ISSN 1063-7842, Technical Physics, 2016, Vol. 61, No. 1, pp. 33–41. © Pleiades Publishing, Ltd., 2016.
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GASES AND LIQUIDS

Linear (in Oscillation Dimensionless Amplitude) Interaction between the Modes of a Nonspherical Charged Drop in an External Electrostatic Field

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Received March 31, 2015

Abstract—The stability of a heavily charged drop in a weak uniform electrostatic field (in which the equilibrium shape of the drop can be represented by a prolate spheroid) is calculated in the fourth order of smallness in the eccentricity of the spheroidal drop and in the first order of smallness in the drop oscillation dimensionless amplitude. It is found that as the order of approximation in eccentricity grows, so does the number of modes interacting with the initially excited mode. In the given order of smallness, the preferred (initially excited) mode is shown to interact with the nearest eight modes. The drop becomes unstable if such is the second mode.

DOI: 10.1134/S1063784216010187

1. INTRODUCTION

The stability of the drop's surface is a subject of much scientific and applied interest. Charged drops subjected to an electrostatic field are observed in thunder clouds and ion–cluster–drop beams resulting from electrodispersion of liquids in liquid-metal ion sources and mass-spectrometers analyzing thermally unstable and volatile liquids. They also appear at the electrospraying of paint-and-lacquer materials, fuels, and insecticides (see, for example, [1–3]). Of special interest is the stability of the drop's surface (hereinafter surface, unless otherwise stated) in different (acoustic, aerodynamic, electromagnetic, electrostatic) levitators (contactless suspensions) [4–10]. Levitators are finding wide application in advanced technologies of high-purity materials. Also, they are applied in attempts to check the Rayleigh criterion for the stability of a drop against its charge [4–11].

The equilibrium shape of a charged drop in an electrostatic suspension (when an external uniform electrostatic field and the gravity field collinear to it keep the drop suspended) was found in [12]. Stability conditions for a charged drop in uniform electrostatic and gravitational fields were analytically obtained in [13], and nonlinear corrections to the oscillation frequency of the fundamental mode were analytically derived in a quadratic (in amplitude) approximation in [14].

Our goal is to study the stability of oscillations of a heavily (maximally) charged drop placed in a weak electrostatic field that merely sets a preferred direction in which polarization of the drop can be neglected.

2. PROBLEM DEFINITION

Consider a drop of an ideal incompressible conducting liquid with density ρ and surface tension coefficient σ that bears charge Q. The drop is in an electrostatic suspension; that is, oppositely directed collinear electrostatic field $\mathbf{E}_0$ and gravitational field $\mathbf{g}$ keep it quiescent. In the absence of the fields, the drop is a sphere with radius R.

Let us calculate the oscillations of the drop in a spherical coordinate system centered at the center of mass of the drop. The problem will be solved in terms of dimensionless variables $\rho = \sigma = R = 1$.

Let a heavily charged drop be placed in a weak electrostatic field that is collinear to the gravitational field. Under such conditions, the equilibrium shape of the drop is close (in an approximation linear in eccentricity squared) to a spheroid extended along the electrostatic field. In the dimensionless variables adopted, the free surface of the drop is described by the expression [12]

$$r(\theta) = 1 + a_0^{(4)}P_0(\mu) + (a_2^{(2)} + a_2^{(4)})P_2(\mu) + a_3^{(3)}P_3(\mu) + a_4^{(4)}P_4(\mu), \tag{1}$$

where

$$a_2^{(2)} = \frac{3w}{1-W}, \quad a_3^{(3)} = \frac{108\sqrt{Ww^3}}{5(1-W)(5-4W)},$$

$$a_0^{(4)} = -\frac{9w^2}{5(1-W)^2},$$

$$a_2^{(4)} = \frac{9w^2(79 - 84W + 8W^2)}{7(1-W)^3(5-4W)},$$

$$a_4^{(4)} = \frac{54w^2(65 - 33W - 28W^2)}{35(1-W)^2(3-2W)(5-4W)},$$

$$w = \frac{E_0^2}{16\pi}, \quad w = \frac{Q^2}{16\pi}, \quad \mu \equiv \cos\theta.$$

Here, w and W are Rayleigh and Taylor parameters, respectively.

Mathematically, the problem of the drop's oscillations under such conditions is stated as

$$\partial_t \mathbf{u} + (\mathbf{u}\nabla)\mathbf{u} = -\nabla p_{\text{in}} + \mathbf{g},$$
$$\text{div}\mathbf{u}(\mathbf{r},t) = 0, \quad \text{div}\mathbf{E}(\mathbf{r},t) = 0,$$
$$r \to \infty: \quad \mathbf{E}(\mathbf{r},t) \to \mathbf{E}_0,$$
$$r \to 0: \quad \mathbf{u}(\mathbf{r},t) \to 0,$$
$$r = 1 + h(\theta) + \xi(\theta,t): \quad \frac{dF(\mathbf{r},t)}{dt} = 0,$$
$$F(\mathbf{r},t) \equiv r - 1 - h(\theta) - \xi(\theta,t),$$
$$\Phi(\mathbf{r},t) = \Phi_s, \quad p_q(\mathbf{r},t) + p_{\text{in}}(\mathbf{r},t) - p_0 - p_\sigma(\mathbf{r},t) = 0,$$

where

$$p_q = \frac{E^2}{8\pi}, \quad p_\sigma = \text{div}\mathbf{n}(\mathbf{r},t), \quad \mathbf{n}(\mathbf{r},t) \equiv \frac{\nabla F}{|\nabla F|}.$$

Here, $\mathbf{u}(\mathbf{r}, t)$ and $p_{\text{in}}(\mathbf{r}, t)$ are, respectively, the velocity field and pressure in the drop; $\mathbf{E}(\mathbf{r}, t)$ is the electric field strength near the drop; $\Phi(\mathbf{r}, t)$ is the electric field potential ($\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t)$; $p_\sigma(\mathbf{r}, t)$ is the capillary pressure on the surface; $p_q(\mathbf{r}, t)$ is the pressure of the electrostatic field plus the pressure of the drop's charge electrostatic field; p_0 is the constant external pressure; and

$$h(\theta) = a_0^{(4)}P_0(\mu) + (a_2^{(2)} + a_2^{(4)})P_2(\mu) + a_3^{(3)}P_3(\mu) + a_4^{(4)}P_4(\mu)$$

is the deviation of the equilibrium spheroidal shape of the drop from the initially spherical one Eq. (1).

