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Algebraic and Statistical Methods in Quantum and Gravitational Physics

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Abstract—A scheme for constructing quantum mechanics is described, in which the Hilbert space and linear operators are not the primary notions of the theory. Instead, a variant of the algebraic approach is considered. The noncommutative algebra elements (observables) and the functionals on this algebra (elementary states), which are associated with the results of individual measurements, are used as primary notions. This scheme makes it possible, on the one hand, to use the apparatus of the classical (Kolmogorov) probability theory, and on the other hand, to reproduce the standard mathematical apparatus of quantum mechanics and determine the limits to its applicability.

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INTRODUCTION

In recent decades, quantum field theory has made significant advances. The success is associated with the construction of non-Abelian (noncommutative) gauge models. On their basis, the newest quark physics arose. The Abelian gauge model, quantum electrodynamics, has been known for a long time. The transition to non-Abelian models was a qualitative leap forward in the development of quantum field theory.

At the same time, this transition did not entail any significant revision of the basic concepts of quantum field theory. Moreover, it did not demand any changes in logic and mathematics.

This paper will use the idea that the transition from classical to quantum physics is similar to the transition from Abelian gauge models to non-Abelian ones. Definitely, quantum physics is a qualitatively new theory. However, for the successful development of quantum theory, it is not at all necessary to withdraw from the basic concepts of classical theory (formal logic, classical probability theory, the principle of causality, the representation of objective physical reality) or, as philosophers say, from the classical paradigm. It is generally believed that the classical paradigm is incompatible with the mathematical apparatus used in quantum mechanics. The opposite will be proved here.

The basic concepts of the modern standard approach to quantum mechanics are the Hilbert space and linear operators in this space. Von Neumann formulated quantum mechanics in a mathematically clear manner on the basis of these concepts. Heisenberg's matrix mechanics and Schrödinger's wave mechanics are specific representations of the abstract Hilbert space method of von Neumann.

The apparatus of Hilbert space became the mathematical basis for the tremendous advances achieved by quantum mechanics. However, these successes also have a reverse side. A definite fetishization of Hilbert space occurred. Physicists stopped paying attention to the fact that Hilbert space is a very specific mathematical object. It turned out to be an excellent basis for calculating the mean values of the observed quantities and their probability distributions. At the same time, the idea that observables are operators in Hilbert space is by no means self-evident.

Attempts to use the apparatus of Hilbert space to describe single physical phenomena are far from being so successful. The reasoning used in this case often proves to be controversial.

So, in order to reconcile the concept of Hilbert space with the results of individual measurements, von Neumann resorts to a very dubious idea of the "inner self." The rejection of the principle of causality is also perceived with great difficulty.

The same refers to the widely accepted concept of the determining the influence of the observer on quantum mechanical processes. In this case, it is no longer very surprising that, in this respect, quantum mechanics turned out to be a unique branch of not only physics, but also science in general. Ideas are put forward about the active role of consciousness in quantum mechanics.

All this indicates that although Hilbert space is a very useful mathematical object, its basic role is not at all indisputable. The idea that Hilbert space and linear

operators should not be the primary elements of quantum theory is not new. It was this idea that became the basis of the algebraic approach to quantum theory. The entire content of this review is devoted to the development of this idea.

1. AXIOMS OF QUANTUM MECHANICS AND PHYSICAL INTUITION

The development of quantum theory produced a revolution in physics. This statement has long become a commonplace but has not ceased to be true for it. Indeed, within quantum physics, it was possible to quantitatively describe a huge number of phenomena that could not be described in the context of classical physics. A huge number of new technologies have developed on the basis of quantum physics.

However, like any revolution, the quantum revolution had not only positive, but also negative consequences. In physics, the substitution of concepts somehow imperceptibly occurred. Under the words "to explain some phenomenon" in quantum physics, "to give a mathematical description of this phenomenon" is usually meant.

This substitution has a completely understandable origin. Modern quantum physics is built as an axiomatic theory based on the mathematical axioms of von Neumann [1]. These axioms are very convenient for constructing a powerful mathematical apparatus. At the same time, their connection with our intuitive ideas is almost completely absent. As noted by Segal [2], "these axioms are technically simple, but intuitively completely unclear and seem to have arisen ad hoc. Among the majority of theoretical physicists, the opinion is firmly established that the physical intuition based on classical concepts is useless in quantum theory. Therefore, a theory can be built on the basis of a more or less arbitrary set of mathematical axioms. If only they were not internally contradictory, and their consequences well describe quantitatively a fairly wide range of experimental results. Thus, the replacement of the explanation of a physical phenomenon with its mathematical description became one of the consequences of the quantum revolution.

Current standard quantum mechanics is based on the following postulates:

(I) The state of a physical system is described by a vector $|\Psi\rangle$ of some Hilbert space or a statistical operator (density matrix) in this space.

(II) The observed $\hat{D}$ systems are described by self-adjoint operators $\hat{\mathcal{D}}$.

(III) The mean value of the observed $\hat{D}$ in the state $|\Psi\rangle$ is equal to the mathematical expectation $\langle\Psi|\hat{\mathcal{D}}|\Psi\rangle$.

Why does the Hilbert space have something to do with the state of the physical system, why does the operator $\hat{\mathcal{D}}$ correspond to the observable $\hat{D}$, and finally, why does the mean value of the observable equal $\langle\Psi|\hat{\mathcal{D}}|\Psi\rangle$? All of these questions are considered irrelevant.

In the standard approach to quantum mechanics, the slogan "Winners are not judged!" has triumphed. The great number of excellent results obtained on the basis of these postulates allows the irrelevant "why" to be ignored with a safe conscience.

Nevertheless, deep down, a worm of doubt is stirring. On the other hand, these excellent results cannot be of random nature. Or maybe statements I–III need not be accepted as primary postulates, perhaps they follow from more fundamental provisions that are more directly related to physics?

If this is so, then there is a hope to identify the conditions under which statements I–III are true, in other words, to establish the limits of applicability of the standard quantum mechanics. In turn, this can help to put an end to the debate over quantum paradoxes that excite the physical community almost since the very dawn of quantum mechanics.

Postulates I–III have another significant drawback. They break a link between quantum physics and classical physics. In the latter, states and observables are described using completely different mathematical concepts. In general, the link between classical physics and quantum physics turns out to be somewhat strange. On the one hand, classical physics is considered the limiting case of quantum physics, i.e., classical physics is a secondary theory. On the other hand, to formulate the main provisions of quantum physics, an idea of the interaction of a quantum object with a device is required, which is described using classical physics [3]. In logic, this situation is well known and is called a vicious circle. As a way out of this situation, the statement is made that classical logic does not work in quantum physics, while the special quantum logic is required there.

Thus, quantum theory also implies a revolution in logic. However, unlike physics, few significant positive results are observed here from such a revolution. In addition, the quantum logic, as a kind of sequential complete scheme, has not been created. Separate formulated statements are actually a reformulation of postulates I–III or consequences from them. No wonder that the overwhelming majority of actual practicing physicists do not refer to the statements of quantum logic but prefer to appeal directly to the postulates I–III.

There is another embarrassment in the standard formulation of quantum mechanics. Quantum mechanics is essentially a statistical theory. Therefore, it should be based on the theory of probability. Probability theory (as formulated by Kolmogorov [4]) is cur-

rently a fully formed branch of mathematics. However, it is believed that such a theory of probability is not suitable for quantum mechanics and a special quantum theory of probability is required. In other words, quantum theory still requires a revolution in mathematics. Here too, no particular positive results have been achieved. Only individual statements of this new theory of probability have been formulated, which again actually reduce to postulates (see, e.g., [5]).

Thus, a gap between quantum theory and mathematics is observed; the latter remains in the traditional mainstream of classical logic and classical probability theory. It follows from what has been said that it would be highly desirable to construct a theoretical-mathematical scheme that would be suitable for both classical physics and quantum physics. It would be very good if rules of the game in this scheme, or, as the philosophers say, the paradigm, are classical. The classical paradigm here will imply, first of all, classical formal logic, and the idea of the presence of a causal relationship between both physical phenomena and logical statements. Further, it assumes the existence of physical realities, which are carriers of the causes of physical phenomena. In addition, it assumes that probabilistic judgments obey the classical Kolmogorov probability theory.

It is usually believed that all of these provisions are incompatible with the mathematical scheme that is accepted in quantum mechanics. An attempt is made here to prove the opposite. In this case, we will not rely on postulates I–III, but we use the algebraic approach [6–8]. Within this approach, it turns out to be possible to formulate axioms, which, firstly, are more fundamental than postulates I–III, and, secondly, are intuitively much more understandable [9–11].

Here, however, a certain psychological barrier will have to be overcome. The point is that the Hilbert space apparatus has become standard in quantum mechanics. Therefore, it seems to be intuitively understandable. However, this is the habit of a certain mathematical apparatus, and not a physical intuition. In contrast to this, the apparatus of the theory of algebras is much less familiar to most physicists. Therefore, statements using the language of algebra theory seem to be more complex than statements using the language of Hilbert space theory. Although, as a rule, algebraic statements are more elementary. To help to overcome this psychological barrier, the next section will provide basic information from the theory of algebras.

2. ELEMENTS OF THE THEORY OF ALGEBRAS

Definitions and statements are borrowed from monographs [6, 12–15].

Definition 1. A set $\mathfrak{L}$ is called a complex (real) linear space if:

(a) For any complex (real) number α and any element $\hat{U} \in \mathfrak{L}$, an element $\alpha\hat{U} \in \mathfrak{L}$ is defined.

