1-\alpha_2^2)^{1/2}(\pi[(1-\alpha_1^2)^2+(1-\alpha_2^2)^2])^{-1/3}, \quad (35)$$
$$H \sim \frac{-(2-\alpha_1^2-\alpha_2^2)}{M},$$
$$h \sim (2\pi)^{-1/3}(2-\alpha_1^2-\alpha_2^2)[(1-\alpha_1^2)^2+(1-\alpha_2^2)^2]^{-1/3}.$$

Figure 5 shows the dependence of quantity H on parameter M for $\alpha_1 = 0.5$ and $\alpha_2 = 0.7$ for four variants of the choice of the roots of quadratic equations (branches (++), (+–), (–+), and (– –)). It can be seen that the behavior of the curves in the vicinity of point $M = 0$ corresponds to the above asymptotic form of the solution.

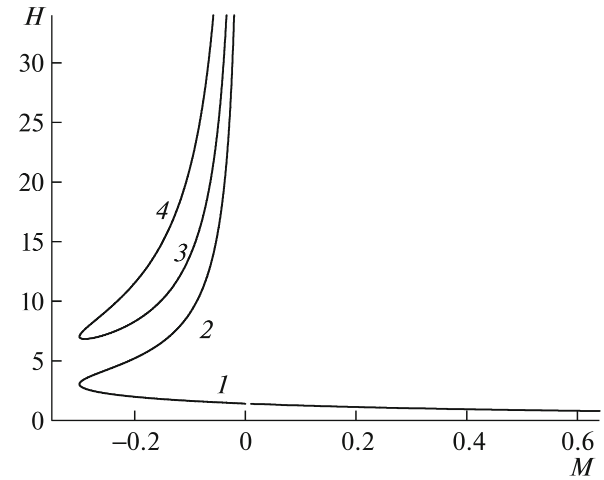

Fig. 5. Dependence of quantity H on parameter M for $\alpha_1 = 0.5$, $\alpha_2 = 0.7$. Curve *1* corresponds to variant (++) of the choice of the roots of Eqs. (24), (25); curves *2–4* correspond to variants (+–), (–+), and (– –), respectively.

Let us now consider the case of strong wetting (small wetting angle) on at least one of the solid surfaces. Let us suppose that, e.g., $\theta_1 \to 0$ and, hence, $\alpha_1 \to 1$. Then, region (a) obviously shrinks to a point, the left boundary of the range of admissible values of parameter M is point $M = 0$, and variant (++) is realized. Relations (31) in this case (for $M = 0$, $\alpha_1 \to 1$) give

$$\xi_* \sim (2)^{2/3}(\pi)^{-1/3}(1-\alpha_1)^{1/3}, \quad H \sim -0.5\ln(1-\alpha_1),$$
$$h \sim -(2\pi)^{-1/3}(1-\alpha_1)^{1/3}\ln(1-\alpha_1),$$
$$\xi_1 \sim (2)^{1/6}(\pi)^{-1/3}(1-\alpha_1)^{-1/6}, \quad (36)$$
$$\xi_2 \sim (2)^{2/3}(\pi)^{-1/3}(1-\alpha_2^2)^{-1/2}(1-\alpha_1)^{1/3}.$$

It can be seen that $\xi_* \to 0$, $\xi_1 \to +\infty$, $\xi_2 \to 0$, $h \to 0$ for $\alpha_1 \to 1$.

Let us now consider the maximal value of dimensionless height h of a liquid bridge between two solid planes. It is clear from general considerations that this value can only be attained for wetting angles θ_1 and θ_2 that tend to $\pi/2$. The above dependence of h on parameter M for $b = 0$ (see Fig. 2) shows that the maximal value of the bridge height is achieved in variant (– –). It should be noted that the same configuration of branches is also observed for $\alpha_1 = \alpha_2 \to 0$. Analyzing the results of numerical calculations for variant (– –) at $b = 0$, $\alpha_1 = \alpha_2 = 0$, and $M \to -1$, we find the maximal value of h, which is $h_{max} \approx 2.325$.