The problem is complemented by the constancy conditions for the volume and charge of the drop and also by the immovability condition for its center of mass,

$$\iiint_V dV = \frac{4}{3}\pi, \quad \iint_S (\mathbf{n}, \mathbf{E})dS = 4\pi Q, \quad \iiint_V \mathbf{r}dV = 0,$$

where

$$S \equiv \{r = 1 + h(\theta) + \xi(\theta,t); 0 \le \theta \le \pi; 0 \le \varphi \le 2\pi\},$$
$$V \equiv \{0 \le r \le 1 + h(\theta) + \xi(\theta,t); 0 \le \theta \le \pi; 0 \le \varphi \le 2\pi\}.$$

3. CALCULATION OF THE DROP'S OSCILLATIONS

Using the model of liquid potential flow, we pass from the velocity vector to scalar hydrodynamic potential $\psi(\mathbf{r}, t)$ and from the field strength vector to electrostatic field potential $\Phi(\mathbf{r}, t)$,

$$\mathbf{u}(\mathbf{r},t) = \nabla\psi(\mathbf{r},t), \quad \mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t).$$

The problem will be solved asymptotically in an approximation linear in the dimensionless amplitude of drop's oscillations, $\zeta \sim |\xi|$, in the fourth order of smallness in eccentricity e of its equilibrium (spheroidal) shape. Terms up to the order of smallness ζe^4 will be left. Since the flow inside the drop is due to a surface perturbation, we have $\psi \sim \zeta$ in the dimensionless variables adopted. It should be borne in mind that the charge by itself does not deform a spherical drop and the eccentricity of a spheroidal drop squared is proportional to the external field strength squared [13]; therefore, when writing solutions, we will take into account that

$$w \sim E_0^2 \sim e^2, \quad W \sim Q^2 \sim e^0.$$

The electrostatic potential is represented as

$$\Phi(\mathbf{r},t) = \Phi_0(\mathbf{r}) + \Phi_1(\mathbf{r},t) + O(\zeta^2),$$

where $\Phi_0(\mathbf{r})$ is the electrostatic potential near the drop with an equilibrium shape and $\Phi_1(\mathbf{r}, t)$ is the correction to the electrostatic potential of the drop with an equilibrium shape. This correction is due to a surface perturbation. Having substituted the expansion of the electrostatic potential into the problem, we divide it into subproblems of the zeroth and first orders of smallness.

Below, the dependences of a number of quantities on azimuthal angle φ will be neglected. Such an assumption allows us to cut the amount of computation without loss of generality.

The subproblem of the zeroth order of smallness refers to the undisturbed (equilibrium) surface, and its essence is to calculate the electrostatic field near the equilibrium drop,

$$\Delta\Phi_0(r,\theta) = 0,$$
$$r \to \infty: \quad -\nabla\Phi_0(r,\theta) \to E_0 - \mathbf{e}_z,$$
$$r \to 1 + h(\theta): \quad \Phi(r,\theta,t) = \Phi_s,$$
$$\iint_S (\mathbf{n}, \nabla\Phi_0)dS = -4\pi Q,$$
$$S \equiv \{r = 1 + h(\theta); 0 \le \theta \le \pi; 0 \le \varphi \le 2\pi\}.$$

The electric field potential in the zeroth approximation in oscillation dimensionless amplitude is given by (up to terms $\sim e^4$),

$$\Phi_0(r,\theta) = \frac{4\sqrt{\pi W}}{r} + \sqrt{w}A_1 + wA_2 + \sqrt{w^3}A_3 + w^2A_4,$$

where

$$A_1 \equiv 4\sqrt{\pi}\left(\frac{1}{r^2} - r\right)P_1(\mu), \quad A_2 \equiv \frac{12\sqrt{\pi W}}{r^3(1-W)}P_2(\mu),$$

$$A_3 \equiv \frac{36\sqrt{\pi}(2r^2(5-4W)P_1(\mu) + 15P_3(\mu))}{5r^4(1-W)(5-4W)},$$

$$A_4 \equiv \frac{36\sqrt{\pi W}}{5r(1-W)^2} + \frac{108\sqrt{\pi W}(201 - 272W + 76W^2)}{35r^3(1-W)^3(5-4W)}P_2(\mu) + \frac{216\sqrt{\pi W}(227 - 285W + 68W^2)}{835r^5(1-W)^2(3-2W)(5-4W)}P_4(\mu).$$

For the subproblem of the first order of smallness, we have

$$\Delta\Phi_1(r,\theta,t) = 0, \quad \Delta\psi(r,\theta,t) = 0,$$

$$r \to \infty: \quad \Phi_1(r,\theta,t) \to 0,$$

$$r \to 0: \quad \psi(r,\theta,t) \to 0,$$

$$r = 1 = h(\theta): -\partial_1\xi(\theta,t) - \frac{\partial_\theta h(\theta)\partial_\theta\psi(r,\theta,t)}{r^2} + \partial_r\Psi(r,\theta,t) = 0, \tag{2}$$