(b) For any two elements $\hat{U},\hat{V} \in \mathfrak{L}$, an element $\hat{U} + \hat{V} \in \mathfrak{L}$ is defined.

(c) Operations (a) and (b) have the usual properties of multiplication and addition, respectively.

Definition 2. A complex (real) linear space $\mathfrak{L}$ is called a complex (real) algebra $\mathfrak{A}$ if for any elements $\hat{U},\hat{V},\hat{W} \in \mathfrak{A}$, a multiplication operation is defined that has the following properties:

(a) $\hat{U}\hat{V} \in \mathfrak{A}$.

(b) $(\hat{U} + \hat{V})\hat{W} = \hat{U}\hat{W} + \hat{V}\hat{W}$, $\hat{U}(\hat{V} + \hat{W}) = \hat{U}\hat{V} + \hat{U}\hat{W}$.

(c) $\alpha(\hat{U}\hat{V}) = (\alpha\hat{U})\hat{V} = \hat{U}(\alpha\hat{V})$.

Definition 3. An algebra $\mathfrak{A}$ is called associative if $\hat{U}(\hat{V}\hat{W}) = (\hat{U}\hat{V})\hat{W}$ holds for any $\hat{U},\hat{V},\hat{W} \in \mathfrak{A}$.

Definition 4. An algebra $\mathfrak{A}$ is called commutative if $\hat{U}\hat{V} = \hat{V}\hat{U}$ holds for any $\hat{U},\hat{V} \in \mathfrak{A}$.

Examples:

(a) A set of all real continuous bounded functions of one variable is a real algebra.

(b) A set of all complex continuous bounded functions of one variable is a complex algebra.

(c) A set of bounded linear operators of the Hilbert space is a complex algebra.

(d) A set of mutually commuting bounded Hermitian linear operators of the Hilbert space is a real algebra.

(e) A set of all bounded Hermitian linear operators in the Hilbert space is not an algebra.

Definition 5. A mapping $\hat{U} \to \hat{U}^*$ of a complex algebra $\mathfrak{A}$ onto itself ($\hat{U},\hat{U}^* \in \mathfrak{A}$) is called an involution if for any complex number α and for all $\hat{U},\hat{V} \in \mathfrak{A}$, the following is true:

(a) $(\hat{U} + \hat{V})^* = \hat{U}^* + \hat{V}^*$.

(b) $(\alpha\hat{U})^* = \alpha^*\hat{U}^*$.

(c) $(\hat{U}\hat{V})^* = \hat{V}^*\hat{U}^*$.

(d) $\hat{U}^{**} = \hat{U}$.

Examples:

(a) If $\mathfrak{A}$ is the set of all complex continuous bounded functions of one variable, then the operation of complex conjugation is an involution.

(b) If $\mathfrak{A}$ is the set of all bounded linear operators of Hilbert space, then the operation of Hermitian conjugation is an involution.

Definition 6. A complex algebra equipped with an involution operation is called an involutive algebra.

Remark. In real commutative algebra, the involution operation can be defined as the identity transformation.

Definition 7. If $\hat{U}^* = \hat{U}$ ($\hat{U} \in \mathfrak{A}$), then the element $\hat{U}$ is called Hermitian.

Definition 8. An element $\hat{I} \in \mathfrak{A}$ is called a unit element of the algebra if $\hat{I}\hat{U} = \hat{U}\hat{I} = \hat{U}$ is true for any $\hat{U} \in \mathfrak{A}$.

Statement 1. Any algebra either contains the unit element or can be supplemented with an element that has the properties of the unit element.

Further, algebras will be considered in which the unit element is included.

Definition 9. An element $\hat{U}^{-1} \in \mathfrak{A}$ is called the inverse of an element $\hat{U}$ if $\hat{U}^{-1}\hat{U} = \hat{U}\hat{U}^{-1} = \hat{I}$.

Definition 10. The set of all such numbers λ, for which the element $\lambda\hat{I} - \hat{U}$ has no inverse in the algebra $\mathfrak{A}$, is called the spectrum $\sigma(\hat{U};\mathfrak{A})$ of an element $\hat{U}$ in the algebra $\mathfrak{A}$ ($\hat{U} \in \mathfrak{A}$).

Definition 11. The spectral radius of an element $\hat{U}$ is called the number $r = \sup\{|\lambda|;\ \lambda \in \sigma(\hat{U};\mathfrak{A})\}$.

Definition 12. A subset $\mathfrak{O}$ of an algebra $\mathfrak{A}$ is called a subalgebra if $\mathfrak{O}$ is an algebra with the same definition of the addition and multiplication operations.

Definition 13. Let $\mathfrak{O}$ be a real commutative subalgebra of algebra $\mathfrak{A}$. The subalgebra $\mathfrak{O}$ is called a maximal real commutative subalgebra if it is not a subalgebra of any other similar subalgebra of algebra $\mathfrak{A}$.

In the general case, the spectrum $\sigma(\hat{U};\mathfrak{O})$ of an element $\hat{U}$ in the subalgebra $\mathfrak{O}$ may not coincide with the spectrum $\sigma(\hat{U};\mathfrak{A})$ of the same element in the algebra $\mathfrak{A}$. However, the following assertion is correct.

Statement 2. If $\mathfrak{O}$ is a maximal real commutative subalgebra of the algebra $\mathfrak{A}$ and $\hat{U} \in \mathfrak{O}$, then $\sigma(\hat{U};\mathfrak{O}) = \sigma(\hat{U};\mathfrak{A})$.

Definition 14. A set $\mathfrak{I}_l$ of elements of an algebra is called its left ideal if:

(a) $\mathfrak{I}_l \neq \mathfrak{A}$.

(b) $\mathfrak{I}_l$ is the linear subsplace of .

(c) From $\hat{U} \in \mathfrak{I}_l$, $\hat{V} \in \mathfrak{A}$ it follows that $\hat{V}\hat{U} \in \mathfrak{I}_l$.

A right ideal is defined similarly. A set of elements of the algebra $\mathfrak{A}$ that is simultaneously both a left and a right ideal is called a two-sided ideal.

Definition 15. Let $\mathfrak{I}$ be a two-sided ideal of the algebra $\mathfrak{A}$. Elements $\hat{U},\hat{V}$ are called to be equivalent with respect to $\mathfrak{I}$ if $\hat{U} - \hat{V} \in \mathfrak{I}$. The set of all mutually equivalent elements is called the residue class of the algebra $\mathfrak{A}$.

Definition 16. The set of all residue classes of the algebra $\mathfrak{A}$ is called the quotient algebra and is denoted $\mathfrak{A}/\mathfrak{I}$.

Statement 3. If in the set $\mathfrak{A}/\mathfrak{I}$, the operations of the multiplication of classes by a number, the addition of classes and their multiplication are introduced as the corresponding actions on representatives of these classes, then the set $\mathfrak{A}/\mathfrak{I}$ will acquire the structure of an algebra, i.e., a quotient algebra is an algebra.

Definition 17. An involutive algebra is called normalized if in it for each element $\hat{U}$ a norm $\|\hat{U}\|$ is defined which is a nonnegative number that satisfies the conditions:

(a) $\|\alpha\hat{U}\| = |\alpha|\|\hat{U}\|$;

(b) $\|\hat{U} + \hat{V}\| \le \|\hat{U}\| + \|\hat{V}\|$;

(c) $\|\hat{U}^*\| = \|\hat{U}\|$;

(d) $\|\hat{U}\hat{V}\| \le \|\hat{U}\|\|\hat{V}\|$;

(e) if $\|\hat{U}\| = 0$, then $\hat{U} = 0$.

Definition 18. The value $\|\hat{U}\|$ for which all the conditions of the previous definition are satisfied, except for item (e), is called a seminorm.

Definition 19. A sequence $\{\hat{U}_n\}$ of elements of a normalized space is called fundamental if for any $\varepsilon > 0$ a number $N(\varepsilon)$ can be specified such that for $n > N(\varepsilon)$ and $m > N(\varepsilon)$, it is true that $\|\hat{U}_n - \hat{U}_m\| < \varepsilon$.

Definition 20. A normalized space, in which every fundamental sequence converges in norm to some element of this space, is called complete.

Definition 21. A complete normalized space is called a Banach space.

Statement 4. Any normalized space can be completed to a Banach space.

Definition 22. An involutive associative algebra, which is a Banach space (Banach algebra), in which the norm satisfies the additional condition $\|\hat{U}^*\hat{U}\| = \|\hat{U}\|^2$, is called a C*-algebra.

Definition 23. A mapping $\hat{U} \to \hat{U}'$ from an involutive algebra $\mathfrak{A}$ ($\hat{U} \in \mathfrak{A}$) to an involutive algebra $\mathfrak{A}'$ ($\hat{U}' \in \mathfrak{A}'$) is called a homomorphism of the algebra $\mathfrak{A}$ into the algebra $\mathfrak{A}'$, if from $\hat{U} \to \hat{U}'$, $\hat{V} \to \hat{V}'$, it follows $\hat{U}^* \to \hat{U}'^*$, $\alpha\hat{U} \to \alpha\hat{U}'$, $\hat{U} + \hat{V} \to \hat{U}' + \hat{V}'$, $\hat{U}\hat{V} \to \hat{U}'\hat{V}'$.

Under a homomorphism, one element of the algebra $\mathfrak{A}'$ can correspond to several elements of the algebra $\mathfrak{A}$.