The integrals in expressions (21) and (29) can be expressed in terms of elliptic integrals of the first and second kind as follows:

$$\int_1^\eta \frac{\Phi(s)ds}{\sqrt{1-(\Phi(s))^2}} = F\left(\frac{\pi}{2},k\right) - F(\tau(\eta),k) + \frac{1+\delta}{1-\delta}\left(E\left(\frac{\pi}{2},k\right) - E(\tau(\eta),k)\right),$$

$$\int_1^\eta \frac{\Phi(s)s^2ds}{\sqrt{1-(\Phi(s))^2}} = -\frac{1+\delta}{3(1-\delta)}\left\{F\left(\frac{\pi}{2},k\right) - F(\tau(\eta),k) - \frac{7+\delta^2}{(1-\delta)^2}\left[E\left(\frac{\pi}{2},k\right) - E(\tau(\eta),k)\right] + \eta(\eta^2-1)^{1/2}\left(1-\frac{(1-\delta)^2\eta^2}{(1+\delta)^2}\right)^{1/2}\right\}, \quad (37)$$

where

$$\delta = M + 1,$$

$$\tau(\eta) = \arcsin\left(\frac{1-(1-\delta)^2(1+\delta)^{-2}\eta^2}{k}\right),$$

$F(\varphi, k)$, $E(\varphi, k)$ are the elliptic integrals of the first and second kind in the Legendre form, and k is their modulus (in our case, $k = 2\sqrt{\delta}/(1 + \delta)$ [10]).

Using these representations, we can find the expression for the maximal height of the liquid bridge, which was determined numerically as follows:

$$h_{\max} = (2)^{2/3}(\pi)^{1/3}.$$

CONCLUSIONS

We have proposed a variational formulation of the problem of a vertical liquid bridge between two parallel solid planes taking into account the force of gravity in the axisymmetric case (in cylindrical system of coordinates). We have constructed an iterative process for obtaining an approximate solution to this problem under the assumption of smallness of the Bond number. It is shown that the inclusion of the second iteration step for small values of the modified Bond number does not change significantly the shape of the liquid bridge profile.

We have discovered that there is no unique solution: for a fixed liquid bridge height (distance between the bottom and top), several solutions can exist (several different profiles of the lateral surface of the drop), the maximal number of such solutions being four. For liquid bridge heights exceeding the value $h = h_{\max}$, solutions do not exist. For heights below a certain value, there is only one profile of the surface.

In the case of strong wetting, we have constructed the asymptotic expression for shape of the surface of the liquid bridge in the vicinity of the left boundary of admissible values region of parameter M. We have determined the maximal height of the vertical bridge at the first step of the iterative process.

REFERENCES

1. R. Finn, *Equilibrium Capillary Surfaces* (Springer, New York, 1986).
2. E. V. Galaktionov, N. E. Galaktionova, and E. A. Tropp, Tech. Phys. **61**, 1781 (2016).
3. P. I. Antonov, L. M. Zatulovskii, A. S. Kostygov, et al., *Fabrication of Profiled Single Crystals and Articles by the Stepanov Method*, Ed. by V. R. Regel' and S. P. Nikanorov (Nauka, Leningrad, 1981).
4. Ya. E. Geguzin, *Droplet* (Nauka, Moscow, 1973).
5. E. V. Galaktionov, N. E. Demidovich, and E. A. Tropp, in *Issues of Mathematical Physics and Applied Mathematics*, Ed. by E. V. Galaktionov (Ioffe Institute, St. Petersburg, 2010).
6. M. A. Fortes, J. Colloid Interface Sci. **88**, 338 (1982).
7. L. G. Fel and B. Y. Rubinstein, Z. Angew. Math. Phys. **66**, 3447 (2016).
8. T. Vogel, Pac. J. Math. **224**, 367 (2006).
9. M. Haynes, S. B. G. O'Brien, and E. S. Benilov, Phys. Fluids **28**, 042107 (2016).
10. G. M. Fichtenholz, *Differential and Integral Calculus* (GIF-ML, Moscow, 1959), Vol. 2.

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