$$\Phi_1(r,\theta,t) + \xi(\theta,t)\partial_r\Phi_0(r,\theta) = 0, \tag{3}$$

$$p_{\text{in}}^{(1)} + p_{EQ}^{(1)} + p_g^{(1)} - p_\sigma^{(1)} = 0, \tag{4}$$

where

$$p_{\text{in}}^{(1)} = f_1(t) - \partial_t\psi(r,\theta,t),$$

$$p_g^{(1)} = -g(\xi(0,t) - \xi(\theta,t)\cos\theta),$$

$$p_{EQ}^{(1)}(r,\theta,t) = r\partial_\theta\Phi_0(r,\theta)\partial_\theta\Phi_1(r,\theta,t) + r^3\partial_r\Phi_3(r,\theta)\partial_r\Phi_1(r,\theta,t) + \frac{\xi(\theta,t)}{4\pi r^3}(-(\Phi_0(r,\theta))^2 + r\partial_\theta\Phi_0(r,\theta)\partial_{r,\theta}\Phi_0(r,\theta) + r^3\partial_r\Phi_0(r,\theta)\partial_{r,r}\Phi_0(r,\theta));$$

$$p_\sigma^{(1)} = \frac{1}{r^2\sqrt{r^2 + (\partial_\theta h(\theta))^2}} \times [(2r^5 + 5r^3(\partial_\theta h(\theta))^2 + 2r^4\partial_{\theta\theta}h(\theta) - r^2(\partial_\theta h(\theta))^2\partial_{\theta,\theta}h(\theta) + (2r^4\partial_\theta h(\theta) + 3r^2(\partial_\theta h(\theta))^3 + (\partial_\theta h(\theta))^5)\cot\theta)\xi(\theta,t) + r^3(r^2 + (\partial_\theta h(\theta))^2)\partial_{\theta,\theta}\xi(\theta,t) + (r^3(r^2 + (\partial_\theta h(\theta))^2)\cot\theta - 3r^2((\partial_\theta h(\theta))^2 + r\partial_{\theta,\theta}h(\theta))\partial_\theta h(\theta))\partial_\theta\xi(\theta,t)] = 0.$$

Symbols ∂_z and $\partial_{z,\,x}$ designate the first and second derivatives with respect to argument z and z and x, respectively., and $f_1(t)$ is a constant of integration over the spatial coordinates, which is a function of time in the general case.

In a spherical coordinate system with the origin at the center of mass of the drop, the solution to the Laplace equation has the form

$$\Phi_1(r,\theta,t) = \sum_{n=2}^{\infty} r^{-(1+n)}A_n(t)P_n(\mu), \tag{5}$$

$$\psi(r,\theta,t) = \sum_{n=2}^{\infty} r^n B_n(t)P_n(\mu), \tag{6}$$

where $P_n(\mu)$ are axisymmetric Legendre polynomials of order n [15].

Surface perturbation $\xi(\theta, t)$ should be sought in the form [14]

$$\xi(\theta,t) = \sum_{n=2}^{\infty}\alpha_n(t)P_n(\mu), \tag{7}$$

were $A_n(t)$, $B_n(t)$, and $\alpha_n(t)$ are unknown functions of time and other physicochemical parameters of the problem.

Having substituted projects (5)–(7) of solutions into Eqs. (2)–(4), we rearrange terms so that the equations represent expansions in Legendre polynomials. Then, Eqs. (2)–(4) can be satisfied by setting the coefficients at the Legendre polynomials equal to zero. Leaving the terms of the order of smallness $\sim e^4$, we arrive at the following relationships between the coefficients of Eqs. (5)–(7) and kinematic condition (2):

$$e^4K_{n\pm4}^{(1)}B_{n\pm4}(t) + e^3K_{n\pm3}^{(2)}B_{n\pm3}(t) + (e^2K_{n\pm2}^{(3)} + e^4K_{n\pm2}^{(4)})B_{n\pm2}(t) + e^3K_{n\pm1}^{(5)}B_{n\pm1}(t) + (K_n^{(6)} + e^2K_n^{(7)} + e^4K_n^{(8)})B_n(t) - \partial_1\alpha_n(t) = 0. \tag{8}$$

From equipotentiality condition (3), we obtain

$$\begin{gathered}
e^4E^{(1)}_{n\pm4}A_{n\pm4}(t)+e^3E^{(2)}_{n\pm3}A_{n\pm3}(t)\\
+(e^2E^{(3)}_{n\pm2}+e^4E^{(4)}_{n\pm2})A_{n\pm2}(t)+e^3E^{(5)}_{n\pm1}A_{n\pm1}(t)\\
+(E^{(6)}_{n}+e^2E^{(7)}_{n}+e^4E^{(8)}_{n})A_{n}(t)+e^4E^{(9)}_{n\pm4}\alpha_{n\pm4}(t)\\
+(e^2E^{(11)}_{n\pm2}+e^4E^{(12)}_{n\pm2})\alpha_{n\pm2}(t)+e^3E^{(10)}_{n\pm3}\alpha_{n\pm3}(t)\\
+(eE^{(13)}_{n\pm1}+e^3E^{(14)}_{n\pm1})\alpha_{n\pm1}(t)\\
+(E^{(15)}_{n}+e^2E^{(16)}_{n}+e^4E^{(17)}_{n})\alpha_{n}(t)=0.
\end{gathered}\tag{9}$$

From dynamic equation (4) it follows that

$$\begin{gathered}
D^{(11)}_{n}A_n(t)+D^{(1)}_{n\pm1}A_{n\pm1}(t)+D^{(1)}_{n\pm2}A_{n\pm2}(t)\\
+D^{(1)}_{n\pm3}A_{n\pm3}(t)+D^{(1)}_{n\pm4}A_{n\pm4}(t)+D^{(2)}_{n}\alpha_n(t)\\
+D^{(2)}_{n\pm1}\alpha_{n\pm1}(t)+D^{(2)}_{n\pm2}\alpha_{n\pm2}(t)+D^{(2)}_{n\pm3}\alpha_{n\pm3}(t)\\
+D^{(2)}_{n\pm4}\alpha_{n\pm4}(t)+D^{(3)}_{n}\partial_tB_n(t)+D^{(3)}_{n\pm1}\partial_tB_{n\pm1}(t)\\
+D^{(2)}_{n\pm2}\partial_tB_{n\pm2}(t)+D^{(3)}_{n\pm3}\partial_tB_{n\pm3}(t)+D^{(3)}_{n\pm4}\partial_tB_{n\pm4}(t)=0,
\end{gathered}\tag{10}$$

where $K^{(j)}_n$, $E^{(j)}_n$, and $D^{(j)}_n$ are some functions the form of which is omitted here because of awkwardness.