Definition 24. If a homomorphism is a one-to-one mapping, then it is called an isomorphism.

Definition 25. An isomorphic mapping of an algebra onto itself is called an automorphism.

Definition 26. A homomorphism of a commutative associative real (complex) algebra $\mathfrak{A}$ into the set of real (complex) numbers is called a character of this algebra.

Definition 27. A homomorphism of an algebra $\mathfrak{A}$ into a set of linear operators of some Hilbert space $\mathfrak{H}$ is called a representation of this algebra.

Definition 28. A mapping from a normalized algebra $\mathfrak{A}$ to a normalized algebra $\mathfrak{A}'$ is called isometric if $\|\hat{U}\| \to \|\hat{U}'\|$ follows from $\hat{U} \to \hat{U}'$.

Definition 29. The mapping $\hat{U} \to \varphi(\hat{U})$ of the algebra $\mathfrak{A}$ into the set of complex numbers is called a linear functional if $\varphi(\alpha\hat{U}) = \alpha\varphi(\hat{U})$, $\varphi(\hat{U} + \hat{V}) = \varphi(\hat{U}) + \varphi(\hat{V})$. Here, $\hat{U}, \hat{V} \in \mathfrak{A}$, while α and $\varphi(\hat{U})$ are complex numbers.

Definition 30. A linear functional φ on an involutive algebra is called positive if $\varphi(\hat{U}\hat{U}^*) \geq 0$ for each $\hat{U} \in \mathfrak{A}$.

Statement 5. If $\varphi(\hat{U})$ is a positive functional, then

(a) $\varphi(\hat{U}^*) = \varphi^*(\hat{U})$,

(b) $|\varphi(\hat{U}^*\hat{V})|^2 \leq \varphi(\hat{U}^*\hat{U})\varphi(\hat{V}^*\hat{V})$.

Statement 6. A positive functional on a Banach algebra is continuous.

Statement 7. If $\varphi(\hat{U})$ ($\hat{U} \in \mathfrak{A}$) is a character of a commutative associative algebra $\mathfrak{A}$, then

(a) $\varphi(0) = 0$,

(b) $\varphi(\hat{I}) = 1$,

(c) $\varphi(\hat{U}\hat{U}^*) \geq 0$.

Thus, a character is the positive functional on the algebra $\mathfrak{A}$.

Statement 8. If, in addition, the algebra $\mathfrak{A}$ is a Banach one, while $\{\varphi(\hat{U})\}$ is the set of all its characters, then

(a) $\lambda = \varphi(\hat{U}) \in \sigma(\hat{U}; \mathfrak{A})$;

(b) If $\lambda \in \sigma(\hat{U}; \mathfrak{A})$, then $\lambda = \varphi(\hat{U})$ for some $\varphi(\hat{U}) \in \{\varphi(\hat{U})\}$.

Definition 31. An element $\hat{p}$ of an algebra $\mathfrak{A}$ is called a projector if $\hat{p}^* = \hat{p}$, $\hat{p}^2 = \hat{p}$.

Definition 32. The projector $\hat{p}_\lambda \neq 0$ is called minimal, if from $\hat{p}_\lambda\hat{p}_\mu = \hat{p}_\mu\hat{p}_\lambda = \hat{p}_\mu$ it follows that either $\hat{p}_\mu = 0$, or $\hat{p}_\mu = \hat{p}_\lambda$.

Statement 9. If $\mathfrak{A}$ is an algebra of linear bounded operators in the Hilbert space $\mathfrak{H}$, then a minimal projector is a projector on the one-dimensional subspace of the space $\mathfrak{H}$.

Definition 33. We will say that a sequence $\{\hat{U}_n\}$ of elements of the algebra $\mathfrak{A}$ converges in the weak topology to an element $\hat{U}$ if for any linear bounded positive functional φ, it follows that $\varphi(\hat{U}_n) \to \varphi(\hat{U})$.

Definition 34. A set G of elements of a Banach algebra $\mathfrak{A}$ is called a system of generators of this algebra if the smallest closed subalgebra containing G coincides with $\mathfrak{A}$.

Definition 35. The Boolean algebra of a set Ω is a collection of subsets of the set Ω, in which the following algebraic operations are defined:

(a) The operation of logical addition—the union of subsets.

(b) The operation of logical multiplication—the intersection of subsets.

(c) The operation of logical negation—the complement of a subset to a set Ω.

Definition 36. A Boolean algebra is called closed under some algebraic operation if this operation results in an element of the initial algebra.

Definition 37. A Boolean algebra is called a σ-algebra if:

(a) It contains the set Ω itself and the empty set $\varnothing$.

(b) It contains the complement to Ω of each subset included in the algebra.

(c) It is closed under a countable number of unions and intersections of subsets.

Definition 38. The set Ω, in which a definite σ-algebra is chosen, is called a measurable space.

In the future, references to definitions and statements will be made according to the following template: (D.35.b)—Definition 35, item (b); (S.7.c)—Statement 7, item (c).

OBSERVABLES, MEASUREMENTS, STATES

Let us now turn to physical problems.

When studying physical systems, a basic concept is the "observable." Heuristically, the observable is such an attribute of the physical system under study, for which a numerical value can be obtained using a definite measurement procedure.

Remark. Further, we will assume that a definite system of units is fixed, and therefore all observables can be considered dimensionless.

The concept of an observable is basic in both classical and quantum physics. However, traditionally, in the mathematical apparatus of classical and quantum

physics, an observable corresponds to different mathematical objects. Let us try to conduct the unification. To this end, it is necessary to single out the really significant mathematical characteristics of the observables, separating them from the characteristics that are usually attributed to the observables for the sake of convenience in constructing a mathematical apparatus.

Remark. Often, for the system under study, the values of some observables are known in advance and remain unchanged. For example, when studying the interaction of electrons with photons, the masses of electrons and photons and the charge of an electron are known a priori. It is convenient to exclude such quantities from a set of observables and consider their values as parameters included in the definition of the physical system under study.

In the measurement process, the physical system is affected by the measuring device. By the nature of this impact, measurements can be divided into two types: reproducible and nonreproducible. Reproducible measurements are characterized by the fact that, despite the perturbation that the system experiences during each measurement, a repeated measurement of the same observable by the same or a different device gives the same result. It is assumed that in the interval between measurements the system was not subjected to external influence, and we can interpret the change in the values of observables due to the free evolution.

Of particular interest is the issue of reproducibility when we measure several observables for the same physical system. Suppose we, e.g., measure the observable $\hat{B}$, then the observable $\hat{C}$, then the observable $\hat{B}$ again (perhaps with a device different from the original one) and, finally, the observable $\hat{C}$. If the results of repeated measurements for each observable coincide with the results of primary measurements, then we will call these measurements compatible. If for the observables $\hat{B}$ and $\hat{C}$ there are devices that allow compatible measurements to be made, then we will call these observables compatible or simultaneously measurable.

Experience shows that all observables of classical physical systems are compatible. In the quantum case, in contrast, there are both compatible and incompatible observables.

In the standard formalism of quantum mechanics, this fact is qualified as an integral part of the "principle of complementarity" [16]. We will consider it simply as evidence that to measure two incompatible observables, the instruments are required which are incompatible with each other [17].

Let us denote a set of all observables by $\mathfrak{N}_+$, and let its maximum subset of compatible observables be denoted by $\mathfrak{O}_\xi$. An index ξ distinguishes one maximum subset of compatible observables from another. In turn, a set of index values ξ will be denoted by Ξ. It is clear that for a classical system the set Ξ consists of a single element. For a quantum system, this set comprises more than one element. Later we will see that in this case the set Ξ is infinite and even has the cardinality of the continuum. The same observable can simultaneously belong to different subsets $\mathfrak{O}_\xi$.

Experience shows that for any two compatible observables $\hat{B}$ and $\hat{C}$ there is a third observable $\hat{D}$, which has the following properties. First, it is compatible with both $\hat{B}$ and $\hat{C}$. Second, the results of simultaneous measurements of the observables $\hat{B}$, $\hat{C}$, and $\hat{D}$ (for one physical system) satisfy the relation

$$B + C = D. \tag{1}$$

Here and below, the measurement results are denoted by the same symbols as the observables, but without a "hat."

In reality, a simultaneity is not very significant. It is enough that the measurements of these observables are compatible. However, in what follows, for brevity in such a situation, we will say that the observables are measured simultaneously.

Relation (1) is always satisfied, regardless of specific measurement results. This allows us to consider that the observables in themselves are linked by the similar relation

$$\hat{B} + \hat{C} = \hat{D}.$$

Hence, it is possible to equip the set $\mathfrak{O}_\xi$ with the operation of addition. The operations of multiplication of elements and multiplication by a real number are introduced in a similar way. The experience shows that each subset $\mathfrak{O}_\xi$ has properties of a real associative commutative algebra. Thus, the characteristic mathematical property of observables is that they can be considered as elements of some algebra. So far, we have justified this statement only for compatible observables. We will see later that it can be extended to incompatible observables as well.

If for a fixed physical system, using compatible measurements, we compare each observable with the measurement result

$$\hat{B} \to B = \varphi_\xi(\hat{B}),$$

then we define the functional on the algebra $\mathfrak{O}_\xi$. By virtue of the definition of algebraic operations in $\mathfrak{Q}_\xi$, this functional will be one of the characters of the algebra $\mathfrak{O}_\xi$ (see (D.26)).