The amplitudes of hydrodynamic and electrodynamic potentials will be sought in the form of expansions in eccentricity e,

$$A_n(t)=\sum_{j=0}^{4}e^jA^{(j)}_n(t)+O(e^5),$$

$$B_n(t)=\sum_{j=0}^{4}e^jB^{(j)}_n(t)+O(e^5).$$

From Eqs. (8) and (9), one can find relationships between amplitudes $A^{(j)}_n(t)$ and $B^{(j)}_n(t)$ and, having substituted these relationships into Eq. (10), obtain a set of coupled evolutionary equations for the amplitudes of modes (accurate to ~e^4),

$$\begin{gathered}
\frac{\partial^2\alpha_n(t)}{\partial t^2}+\omega_n^2\alpha_n(t)=eN^{(1)}_{n\pm1}\alpha_{n\pm1}(t)\\
+e^2\left(N^{(2)}_{n\pm2}\alpha_{n\pm2}(t)+N^{(3)}_{n\pm2}\frac{\partial^2\alpha_{n\pm2}(t)}{\partial t^2}\right)\\
+e^3\left(N^{(4)}_{n\pm1}\alpha_{n\pm1}(t)+N^{(5)}_{n\pm3}\alpha_{n\pm3}(t)\right.\\
\left.+N^{(6)}_{n\pm1}\frac{\partial^2\alpha_{n\pm1}(t)}{\partial t^2}+N^{(7)}_{n\pm3}\frac{\partial^2\alpha_{n\pm3}(t)}{\partial t^2}\right)\\
+e^4\left(N^{(8)}_{n\pm2}\alpha_{n\pm2}(t)+N^{(9)}_{n\pm4}\alpha_{n\pm4}(t)\right.\\
\left.+N^{(10)}_{n\pm2}\frac{\partial^2\alpha_{n\pm2}(t)}{\partial t^2}+N^{(11)}_{n\pm4}\frac{\partial^2\alpha_{n\pm4}(t)}{\partial t^2}\right).
\end{gathered}\tag{11}$$

Here, $N^{(j)}_n$ are coefficients depending on physicochemical parameters of the problem and ω_n is the frequency of the nth mode,

$$\omega_n^2=\omega_{n,0}^2+e^2\delta\omega_n^2,\quad \omega_{n,0}^2=n(n-1)(2+n-4W).$$

Nonlinear frequency correction $e_n^{\delta\omega^2}$ arising when the equilibrium shape of the drop is nonspherical has the form

$$e^2\delta\omega_n^2\equiv e^2(\omega_{n,1}^2+e^2\omega_{n,2}^2),$$

where

$$\omega_{n,1}^2=\frac{9nw}{(3+2n)(2n+1)(2n-1)(1-W)}(2-4W+n(7-8W)+n^2(17-8W)$$
$$-3n^3(7-8W)-n^4(23-16W)-2n^5),$$

$$\omega_{n,2}^2=\frac{9nw^2}{70(2n-1)^2(2n+1)^2(2n-3)(2n+3)^2(2n+5)(1-W)^2(3-2W)(4W-5)}$$
$$\times(F_0+nF_1+n^2F_2+n^3F_3+n^4F_4+n^5F_5+n^6F_6+n^7F_7+n^8F_8+n^9F_9+n^{10}F_{10}).$$

Here,

$$F_0=-113400+59940W+1167480W^2$$
$$-1866240W^3+760320W^4,$$

$$F_1=-2044980+4149702W-677232W^2$$
$$-2747592W^3+1327392W^4,$$

$$F_2=-9724680+27254148W-33657954W^2$$
$$+22609272W^3-6551616W^4,$$

$$F_3=648270+2188407W-15970115W^2$$
$$+20836068W^3-7761760W^4,$$

$$F_4=358002900-94550217W+100229475W^2$$
$$-55488828W^3+14199360W^4,$$

$$F_5=25382070-68637261W+80457363W^2$$
$$-51529884W^3+14478912W^4,$$

$$F_6 = -17623620 + 35928201W - 26025201W^2 + 11511204W^3 - 3953664W^4,$$

$$F_7 = -16873920 + 35203248W - 28142160W^2 + 14271552W^3 - 4592640W^4,$$

$$F_8 = 1360800 - 2111688W + 771240W^2 - 376992W^3 + 384000W^4,$$

$$F_9 = 3094560 - 5336496W + 2646224W^2 - 788544W^3 + 418816W^4,$$

$$F_{10} = 5040000 - 827184W + 304560W^2 + 24384W^3$$

It was shown [11] that up to the order of smallness ζe^4 (adopted in our consideration), the nth mode is excited on the surface together with eight modes coupled with it. Solution (11) obtained by the method of successive approximations has the form

$$\begin{aligned}\alpha_n(t) = {} & \beta_n \cos(\omega_n t) + \beta_n \Lambda_1 \cos\left(t\sqrt{\omega_{n,0}^2 + e^2\omega_{n,1}^2}\right) \\ & + \beta_{n\pm1}\Lambda_2 \cos\left(t\sqrt{\omega_{n\pm1,0}^2 + e^2\omega_{n\pm1,1}^2}\right) \\ & + \beta_{n\pm2}\Lambda_3 \cos\left(t\sqrt{\omega_{n\pm2,0}^2 + e^2\omega_{n\pm2,1}^2}\right) \\ & + \beta_n\Lambda_4 \cos(\omega_{n,0}t) + \beta_{n\pm1}\Lambda_5 \cos(\omega_{n\pm1,0}t) \\ & + \beta_{n\pm2}\Lambda_6 \cos(\omega_{n\pm2,0}t) + \beta_{n\pm3}\Lambda_7 \cos(\omega_{n\pm3,0}t) \\ & + \beta_{n\pm4}\Lambda_8 \cos(\omega_{n\pm4,0}t),\end{aligned} \qquad (12)$$

where

$$\Lambda_1 = \frac{e^{-2}M_n^{(1)} + M_n^{(2)} + e^2M_n^{(3)} + e^4M_n^{(4)}}{(\omega_{n-1,0}^2 - \omega_{n,0}^2)(\omega_{n+1,0}^2 - \omega_{n,0}^2)},$$

$$\Lambda_2 = \frac{eM_{n\pm1}^{(5)} + e^3M_{n\pm1}^{(6)}}{(\omega_{n\pm2,0}^2 - \omega_{n\pm1,0}^2)(\omega_{n\pm1,0}^2 - \omega_{n,0}^2)},$$

$$\Lambda_3 = \frac{e^2M_{n\pm1}^{(7)} + e^4M_{n\pm1}^{(8)}}{(\omega_{n\pm3,0}^2 - \omega_{n\pm2,0}^2)(\omega_{n\pm2,0}^2 - \omega_{n\pm1,0}^2)(\omega_{n\pm2,0}^2 - \omega_{n,0}^2)},$$

$$\Lambda_4 = \frac{M_n^{(9)} + e^2M_n^{(10)} + e^4M_{11}^{(11)}}{(\omega_{n-1,0}^2 - \omega_{n,0}^2)(\omega_{n+1,0}^2 - \omega_{n,0}^2)(\omega_{n-2,0}^2 - \omega_{n,0}^2)(\omega_{n+2,0}^2 - \omega_{n,0}^2)},$$

$$\Lambda_5 = \frac{eM_{n\pm1}^{(12)} + e^3M_{n\pm1}^{(13)}}{(\omega_{n\pm2,0}^2 - \omega_{n\pm1,0}^2)(\omega_{n\pm1,0}^2 - \omega_{n,0}^2)(\omega_{n\pm1,0}^2 - \omega_{n\pm1,0}^2)},$$

$$\Lambda_6 = \frac{e^2M_{n\pm2}^{(14)} + e^4M_{n\pm2}^{(15)}}{(\omega_{n\pm3,0}^2 - \omega_{n\pm2,0}^2)(\omega_{n\pm2,0}^2 - \omega_{n\pm1,0}^2)(\omega_{n\pm2,0}^2 - \omega_{n,0}^2)(\omega_{n\pm2,0}^2 - \omega_{n\pm1,0}^2)},$$

$$\Lambda_7 = \frac{e^3M_{n\pm3}^{(16)}}{(\omega_{n\pm3,0}^2 - \omega_{n\pm2,0}^2)(\omega_{n\pm3,0}^2 - \omega_{n\pm1,0}^2)(\omega_{n\pm3,0}^2 - \omega_{n,0}^2)},$$

$$\Lambda_8 = \frac{e^4M_{n\pm4}^{(17)}}{(\omega_{n\pm4,0}^2 - \omega_{n\pm3,0}^2)(\omega_{n\pm4,0}^2 - \omega_{n\pm2,0}^2)(\omega_{n\pm4,0}^2 - \omega_{n\pm1,0}^2)(\omega_{n\pm4,0}^2 - \omega_{n,0}^2)}.$$

Here, β_n are constants of integration and $M_n^{(j)}$ are coefficients depending on the physicochemical parameters of the problem (they are omitted because of awkwardness). Compensatory expressions for the amplitudes of the zeroth and first modes can be derived by satisfying the constancy condition for the volume of the drop and the immovability condition for its center of mass,

$$\alpha_0(t) = \frac{18w}{7(9w^2 + 5(1-W)^2)}(\alpha_2(t) + \alpha_4(t)), \qquad (13)$$

$$\alpha_1(t) = m_2\alpha_2(t) + m_3\alpha_3(t) + m_4\alpha_4(t) + m_5\alpha_5(t), \qquad (14)$$

where

$$m_2 = \frac{2916w^{3/2}(1-W)^2\sqrt{W}}{5((126w + 35(1-W))(1-W)^2(5-4W) + 54w^2(89 - 102W + 16W^2))},$$

$$m_3 = \frac{-27w(21(1-W)^2(15-22W+8W^2)+w(3283-6028W+3028W^2-256W^3))}{7(3-2W)((126w+35(1-W))(1-W)^2(5-4W)+54w^2(89-102W+16W^2))},$$

$$m_4 = \frac{432w^{3/2}(1-W)^2\sqrt{W}}{(126w+35(1-W))(1-W)^2(5-4W)+54w^2(89-102W+16W^2)},$$

$$m_5 = \frac{-270w^3(110-209W+95W^2+4W^3)}{11(3-W)((126w+35(1-W))(1-W)^2(5-4W)+54w^2(89-102W+16W^2))}.$$

4. RESULTS AND DISCUSSION

Coefficients β_n in expression (12), being constants of integration, are found from initial conditions. They determine the type of interaction between modes and the contribution of each mode to the shape of the drop.

As has been mentioned above, each excited mode generates eight other modes (accurate to order of smallness $\sim\zeta e^4$). In calculations of order of smallness ζe^2 performed for physically similar problems (problem of electrostatic stability of a solitary charged spheroidal drop or the problem of stability of an uncharged drop in an external uniform electrostatic field), it was shown that the initially excited mode interacts with two or four nearest modes depending on the problem.