In any real measurement, for any observable, the finite value is always obtained. This fact can be reversed and only those observables $\hat{B}$ are considered physical (i.e., fixed in a real experiment) for which

$$\sup_\xi \sup_{\varphi_\xi} \left|\varphi_\xi(\hat{B})\right| < \infty. \tag{2}$$

In what follows, we will see that a boundedness of functionals $\varphi_\xi(\cdot)$ is not an insurmountable obstacle to consideration in the theory of observables that are not bounded. In the standard formalism of quantum mechanics, these observables are often considered.

As a result of the previous reasoning, we formulate four postulates.

Postulate 1. The set $\mathfrak{O}_\xi$ of compatible observables can be equipped with the structure of a real associative commutative algebra. Conversely, if the observables belong to the same real associative commutative algebra, then they are compatible.

Postulate 2. For a classical system, all observables are compatible.

Postulate 3. The results of simultaneous measurement of observables belonging to the algebra $\mathfrak{O}_\xi$ are described by a real bounded (in the sense of inequality (2)) functional $\varphi_\xi(\cdot)$, which is a character of the algebra $\mathfrak{O}_\xi$.

Postulate 4. Observables of a physical system are Hermitian elements of some algebra $\mathfrak{A}$.

With each region $\mathbb{O}$ of the four-dimensional spacetime $\mathfrak{M}$, we associate a set of observables for which numerical values can be obtained by making measurements in the region $\mathbb{O}$. These observables are called (see [6] and [7]) local (localized in the domain $\mathbb{O}$). Strictly speaking, all observables should be considered local, but global (quasi-local) observables as a rule are used in theory considering definite limits of sequences of local observables.

What is a physical system is more or less intuitively clear. However, below it will be more convenient for us to go to the formal level and by the term "physical system" to mean the totality of two sets: $\mathbb{O}$ (system localization region in the space $\mathfrak{M}$) and $\mathfrak{A}_+(\mathbb{O})$ (the set of its observables). This will make it possible to transfer naturally from the physical system to its physical subsystem: $\mathbb{O}, \mathfrak{A}_+(\mathbb{O}) \to \mathbb{O}', \mathfrak{A}'_+(\mathbb{O}')$, where $\mathbb{O}' \subset \mathbb{O}$ and $\mathfrak{A}'_+(\mathbb{O}') \subset \mathfrak{A}_+(\mathbb{O})$. In this case, we will not assume that the subsystems are necessarily isolated from each other. In the general case, systems and subsystems can be open.

In the algebraic approach, points of the space $\mathfrak{M}$ are not considered as observables. Accordingly, the space $\mathfrak{M}$ is not assumed to be linear. This is very essential for the consistency between the quantization procedure and the general theory of relativity. The point is that quantum observables are elements of a linear space, while in general relativity the space $\mathfrak{M}$is nonlinear.

In the algebraic approach, it is accepted:

Postulate 5. The space $\mathfrak{M}$ is a smooth manifold.

A characteristic property of a smooth manifold is that in the vicinity of each point of the manifold, a local Euclidean coordinate system can be introduced.

Let us now discuss the "state of a physical system" concept. We will start the discussion with the classical system. In this case, the state of the system is understood as some attribute of the physical system, which unambiguously determines the result of measuring all observables. Since the time of Newton, the principle of locality has been adopted in physics, which, in particular, assumes that the state of a localized physical system is determined by some internal characteristics of this system and by characteristics of the external field acting on the system which belong to the localization region of the physical system. According to the classical paradigm, there must be some local reality that determines this state.

Mathematically, a state is usually specified using a point in the phase space. It is assumed that the system dynamics is given. Within the approach developed here, this way of specifying the state is not convenient. First, it is difficult to transfer it to the quantum case. Second, it is rigidly connected with a definite way of specifying the dynamics. In particular, it assumes the introduction of canonically conjugate variables. However, it is easy to understand that this way of specifying the state is just a specific version of defining some real functional on the algebra of observables, which is the character of this algebra. If one does not get involved in any specific option, then the state of a classical system can be defined as a character of the algebra of observables of this system.

Let us now turn to the quantum case. The set $\mathfrak{A}_+$ of quantum observables cannot be equipped with the structure of an associative commutative algebra. Therefore, a direct transfer of the definition of a state from a classical system to a quantum one is impossible. However, a set $\mathfrak{A}_+$ can be considered as a collection of subsets $\mathfrak{O}_\xi(\xi \in \Xi)$, each of which has the structure of a real associative commutative algebra. We can consider each subset $\mathfrak{O}_\xi$ as a set of observables corresponding to the classical subsystem of a quantum system. Of course, this classical subsystem will not be isolated from the rest of the quantum system. However, isolation is not a prerequisite for separating a subsystem.

As before, we can describe the state of each such classical subsystem mathematically with the help of a functional $\varphi_\xi(\cdot)$ that is given on an algebra $\mathfrak{O}_\xi$ and is a character of this algebra. In this regard, we introduce a new concept—an elementary state.

Definition 39. Let us call as an elementary state of a physical system the collection $\varphi = [\varphi_\xi]$ $(\xi \in \Xi)$ of functionals φ_ξ, each of which is a definite character of the corresponding algebra $\mathfrak{O}_\xi$. In turn, the sets $\mathfrak{O}_\xi$ are the maximal subsets of the set $\mathfrak{A}_+$ which have a structure of a real associative commutative algebra.

The term "state" is also justified in the quantum case. Indeed, in each individual measurement, and even in a set of compatible measurements, we can at most find the values of some collection of compatible observables. All these observables belong to a definite single algebra $\mathfrak{O}_\xi$. Therefore, their values are determined by the corresponding functional φ_ξ. By specification of an elementary state φ, all these functionals are fixed. Hence, the results of all possible measurements are fixed. In the standard apparatus of quantum mechanics, another mathematical object is called a state. Therefore, for $\varphi = [\varphi_\xi]$ we use the term "elementary state." The standard formulation of quantum mechanics does not use this term. In the case of a classical system, the concepts of elementary state and state are identical.

We conclude this stage of reasoning with the following postulate.

Postulate 6. The result of each individual experiment for measuring the observables of a physical system is determined by the elementary state of this system.

In the case of a classical system, this postulate does not give anything new. On the contrary, in the quantum case this postulate is completely unusual. Moreover, there is much "evidence" that nothing of the sort can happen. In our approach, this postulate will occupy a central place.

Note that no additional assumptions about the properties of the functionals φ_ξ are made. In particular, it is not assumed that

$$\varphi_\xi(\hat{B}) = \varphi_{\xi'}(\hat{B}), \quad \text{if} \quad \hat{B} \in \mathfrak{O}_\xi \cap \mathfrak{O}_{\xi'}. \tag{3}$$

Definitely, Eq. (3) can hold for some functionals φ. We will say that the functional φ is stable on the observable $\hat{B}$ if Eq. (3) is satisfied for all allowed ξ and ξ'.

On the other hand, it seems very natural to require that Eq. (3) be valid. In this regard, the possibility of violating this equality needs a special comment.

Experimentally, the values of the observables manifest themselves as the response of the measuring device to the action of the system under study. In principle, the responses of different instruments to the same action may be different.

The simplest classical example is the measurement of a speed. Instruments moving in different ways for the same system under study will record different speed values. In similar cases, it is usually said that the value of many observables may depend on the chosen coordinate system. Physically, the selection of a definite coordinate system is a choice of a definite collection of measuring instruments. In the general case, the instruments can differ from each other not only in their spatiotemporal characteristics.

From the viewpoint of an experimenter, the dependence of the measurement result on the characteristics of the measuring instrument is highly undesirable. Therefore, the experimenter seeks to unify the response (indications) of various instruments. The instrument calibration procedure serves for this unification.

Schematically, the procedure is as follows. As a template, a measuring instrument is taken, which performs a reproducible measurement of some observable $\hat{B}$. With the help of this device, the observable of some test physical system is measured. A definite value B is obtained. By definition of a reproducible measurement, a repeated measurement of the same observable by an instrument to be calibrated should give the same value. Only an instrument that repeatedly withstands this test deserves the name "measuring instrument." The calibration is destined to exclude the dependence of the measurement result on the uncontrolled effect of the instrument, in particular, on the uncontrolled state of the instrument.

However, for the single parameter, the value of which can be determined by the instrument, the calibration is powerless. This parameter is $\xi \in \Xi$. Let us explain what relation the parameter ξ bears to the measuring instrument. Each device, depending on its design (settings), is intended either to measure one observable $\hat{B}$ or to simultaneously measure a group of observables. This observable (a group of observables) belongs to some algebra $\mathfrak{O}_\xi$. We will assume that the instrument belongs to the type ξ, if, first, it is designed to measure the observable (observables) from the subset $\mathfrak{O}_\xi$, and second, the result of measuring the observable $\hat{B} \in \mathfrak{O}_\xi$ (a group of compatible observables) is $B_\xi = \varphi_\xi(\hat{B})$ (a group of corresponding results).

It is impossible to find out with the help of calibration whether the measurement result depends on the parameter ξ. Indeed, the first stage of calibration is a reproducible measurement, after which the state of the test system becomes stable on the observed $\hat{B}$. Therefore, the result of the subsequent measurement of this observable in any case will not depend on the parameter ξ. In any other way of checking Eq. (3), we must subject the same system under test to two measurements: once with an instrument of type ξ, another time with type ξ' ($\xi \neq \xi'$). These are two different instruments, so we cannot perform both measurements at the same time.

Let us first make a measurement using an instrument of type ξ and get the result $B_\xi = \varphi_\xi(\hat{B})$. If this measurement is irreproducible, then after it, the elementary state φ of the system under test will change in an uncontrolled way. Therefore, the result of the second measurement (with a device of type ξ') will in no way be related to the result of the first measurement. If the first measurement is reproducible, then after it, the

elementary state φ will be replaced by φ'. Since, after a reproducible measurement, the elementary state becomes stable on the corresponding observable, the relation $\varphi'_{\xi'}(\hat{B}) = \varphi_\xi(\hat{B})$ must hold for the state φ' regardless of whether equality (3) is satisfied or not. Thus, in any case, we cannot check equality (3).

Of course, the above reasoning does not guarantee that a dependence of the measurement results on the parameter ξ exists. It only demonstrates a possibility of this dependence. Therefore, any conclusions that assume the validity of equality (3) do not have a probative value. It should be emphasized that the classification of instruments by types ξ is a classification according to the nature of interaction between the instrument and the system under study. Therefore, it is determined not only by properties of the instrument, but also by the system under study (by the set $\mathfrak{A}_+$, by algebras $\mathfrak{O}_\xi$).

The measurement result dependence on the type of the device can be considered as an implementation and concretization of Bohr's concepts [18] of the measurement result dependence on a general context of the experiment. At the same time, the dependence variant proposed here contradicts neither the principle of causality nor the notion of local reality existence. However, local reality is not the definite value of each observable for the physical system under consideration, but a definite cause that produces some reaction of a measuring instrument of the definite type.

In the general case, a value of observable cannot be considered as an objective characteristic of the system under study. It may depend on characteristics of the measuring instrument. On the other hand, nothing prevents us from considering the elementary state as this characteristic (physical reality).

While measuring by a classical instrument, the elementary state of a quantum system cannot be fixed unambiguously. Indeed, since instruments designed to measure incompatible observables are incompatible, in one experiment we can measure observables that belong to the single algebra $\mathfrak{O}_\xi$. As a result, we will determine only the values of the functional φ_ξ. For the rest, the elementary state φ will remain undefined. A repeated measurement using a different type of instrument will give new information but will uncontrollably perturb the elementary state that has arisen after the first measurement. Therefore, the information obtained in the first measurement will become useless.

Figuratively speaking, an elementary state is a holographic image of the system under study. With the help of classical instruments, we can look at it from only one side and get a flat image. In this case, the measurement will perturb the system and change its original holographic image. Therefore, if we subsequently look at the system under study from the other side, we will see a flat part of the new holographic image. Thus, we will never be able to see an integral holographic image.

Using the concept of an elementary state, one can take a fresh look at Everett's idea [19] about the existence of many parallel worlds. The original idea was that quantum systems are located simultaneously in many parallel worlds, while a classical observer happens to be in one of these worlds. Therefore, he sees the version of the quantum system that is presented in his world. The idea as such looks fantastic.

In contrast, the notion that an elementary state is analogous to a holographic picture looks quite plausible. On the other hand, the idea of an elementary state leads to approximately the same consequences as the idea of the existence of many worlds. However, there is a significant difference. In Everett's picture, the randomness inherent in a quantum measurement is associated with the randomness of the entry of an observer into one or another world. In the picture we are considering, a specific measurement result does not unambiguously fix the elementary state of a physical system. Therefore, we actually do not give a description of an individual system but describe some characteristics common to the whole ensemble of quantum systems. In this connection, it is useful to introduce a notion of φ_η-equivalence. Two elementary states $\varphi = [\varphi_\xi]$ and $\varphi' = [\varphi'_\xi]$ are called φ_η-equivalent if $\varphi_\eta = \varphi'_\eta$ holds in them. The rest φ_ξ and φ'_ξ can be in any relationships. For the class of φ_η-equivalent elementary states, we introduce the notation $\{\varphi\}_{\varphi_\eta}$. The most that can be learned about an elementary state φ is that it belongs to some equivalence class: $\varphi \in \{\varphi\}_{\varphi_\eta}$.

After that, we can make the following predictions. If instruments of the η-type are used, then for the observable $\hat{B} \in \mathfrak{O}_\eta$, the result $B = \varphi_\eta(\hat{B})$ will be obtained. If the elementary state φ is stable on observables $\hat{B} \in \mathfrak{O}_\eta$, then this result will be obtained when using instruments of any type ξ. Of course, ξ should be such that $\hat{B} \in \mathfrak{O}_\xi$. Otherwise, the instrument simply will not be designed for measuring this observable. Nothing definite can be said about the results of measuring the observables $\hat{B} \neq \mathfrak{O}_\eta$, since these results will be different for different elementary states $\varphi \in \{\varphi\}_{\varphi_\eta}$. The most we can hope for is that we will be able to predict the probability of obtaining one or another result.

In the standard mathematical apparatus of quantum mechanics, a quantum state, fixed by the definite values of a complete set of mutually commuting observables, possesses the mentioned physical properties. This allows us to give the following definition of a quantum state in the proposed approach.

Definition 40. A quantum state Ψ_{φ_η} is a class $\{\varphi\}_{\varphi_\eta}$ of φ_η-equivalent elementary states that are stable on observables $\hat{B} \in \mathfrak{O}_\eta$.

In fact, this definition of a quantum state is convenient only for systems in which there are no identical particles. The fact is that the measuring instrument cannot determine which of the identical particles has hit it. Therefore, it is convenient to somewhat generalize the equivalence concept. An elementary state φ is said to be weakly φ_ξ-equivalent to an elementary state φ' if the restriction φ_ξ of the elementary state φ to the algebra $\mathfrak{O}_\xi$ coincides with the restriction $\varphi'_{\xi'}$ of the elementary state φ' to the algebra $\mathfrak{O}_{\xi'}$. It is assumed here that the algebra $\mathfrak{O}_{\xi'}$ is obtained from the algebra $\mathfrak{O}_\xi$ by replacing the observables of one of the identical particles by the corresponding observables of the other particle. For systems with identical particles in the quantum state definition, an equivalence should be replaced by a weak equivalence. In what follows, we will assume that, if necessary, this replacement has been made.

If we can consider an elementary state as an objective attribute of a separate physical system, then a quantum state (an equivalence class of elementary states) should be considered an attribute of an ensemble of physical systems. This ensemble will be called a quantum ensemble. In principle, the equivalence class of elementary states can also be associated with a separate physical system. To this end, it suffices to consider that a separate elementary state is an attribute of this system at a fixed point of time, and to compare different elementary states to different points of time.

In this case, time will play a role different from that which it plays in Newtonian mechanics. In the latter, time marks the sequence of states that the system under study goes through in the course of its evolution. Moreover, in this sequence, the time arrow is clearly traced: the cause always precedes the consequence. In the quantum state mentioned in the previous paragraph, time marks the various elementary states that make up the quantum ensemble. All probabilistic characteristics of this ensemble may depend on the ensemble form but cannot depend on the sequence in which the markers of individual elements of the ensemble are located. This can explain why, in contrast to Newton's equations, the equations describing the dynamics of quantum systems are invariant under time reversal.

4. PROBABILITY THEORY AND QUANTUM ENSEMBLE

Most of the quantum theory predictions are probabilistic in nature. Therefore, the quantum theory must be based on the theory of probability. At present, Kolmogorov probability theory is the most mathematically developed [4]. It is generally believed that a special quantum probability theory is required for quantum systems. Here we will defend the thesis that the classical Kolmogorov probability theory is quite sufficient in the quantum case, we only need to consider the peculiarity of quantum measurements [20].

Kolmogorov's theory of probability (see, e.g., [4], [21]) is based on the so-called probability space $(\Omega, \mathcal{F}, P)$.

The first component Ω is a set (space) of elementary events. The physical meaning of elementary events is not specifically stipulated, but it is believed that they are mutually exclusive, and in each test one and only one elementary event occurs. In our case, an elementary state φ can serve as an elementary event.

There is no candidate for this role in the standard mathematical apparatus of quantum mechanics. A quantum state obviously cannot play the role of an elementary event, since two nonorthogonal quantum states are not mutually exclusive. Therefore, indeed, while remaining within the standard formalism of quantum mechanics, one cannot use Kolmogorov's probability theory. The situation is similar with classical formal logic.

In addition to the elementary event, the concept of "random event" or simply "event" is also introduced in the probability theory. Each event F is identified with some subset of the set Ω. It is considered that an event occurred if one of the elementary events belonging to this subset ($\varphi \in F$) was implemented. It is assumed that in each trial we can determine whether an event has occurred or not. This is not required for elementary events.

Collections of subsets of a set Ω (including the set Ω itself and the empty set $\varnothing$) are provided with a structure of Boolean algebras. Accordingly, the second component of the probability space is some Boolean σ-algebra $\mathcal{F}$. Thus, the probability space is equipped with the structure of a measurable space.

Finally, the third component of the probability space is the probability measure P. This is a mapping of a set $\mathcal{F}$ into a set of real numbers (each $F \in \mathcal{F}$ is put in correspondence with a number $P(F)$) that satisfies the following conditions: (a) $0 \le P(F) \le 1$ for all $F \in \mathcal{F}$, $P(\Omega) = 1$; (b) $P\left(\sum_j F_j\right) = \sum_j P(F_j)$ for any countable collection of disjoint subsets $F_j \in \mathcal{F}$.

Note that the probability measure is defined only for events included in the algebra $\mathcal{F}$. For elementary events, the probability, generally speaking, is not defined.