Consider the situation when only one mode is excited at a zero instant of time. Then, the set of initial conditions is as follows:

$$\alpha_n(0) = \zeta, \quad \alpha_{n\pm1}(0) = 0, \quad \alpha_{n\pm2}(0) = 0,$$
$$\alpha_{n\pm3}(0) = 0, \quad \alpha_{n\pm4}(0) = 0, \tag{15}$$
$$\frac{d\alpha_n}{dt}(0) = 0, \quad \frac{d\alpha_{n\pm j}}{dt}(0) = 0 \quad (j = 1,\ldots,4).$$

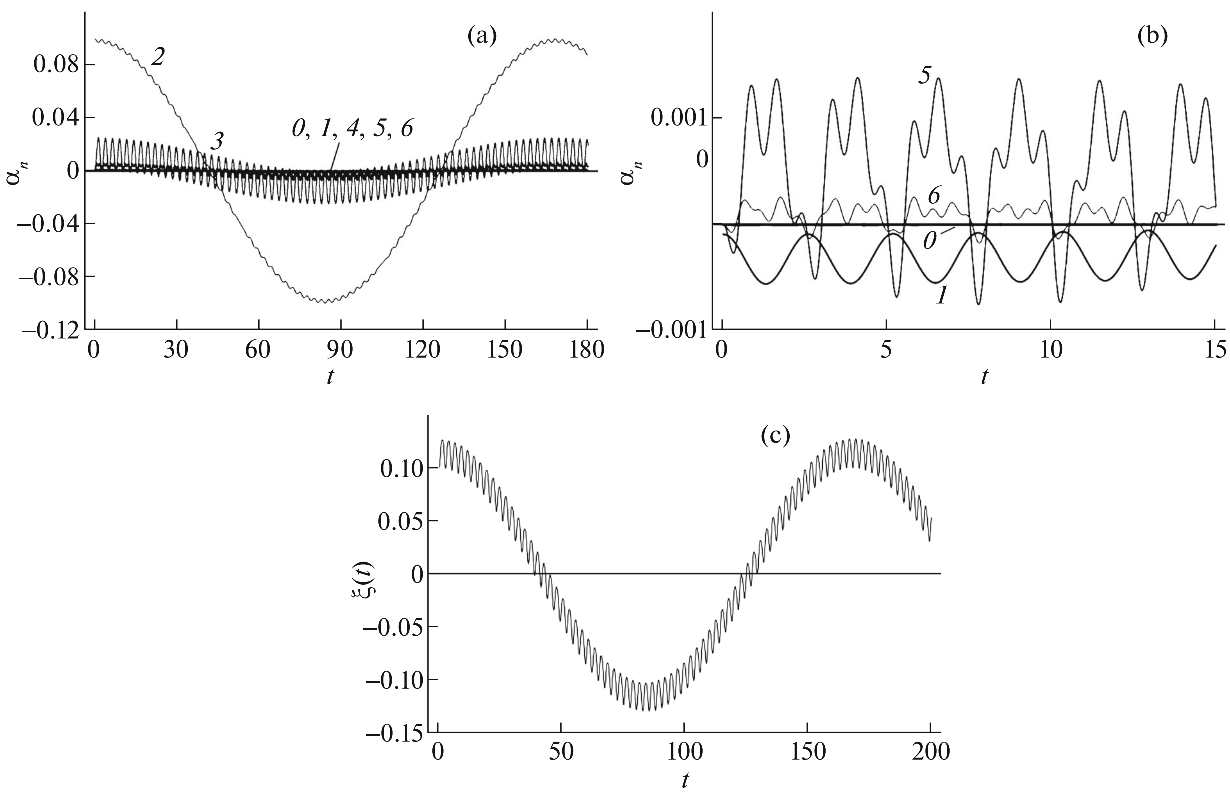

Fig. 1. Time evolution of the amplitudes of the initially excited (second) mode and those coupled with it: (a) general pattern of the mode amplitude evolution, (b) amplitude evolution of only those modes coupled with the initially excited one, and (c) general pattern of surface perturbation $\zeta(t)$ when the second mode is initially excited (hereinafter, the number by the curves is the mode number).

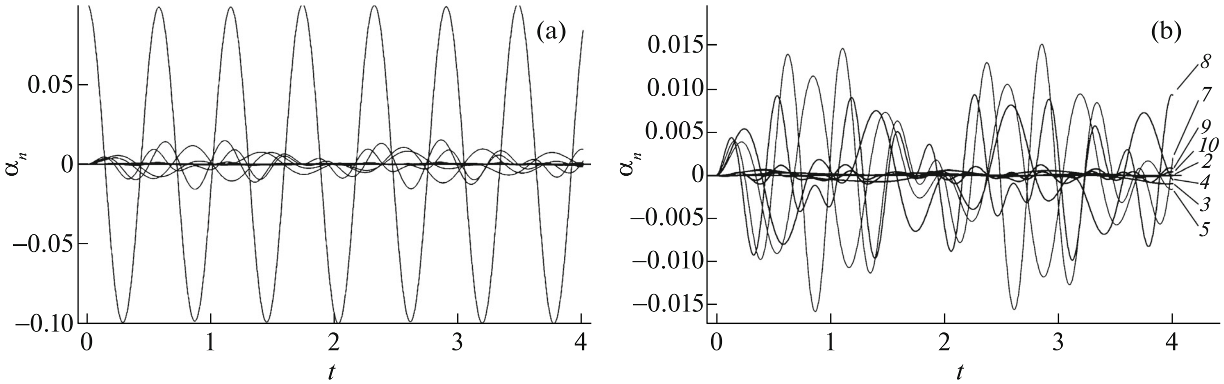

Fig. 2. Time evolution of the amplitudes of the initially excited (sixth) mode and those coupled with it: (a) general pattern of the mode amplitude evolution and (b) amplitude evolution of only those modes coupled with the initially excited one. All other parameters are the same as in Fig. 1 (hereinafter, figures by the curves are mode numbers).

Substituting Eqs. (12) of the evolutionary system into initial conditions (15) and leaving terms $\sim\zeta e^4$, we arrive at a set of nine equations (for the zeroth and first modes, we use expressions (13) and (14), respectively). Having found coefficients β_n from the resulting set of equations and having substituted them into (12), we obtain final expressions for amplitudes $\alpha_n(t)$, which determine the time evolution of the nine coupled modes.