Let us illustrate the last statement using a simple example. Let the space of elementary events be a set of rational numbers lying between zero and one. A test is the guessing of one of these numbers, guessed by the

interlocutor. Obviously, the probability of guessing any of the numbers cannot have any numerical value other than zero. However, it also cannot be equal to zero. Indeed, the probability that the hidden number lies between zero and one is equal to one. The set of rational numbers is countable. Therefore, according to the properties of the probability measure, the unit would have to be equal to a countable sum of zeros. No contradiction arises if we choose as $\mathcal{F}$ the set of all intervals (and their unions) and assign to each interval a probability equal to its length.

Thus, measurability is a very essential property of a probability space. Further, we will see that in the quantum case the role of measurability is even more important. In addition, the measurability property carries not only a mathematical, but also a very significant physical load.

Let us now discuss the specifics of applying the basic principles of probability theory to quantum measurements. Most quantum measurements are associated with finding the probability distributions of particular observed quantities. Using the definite measuring equipment, we can obtain this distribution for some set of compatible observables. From the viewpoint of probability theory, by choosing the definite measuring equipment, we choose the definite σ-algebra $\mathcal{F}$.

For greater clarity, the further discussion will be carried out on a specific example. Let our system under study be a particle that can move in a fixed plane. Suppose that first we want to find the probability distribution of the X-th coordinate of this particle. To do this, we must break the plane of motion into strips perpendicular to the axis X. The width of the strips must be consistent with the sensitivity of the measuring instrument used. These strips will play a role of elements F_i^X of the σ-algebra $\mathcal{F}_X$. With the help of a measuring device, we will be able to determine the probability of a particle falling into the definite strip. A similar experiment can be carried out to find the probability distribution along the axis Y. In this case, the strips will be denoted by F_j^Y while the σ-algebra, by $\mathcal{F}_Y$.

We can conduct a more detailed study and find the probability distribution of particle coordinates along both axes at the same time. To this end, it is necessary to divide the plane of motion into rectangles, obtained by the intersections of various strips: $F_{ij}^{XY} = F_i^X \cap F_j^Y$. The rectangles F_{ij}^{XY} will be elements of the σ-algebra $\mathcal{F}_{XY}$. An algebra $\mathcal{F}_{XY}$ is said to be generated by the algebras $\mathcal{F}_X$ and $\mathcal{F}_Y$. So far, there is no difference in the classical and quantum considerations.

Now we want to know the probability distribution not only for coordinates, but also for momenta. If we are interested in probability distributions over coordinates and separately over momenta, then the experiment can be set according to the previous scheme. Only the strips will have to be traced in the plane of momenta.

However, the situation will change radically if we want to find a probability distribution that is compatible with respect to the Xth coordinate and the K_x-th projection of the momentum. Formally, purely mathematically (see, e.g., [21]), we can construct a σ-algebra $\mathcal{F}_{XK_x}$, which is generated by the algebras $\mathcal{F}_X$ and $\mathcal{F}_{K_x}$. The rectangles (and their possible combinations) in the two-dimensional plane (XK_x) of the four-dimensional phase space will be the elements of this algebra. In the classical case, we can organize an experiment to find the probability of a particle hitting this rectangle. However, in the quantum case, this experiment is impossible in principle because the measuring instruments designed to determine the Xth coordinate and the K_x-th projection of the momentum are incompatible. This means that no probability measure can be assigned to this rectangle. In other words, for an event that implies a particle falling into this rectangle, there is no concept of "probability" at all.

The general conclusion from the above example should be formulated as follows. Not every mathematically possible (and admissible in the classical case) σ-algebra is admissible as an σ-algebra of a probability space in the quantum case.

Thus, an element of a measurable space in an experiment $(\Omega, \mathcal{F})$ is matched by a pair: the object under study and the definite type of measuring equipment, which makes it possible to record an event corresponding to a definite collection of compatible observed quantities, i.e., belonging to the same algebra $\mathfrak{O}_\xi$. Therefore, the σ-algebra $\mathcal{F}$ can also be indexed by the parameter ξ: $\mathcal{F} = \mathcal{F}_\xi$.

A specific feature of a quantum experiment requires a careful consideration in defining one of the basic concepts of probability theory—a real random variable. Typically, a real random variable is defined as a mapping from the space Ω of elementary events to the extended real line $\mathcal{R} = [-\infty, +\infty]$. However, this definition does not consider the specific features of a quantum experiment, in which the result may depend on the type of a measuring device. Therefore, we will adopt the following more detailed definition.

Definition 41. A real random variable is a mapping of the measurable space $(\Omega, \mathcal{F}_\xi)$ of elementary events into the extended real line.

When applied to the observable $\hat{B}$, it will look as follows

$$\varphi \xrightarrow{\hat{B}} B_\xi(\varphi) \equiv \varphi_\xi(\hat{B}) \in \mathcal{R}.$$

Let us call as a quantum ensemble a set of identical (i.e., described by a single set $\mathfrak{A}_+$ of observables and a fixed set $\{\mathfrak{O}_\xi\}$ of commutative algebras $\mathfrak{O}_\xi$ $(\xi \in \Xi)$) physical systems that are in a definite quantum state. A mixture of quantum ensembles, into which each of these ensembles enters with a multiplicity C_i $(C_i \geq 0, \sum C_i < \infty)$ is called a mixed quantum ensemble. An experiment testifies in favor of the following postulate.

Postulate 7. A quantum (in the general case, mixed) ensemble can be equipped with the structure of a probability space. As a result of a reproducible measurement, the quantum ensemble transforms to a quantum ensemble, generally speaking, with a different probability distribution of observables.

Consider an ensemble of physical systems that are in a quantum state Ψ_{φ_η} $(\eta \in \Xi)$. Accordingly, we will consider the equivalence class $[\varphi]_{\varphi_\eta}$ as the space $\Omega(\varphi_\eta)$ of elementary events φ. Let a value of the observable $\hat{B} \in \mathfrak{O}_\xi$ be measured in the experiment and an instrument of the type ξ be used. Denote by $(\Omega(\varphi_\eta), \mathcal{F}_\xi)$ the corresponding measurable space. Let P_ξ be the probability measure on this space, i.e., $P_\xi(F)$ is a probability of the event $F \in \mathcal{F}_\xi$.

Let us assume that the event F_B occurs in the experiment if the recorded observable value $\hat{B}$ is no greater than B. Denote the probability of this event as $P_\xi(F_B) = P(\varphi : \varphi_\xi(\hat{B}) \leq B)$. By knowing the probabilities $P_\xi(F)$, with the help of appropriate summations and integrations, we can find the probability $P_{\xi'}(F_B)$; the distribution $P_{\xi'}(F_B)$ is marginal for probabilities $P_\xi(F)$ (see, e.g., [22]).

An observable $\hat{B}$ may belong not only to an algebra $\mathfrak{O}_\xi$, but also to another maximal algebra $\mathfrak{O}_{\xi'}$. Therefore, to determine the probability of an event F_B, we can use an instrument of the type ξ'. In this case, we could get a different value for the probability: $P_{\xi'}(F_B)$. However, the experience shows that probabilities do not depend on the measuring instrument used. Therefore, we must accept one more postulate.

Postulate 8. Let the observable $\hat{B} \in \mathfrak{O}_\xi \cap \mathfrak{O}_{\xi'}$, then the probability of detecting an event F_B for a system in a quantum state Ψ_{φ_η} does not depend on the type of instrument used, i.e., $P(\varphi : \varphi_\xi(\hat{B})) \leq B) = P(\varphi : \varphi_{\xi'}(\hat{B}) \leq B)$.

Therefore, despite the fact that an elementary state φ is a collection of functionals φ_ξ, we have the right to use the notation $P(\varphi : \varphi(\hat{B}) \leq B)$ for the probability of an event F_B.

Let us introduce in addition the notation

$$P_{\hat{B}}(d\varphi) = P(\varphi : \varphi(\hat{B}) \leq B + dB) - P(\varphi : \varphi(\hat{B}) \leq B)$$

and consider an ensemble of quantum systems that are in a quantum state Ψ_{φ_η}. According to probability theory (see, e.g., [21]), the expectation of an observable $\hat{B}$ in this state is given by the formula

$$\langle \hat{B} \rangle = \int\limits_{\varphi \in \Psi_{\varphi_\eta}} P_{\hat{B}}(d\varphi)\varphi(\hat{B}). \quad (4)$$

On the other hand, Khinchin's theorem is valid (see the law of large numbers, e.g., in [21]):

Theorem. Let $B_i = \varphi_i(\hat{B})$ $(1 \leq i \leq n, \varphi_i \in \Psi_{\varphi_\eta})$ be a sequence of mutually independent randomly chosen values having the same probability distribution with a finite mathematical expectation $\langle \hat{B} \rangle$. Then, for $n \to \infty$, the quantity $(B_1 + \ldots + B_n)/n$ converges in probability to $\langle \hat{B} \rangle$. Thus,

$$\Psi_{\varphi_\eta}(\hat{B}) \equiv \lim_{n\to\infty} P[n^{-1}(\varphi_1(\hat{B}) + \ldots + \varphi_n(\hat{B}))] = \langle \hat{B} \rangle. \quad (5)$$

Equation (5) defines the functional (quantum mean) on the set $\mathfrak{A}_+$. We denoted this functional by the symbol $\Psi\varphi_\eta(\cdot)$ and will also call it a quantum state. It immediately follows from Eq. (5) and the properties of the functionals $\varphi_i(\cdot)$ that $\Psi\varphi_\eta(\cdot)$ is linear on each subset $\mathfrak{O}_\xi$ of compatible observables. In other words, the restriction of the functional $\Psi\varphi_\eta(\cdot)$ to each subset $\mathfrak{O}_\xi$ is a linear functional. The linearity property of the functional $\Psi\varphi_\eta(\cdot)$ can be extended to the entire set $\mathfrak{A}_+$. However, the set $\mathfrak{A}_+$ should be equipped beforehand with a structure of the real linear space.