In numerical calculations, the Rayleigh parameter and the Taylor parameter were set equal to $W = 0.9772$ and $w = 1.283 \times 10^{-4}$, respectively, and the radius of the drop was taken to be $R = 0.1$ cm. It is at such values of these parameters that the drop is stable and quiescent in suspension [13]. Note that the critical value of the Rayleigh parameter for a solitary charged drop is $W = 1$ and the Taylor parameter for an uncharged drop in a uniform electrostatic field is $w \approx 0.05$ [13]. The density of the liquid was set equal to $\rho = 1$ g/cm^3; the surface tension coefficient, $\sigma = 73$ dyn/cm (both values are for water); and the initial perturbation amplitude, $\zeta = 0.1R$. The eccentricity of the equilibrium shape of the drop was calculated by the formula $e^2 = 9w(1 - W)$ and was found to be $e^2 \approx 0.0506$. Dimensionless free fall acceleration g was estimated as $g = 980.7(\rho R^2/\sigma \approx 0.134)$ (for the nondimensionalization adopted here).

Figure 1a shows how the amplitudes of the initially excited mode (α_2) and other modes coupled with it (α_0, α_1, α_3, α_4, α_5, α_6) vary with time. Rayleigh parameter W is 0.0002 smaller than the critical value ($W = 0.9774$), and Taylor parameter w is taken so as to satisfy the suspension condition at the given value of W ($w = 1.283 \times 10^{-4}$). It is seen that the amplitude of the third mode is three times smaller than that of the second one and the amplitude of the fourth mode is roughly three times lower than that of the third mode. The amplitudes of the other modes are very small and almost merge with the abscissa axis. In Fig. 1b, the evolution of these modes is shown on a two orders of magnitude larger scale on the vertical axis and within a shorter time interval (abscissa axis). Even in this case, the amplitude of the zeroth mode is hardly discernible from the abscissa axis. It should be noted that the excitation of the zeroth and first modes is a compensatory process; therefore, their amplitudes are negative. Figure 1c illustrates the time evolution of surface perturbation $\zeta(t)$ of the drop when the second mode is initially excited. It is obvious that interaction between the second and neighboring modes redistributes the initial deformation energy among all modes involved in interaction.

Figure 2a shows how the amplitudes of the initially excited sixth mode α_6 and other modes coupled with it (α_2, α_3, α_4, α_5, α_7, α_8, α_9, α_{10}) vary with time for the same Rayleigh and Taylor parameters as in Fig. 1a. The sixth mode has a maximal amplitude, whereas the amplitudes of the other modes are small. This is shown in Fig. 2b, where the evolution of modes 2–5 and 7–10 coupled with the initially excited sixth mode is presented on an enlarged (by 10 times) scale. The farther a given mode is from that excited at an initial instant of time, the weaker is their coupling and the smaller is the contribution of the former to the shape of the surface, as expected from solution (12) and is observed in Fig. 1a.

The onset of instability of coupled modes is of most interest. Figure 3a shows how the amplitudes of the initially excited second mode α_2 and other modes coupled with it (α_0, α_1, α_3, α_4, α_5, α_6) vary with time. Rayleigh parameter W is $W = 0.9774$, and Taylor parameter w is $w = 1.282 \times 10^{-4}$. At such values, the drop loses stability. It is seen from Fig. 3 that the modes coupled with the second mode also become unstable, although the charge of the drop is subcritical for them, their instability being oscillatory. This is because the amplitudes of the modes coupled with the fundamental (second) mode are expressed through the amplitudes of the latter. Figure 3b shows the time

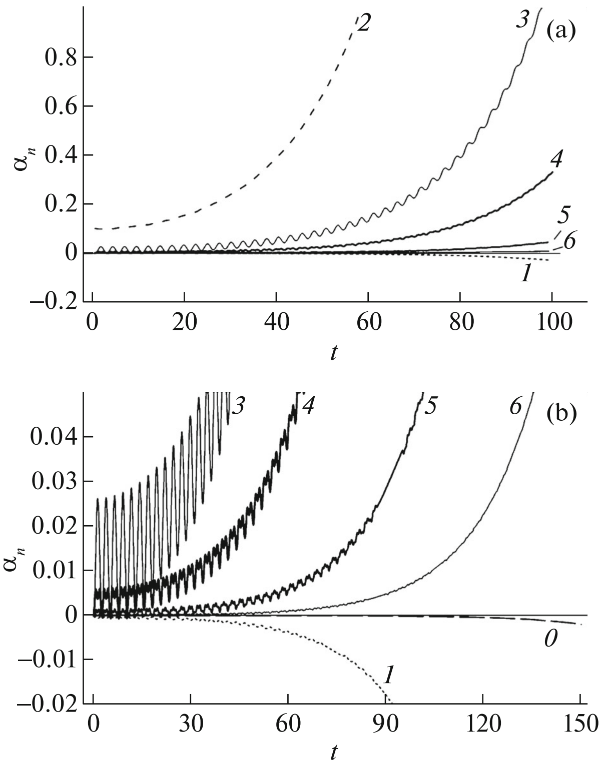

Fig. 3. (a) Same as in Fig. 1 but for the combination of the Rayleigh and Taylor parameters ($W = 0.9774$, $w = 1.283 \times 10^{-4}$) that are overcritical for the second mode and subcritical for other modes coupled with the initially excited (second) one and (b) same as in Fig. 3a on an enlarged scale in the vertical axis.