Since each element $\hat{B}$ of the set $\mathfrak{A}_+$ belongs to some linear subset $\mathfrak{O}_\xi$, then for it the operation of multiplication by a real number is defined. With the operation of adding elements $\hat{B}$ and $\hat{C}$, the situation is more complicated, since these elements can belong to different linear subsets $\mathfrak{O}_\xi$ and $\mathfrak{O}_{\xi'}$. However, the entire totality of quantum experiments indicates that for any $\hat{B}$ and $\hat{C}$ belonging to $\mathfrak{A}_+$, there is an element $\hat{D} \in \mathfrak{A}_+$ such that for each quantum state $\Psi\varphi_\eta(\cdot)$ it is true that

$$\Psi\varphi_\eta(\hat{B}) + \Psi\varphi_\eta(\hat{C}) = \Psi\varphi_\eta(\hat{D}).$$

By definition, we can consider this element $\hat{D}$ as the sum of elements $\hat{B}$ and $\hat{C}$, i.e., $\hat{D} = \hat{B} + \hat{C}$. With these considerations in mind, we accept the following postulate.

Postulate 9. A set $\mathfrak{A}_+$ can be equipped with a structure of the real linear space, and the functionals $\Psi\varphi_\eta(\cdot)$ are linear on this space.

It means that

$$\Psi\varphi_\eta(\hat{B}) + \Psi\varphi_\eta(\hat{C}) = \Psi\varphi_\eta(\hat{B} + \hat{C})$$

also in the case when $\hat{B}$ and $\hat{C}$ belong to different subsets $\mathfrak{O}_\xi$ and $\mathfrak{O}_{\xi'}$.

The set $\mathfrak{A}_+$ can be equipped with the structure of a real algebra. To this end, the product of elements $\hat{B}$ and $\hat{C}$ should be determined by the formula

$$\hat{B} \circ \hat{C} = 1/2\left((\hat{B} + \hat{C})^2 - \hat{B}^2 - \hat{C}^2\right). \quad (6)$$

This product is obviously commutative, but in the general case it is not associative (see (D.3)), i.e., the associator $\{\hat{B},\hat{C},\hat{D}\} = (\hat{B} \circ \hat{C}) \circ \hat{D} - \hat{B} \circ (\hat{C} \circ \hat{D})$ is not necessarily equal to zero. It can be shown (see [6]) that for the product $\hat{B} \circ \hat{C}$ to be distributive (see (D.2.b,c)) it is necessary and sufficient that the associator vanishes for any $\hat{B}$ and $\hat{C}$, and for any real number α. Under this condition, a real algebra with product (6) is called a real Jordan algebra [6, 23].

In principle, we can try to build a quantum theory on the basis of this algebra. However, only a few successes have been achieved along this path (see [6]). A direction based on complex associative algebra, for which the Jordan algebra is in some sense a real part, proved to be much more successful.

All Jordan algebras are divided into two classes: special and exceptional. The special Jordan algebra is defined as follows. Let there be a real or complex algebra $\mathfrak{A}$ with the "ordinary" product $\hat{U}\hat{V}(\hat{U} \in \mathfrak{A}, \hat{V} \in \mathfrak{A}, \hat{U}\hat{V} \in \mathfrak{A})$. With respect to this product, the algebra is associative, but not necessarily commutative. In the set $\mathfrak{A}$, we can introduce a "symmetrized" product

$$\hat{U} \circ \hat{V} = 1/2\,(\hat{U}\hat{V} + \hat{V}\hat{U}). \quad (7)$$

Relative to this product, the set $\mathfrak{A}$ will be a Jordan algebra. Any Jordan algebra that is isomorphic to this algebra (or to its subalgebra) is called special. Otherwise, the Jordan algebra is called exceptional. Not every Jordan algebra is special. Therefore, for the Jordan algebra to be special, its elements must satisfy definite identities, which, in principle, could be verified experimentally. However, at present, the list of these identities is not known. On the other hand, in any of the quantum models considered so far, a set of observables can be equipped with the structure of a special Jordan algebra.

We will remain within this tradition and accept the following hypothesis.

Hypothesis. The Jordan algebra of observables is special and real.

In what follows, the elements of an algebra $\mathfrak{A}$ will be called dynamic quantities. The set $\mathfrak{A}$ can be equipped with a structure of the Jordan algebra by defining the product of its elements using Eq. (7). It follows from the hypothesis that dynamic quantities can be added and multiplied using the usual rules of addition and multiplication (except for commutation). This seems so obvious that it is almost never specifically mentioned. Nevertheless, we called the corresponding statements a hypothesis, not a postulate, because we cannot indicate an experimental way to verify *the necessity* of this statement.

It should be emphasized that in the standard approach to quantum mechanics, the statement of the hypothesis is taken in a much stronger form. It assumes that the observables are self-adjoint operators in some Hilbert space. This assumption can hardly be considered self-evident.

Further, a physical system will be considered to be given if the algebra $\mathfrak{A}$ of its dynamical quantities is given. By virtue of the first postulate, the algebras $\mathfrak{O}_\xi$ of compatible observables are the maximal real commutative subalgebras of the algebra that belong to $\mathfrak{A}_+$. This, in turn, implies that compatible observables are mutually commuting elements of the algebra $\mathfrak{A}$, while incompatible observables do not commute with each other.

It was mentioned earlier that in the quantum case the set Ξ of subalgebras $\mathfrak{O}_\xi(\xi \in \Xi)$ has the cardinality of the continuum. Indeed, even if the algebra $\mathfrak{A}$ is an algebra with two noncommuting Hermitian generators $\hat{B}_1$ and $\hat{B}_2$, then a commutative algebra $\mathfrak{O}_\alpha$ with a generator $\hat{B}_\alpha = \hat{B}_1 \cos\alpha + \hat{B}_2 \sin\alpha$ is an algebra of type $\mathfrak{O}_\xi$ for any real α.

5. C^*-ALGEBRA AND HILBERT SPACE

Any element $\hat{U}$ of the algebra $\mathfrak{A}$ is uniquely represented as $\hat{U} = \hat{B} + i\hat{C}$, where $\hat{B},\hat{C} \in \mathfrak{A}_+$. Therefore, the functional $\Psi_{\varphi_\eta}(\cdot)$ can be uniquely extended to a linear functional on the algebra $\mathfrak{A}$: $\Psi_{\varphi_\eta}(\hat{U}) = \Psi_{\varphi_\eta}(\hat{B}) + i\Psi_{\varphi_\eta}(\hat{C})$.

We define the seminorm of an element $\hat{U}$ by the equality

$$\|\hat{U}\|^2 = \sup_\xi \sup_{\varphi_\xi} \varphi_\xi(\hat{U}^*\hat{U}) = r(\hat{U}^*\hat{U}), \quad (8)$$

where $r(\hat{U}^*\hat{U})$ is the spectral radius of an element $\hat{U}^*\hat{U}$ in the algebra $\mathfrak{A}$.

This definition is acceptable. Firstly, $\|\hat{U}\|^2 \geq 0$ owing to the property (S.7.c). Further, by virtue of the

definition of a probability measure for any $\eta \in \Xi$, we have

$$\Psi_{\varphi_\eta}(\hat{U}^*\hat{U}) = \int\limits_{\varphi\in\{\varphi\}_{\varphi_\eta}} P_{\hat{U}^*\hat{U}}(d\varphi)[\hat{U}^*\hat{U}](\varphi) \leq \sup_{\xi}\sup_{\varphi_\xi}\varphi_\xi(\hat{U}^*\hat{U}) = r(\hat{U}^*\hat{U}). \quad (9)$$

For $\eta \in \Xi$ such that $\hat{U}^*\hat{U} \in \mathfrak{O}_\eta$, $\Psi_{\varphi_\eta}(\hat{U}^*\hat{U}) = \varphi_\eta(\hat{U}^*\hat{U})$ is true. Therefore, for this η,

$$\sup_{\varphi_\eta}\Psi_{\varphi_\eta}(\hat{U}^*\hat{U}) = \sup_{\varphi_\eta}\varphi_\eta(\hat{U}^*\hat{U}) = r_\eta(\hat{U}^*\hat{U}), \quad (10)$$

where $r_\eta(\hat{U}^*\hat{U})$ is the spectral radius in $\mathfrak{O}_\eta$. Since the subalgebra $\mathfrak{O}_\eta$ is maximal, then (see (S.2)) $r_\eta(\hat{U}^*\hat{U}) = r(\hat{U}^*\hat{U})$. Hence, taking into account equalities (8), (9), and (10), we obtain

$$\|\hat{U}\|^2 = \sup_{\xi}\sup_{\varphi_\xi}\varphi_\xi(\hat{U}^*\hat{U}) = \sup_{\xi}\sup_{\varphi_\xi}\Psi_{\varphi_\xi}(\hat{U}^*\hat{U}). \quad (11)$$