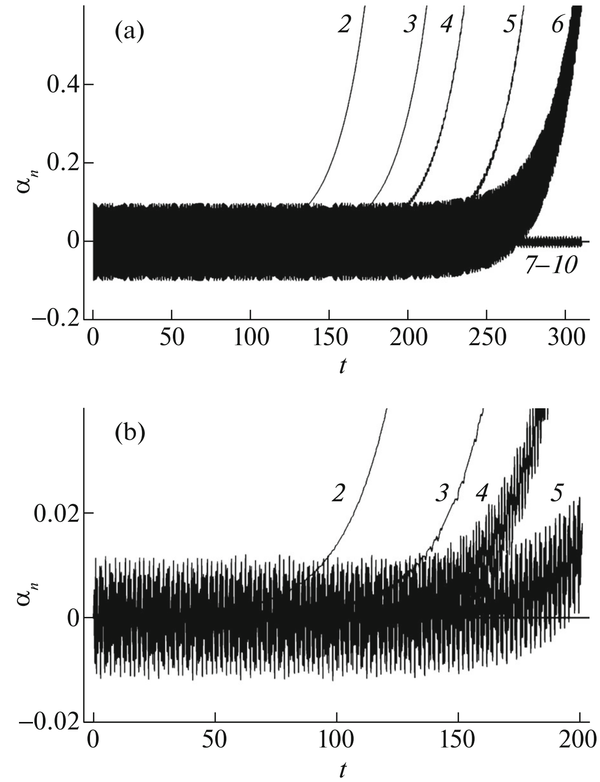

Fig. 4. (a) Same as in Fig. 2 but for the combination of the Rayleigh and Taylor parameters ($W = 0.9774$, $w = 1.283 \times 10^{-4}$) that are overcritical for the second mode and subcritical for the modes coupled with the initially excited (sixth) one and (b) same as in Fig. 4a on an enlarged scale in the vertical axis for the second, third, fourth, and fifth modes separately.

evolution of amplitudes α_0, α_1, α_3, α_4, α_5, and α_6 on an enlarged scale.

In Fig. 4a, the amplitudes of the initially excited (sixth) mode and of the modes coupled with it are shown versus the time for the same Rayleigh and Taylor parameters as in Fig. 3 ($W = 0.9774$, $w = 1.282 \times 10^{-4}$). It is expected that the second mode will be unstable and the others will be stable at such values of the parameters. However, as follows from Fig. 4, the amplitudes of the modes (from the second to the sixth) build up in an oscillatory manner. The initially excited sixth mode loses stability at the given values of the Rayleigh and Taylor parameters, which are subcritical for it. This is because the sixth mode interacts with the second (unstable mode). Thus, in the order of approximation adopted ($\sim\zeta e^4$), the instability of the second mode makes all the modes coupled with it unstable (modes $n = 0$–6). The rest of the modes ($n = 7$–10) remain stable. In Fig. 4b, the time evolution of amplitudes α_2, α_3, α_4, and α_5 under the same conditions as in Fig. 4a are shown on an enlarged scale.

The same behavior of interacting surface modes near the instability threshold were described in [16] for a spheroidal charged drop in the absence of an electrostatic field.

It should be noted that the number of modes coupled with the initially excited one grows with the field nonuniformity around the drop (that is, with increasing order of smallness in eccentricity e of a spheroidal drop in a uniform electric field) [17]. The mode coupling coefficient decreases as the number of the mode under observation moves away from the number of the initially excited mode [18].

CONCLUSIONS

It is shown that when the calculation accuracy (order of smallness in the eccentricity) rises, so does the degree of coupling between modes (i.e., the number of interacting modes). The intensity of mode interaction drops with "distance" between the number of a given mode and the number of a mode excited at an initial instant of time. In drop instability calculations, it is necessary to take into account all coupled modes, since at the instant of disintegration of the drop (at the instability threshold), even those slightly contributing to its shape may influence disintegration. The drop becomes unstable when the lowest of modes involved in interaction becomes unstable.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research, grant nos. 14-01-00170 and 14-08-00240.

REFERENCES

1. M. D. Gabovich, Sov. Phys. Usp. **26**, 447 (1983).
2. A. G. Bailey, Atomization Sprays **2**, 95 (1986).
3. A. I. Grigor'ev, S. O. Shiryaeva, A. N. Zharov, and V. A. Koromyslov, Elektrokhim. Obrab. Met., No. 3, 25 (2005).
4. A. Doyle, D. R. Moffet, and B. Vonnegut, J. Coll. Sci. **19**, 136 (1964).
5. T. G. O. Berg, R. J. Trainor, and U. Vaughan, J. Atmos. Sci. **77**, 1173 (1970).
6. J. D. Schweizer and D. N. Hanson, J. Colloid Interface Sci. **35**, 417 (1971).
7. M. Roulleau and M. Desbois, J. Atmos. Sci. **29**, 565 (1972).
8. D. Duft, H. Lebbeus, and B. A. Huber, Phys. Rev. Lett. **89**, 1 (2002).
9. D. Duft, T. Achtzehn, R. Muller, et al., Nature **421**, 128 (2003).
10. R. L. Grimm and J. L. Beauchamp, J. Phys. Chem. B **109**, 8244 (2005).
11. J. W. Strutt (Rayleigh), Philos. Mag. **14**, 184 (1882).
12. S. O. Shiryaeva, Tech. Phys. **51**, 1284 (2006).
13. A. I. Grigor'ev, S. O. Shiryaeva, and E. I. Belavina, Sov. Phys. Tech. Phys. **34**, 602 (1989).
14. S. O. Shiryaeva, Tech. Phys. **51**, 385 (2006).
15. D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, *Quantum Theory of Angular Momentum* (World Sci., Singapore, 1988).
16. S. O. Shiryaeva, A. I. Grigor'ev, and D. O. Kornienko, Tech. Phys. **55**, 1558 (2010).
17. S. O. Shiryaeva, A. I. Grigor'ev, and A. A. Shiryaev, Tech. Phys. **58**, 664 (2013).
18. M. I. Rabinovich and D. I. Trubetskov, *Introduction to the Theory of Oscillations and Waves* (Kluwer, Dordrecht, 1989).

Translated by V. Isaakayn