Since $\Psi_{\varphi_\xi}(\cdot)$ is a linear positive functional, then the Cauchy–Bunyakovsky–Schwarz inequality is valid (see (S.5.b)):

$$\left|\Psi_{\varphi_\xi}(\hat{U}^*\hat{V})\Psi_{\varphi_\xi}(\hat{V}^*\hat{U})\right| \leq \Psi_{\varphi_\xi}(\hat{U}^*\hat{U})\Psi_{\varphi_\xi}(\hat{V}^*\hat{V}). \quad (12)$$

This implies that for $\|\hat{U}\|$, the axioms of the seminorm of an element $\hat{U}$ hold (see, e.g., [6]):

$$\|\hat{U}+\hat{V}\| \leq \|\hat{U}\| + \|\hat{V}\|, \quad \|\lambda\hat{U}\| = |\lambda|\|\hat{U}\|,$$
$$\|\hat{U}^*\| = \|\hat{U}\|, \quad \|\hat{U}\hat{V}\| \leq \|\hat{U}\|\|\hat{V}\|.$$

Consider now the set $\mathfrak{I}$ of elements $\hat{U}$ of the algebra $\mathfrak{A}$ for which $\|\hat{U}\|^2 = 0$. From inequality (12) it follows that $\mathfrak{I}$ is a two-sided ideal of $\mathfrak{A}$. Therefore, we can form a quotient algebra $\mathfrak{A}' = \mathfrak{A}/\mathfrak{I}$. In the algebra $\mathfrak{A}'$, from $\|\hat{U}\|^2 = 0$, it follows that $\hat{U} = 0$. Therefore, in the algebra $\mathfrak{A}'$, equality (8) defines not a seminorm, but a norm. On the other hand, we can verify that the algebra $\mathfrak{A}'$carries the same physical information as $\mathfrak{A}$.

For this purpose, we consider two observables $\hat{B}$ and $\hat{C}$, which simultaneously either belong or do not belong to each of the subalgebras $\mathfrak{O}_\xi$. Let $\hat{B}$ and $\hat{C}$ satisfy the additional condition $\|\hat{B}-\hat{C}\| = 0$. Then from equality (8), it follows that

$$\varphi_\xi(\hat{B}) = \varphi_\xi(\hat{C}) \quad (13)$$

for all $\mathfrak{O}_\xi$ that contain these observables. Equality (13) means that no experiment can distinguish these observables. Therefore, from the phenomenological viewpoint, these observables should be identified. Mathematically, these observables are $\mathfrak{I}$-ideal equivalent. When passing from algebra $\mathfrak{A}$ to algebra $\mathfrak{A}'$, all equivalent observables are identified mathematically. To deal directly with $\mathfrak{A}'$-type algebra, we can accept the following postulate.

Postulate 10. If $\sup_{\xi}\sup_{\varphi_\xi}\left|\varphi_\xi(\hat{B}-\hat{C})\right| = 0$, then $\hat{B} = \hat{C}$.

Postulate 10 is of a technical nature. At the same time, from the phenomenology viewpoint, it does not impose any additional restrictions. It only simplifies the mathematical description of physical systems. Further, we will assume that the requirement of Postulate 10 is satisfied, and therefore equality (11) determines the norm of the element $\hat{U}$.

It follows from the multiplicativity of the functional φ_ξ that $\varphi_\xi([\hat{U}^*\hat{U}]^2) = [\varphi_\xi(\hat{U}^*\hat{U})]^2$. This means that $\|\hat{U}^*\hat{U}\| = \|\hat{U}\|^2$. Therefore, if we complete the algebra $\mathfrak{A}$ with respect to the norm $\|\cdot\|$, then $\mathfrak{A}$ turns into a C^*-algebra [15]. Thus, the algebra of quantum dynamic quantities can be equipped with the C^*-algebra structure. In the standard algebraic approach to quantum theory, this statement is taken as a starting axiom. Mathematically, this is, of course, very convenient. However, from the phenomenological viewpoint, the necessity of this axiom remains completely unclear.

From a technical viewpoint, it is convenient to replace Postulates 7–10 with a single postulate about the quantum average:

Postulate (QM). The probability distribution on the equivalence class $\{\varphi\}_{\varphi_\eta}$ is such that the right-hand side of Eq. (14) does not depend on ξ and defines a linear functional $\Psi_{\varphi_\eta}(\hat{B})$ on the algebra $\mathfrak{A}$:

$$\Psi_{\varphi_\eta}(\hat{B}) = \int\limits_{\varphi\in\Psi_{\varphi_\eta}} P_{\hat{B}}(d\varphi)\varphi_\xi(\hat{B}), \quad (14)$$

where

$$P_{\hat{B}}(d\varphi) = P(\varphi : \varphi_\xi(\hat{B}) \leq B + dB) - P(\varphi : \varphi_\xi(\hat{B}) \leq B)$$

is the probability measure on the class $\{\varphi\}_{\varphi_\eta}$.

In most of our previous constructions, the elementary state $\varphi = [\varphi_\xi]$ occupied a central place. The elementary state has many properties that are usually attributed to the so-called hidden parameters [24]. In the standard approach to quantum mechanics since the time of von Neumann [1], the opinion has been firmly rooted that there are no hidden parameters in quantum mechanics and cannot be. Therefore, it is necessary to make sure that the elementary states can be introduced in a consistent way.

The assignment of a physical system involves the definition of an algebra $\mathfrak{A}$ of dynamic quantities. As we have just seen, this algebra must have the structure of a C^*-algebra. By defining the algebra $\mathfrak{A}$, we thereby define a set of its maximal commutative associative subalgebras $\mathfrak{O}_\xi$. Each of these subalgebras is a Banach algebra.

It is clear that to construct any elementary state $\varphi = [\varphi_\xi]$ it is necessary and sufficient to construct all its components φ_ξ, while each φ_ξ is a character of the subalgebra $\mathfrak{O}_\xi$. Each functional φ_ξ can be constructed as follows. In the subalgebra $\mathfrak{O}_\xi$, we arbitrarily choose a system $G(\mathfrak{O}_\xi)$ of independent generators. Bearing in mind statement (S.8), we associate each element of the set $G(\mathfrak{O}_\xi)$ with one of the points of its spectrum. This is how we define the functional φ_ξ on the set $G(\mathfrak{O}_\xi)$. By linearity and multiplicativity, the functional φ_ξ is uniquely extended to the entire subalgebra $\mathfrak{O}_\xi$. By sorting out for each element of the set $G(\mathfrak{O}_\xi)$ all points of its spectrum, we construct all the functionals φ_ξ. For another ξ, we build functionals according to the same recipe. This procedure is definitely consistent if for different ξ functionals are constructed independently of each other. If we impose condition (3), then the procedure may turn out to be, and indeed turns out to be contradictory in some cases.

However, it is always possible to construct an elementary state φ that is stable on all observables belonging to any single subalgebra $\mathfrak{O}_\xi$. To this end, it suffices to start constructing the functional φ precisely from this subalgebra, using the procedure just described above. On another subalgebra $\mathfrak{O}_{\xi'}$, we construct a functional $\varphi_{\xi'}$ as follows. Let $\mathfrak{O}_\xi \cap \mathfrak{O}_{\xi'} = \mathfrak{O}_{\xi\xi'}$ and $G(\mathfrak{O}_{\xi\xi'})$ be independent generators of the subalgebra $\mathfrak{O}_{\xi\xi'}$. Let $\tilde{G}(\mathfrak{O}_{\xi\xi'})$ be the complement of these generators to the set of generators of the subalgebra $\mathfrak{O}_{\xi'}$. If $\hat{B} \in \mathfrak{O}_{\xi\xi'}$, then we assume $\varphi_{\xi'}(\hat{B}) = \varphi_\xi(\hat{B})$. If $\hat{B} \in \tilde{G}(\mathfrak{O}_{\xi\xi'})$, then we construct $\varphi_{\xi'}(\hat{B})$ as a mapping of the element $\hat{B}$ to one of the points of its spectrum. The functional $\varphi_{\xi'}$ is extended to the remaining elements of the subalgebra $\mathfrak{O}_{\xi'}$ by linearity and multiplicativity.

Thus, we see that for the elementary state there is no problem of existence. Von Neumann's proof [1] of the impossibility of the existence of hidden parameters for elementary states φ fails for the following reason. Von Neumann assumed that the state is described by a linear functional on the set of observables. An elementary state φ can be considered as a definite functional on the set of observables. However, this functional is linear only on subsets $\mathfrak{O}_\xi$; moreover, this functional is multivalued.

In this proof, von Neumann also has shown that the linearity of the functional describing the system state is incompatible with the assumption of causality at the microscopic level. From this, he concluded that there is no causality at the microlevel, while at the macro level it appears due to averaging over a large number of noncausal events. The approach proposed here allows for a much more plausible assertion. At the microscopic level, the causality is present, while there is no linearity of the state describing an individual quantum system. The linearity of the quantum state arises due to averaging over the quantum ensemble.

Note that the appearance of the property of linearity in averaging is a common phenomenon in probability theory. Therefore, the principles of linearity and superposition, which are usually considered in quantum mechanics as fundamental physical principles, in reality are not of that kind. These properties are just mathematical artifacts that owe their origin to the averaging procedure. In contrast, the causality is indeed a physical principle that is widely used in physics in circumvention of the "official ban." It is the elementary state that can claim the role of a mathematical image of reality, which is the physical carrier of causality.

The mentioned property of superposition owes its origin to the following remarkable feature of the C^*-algebra. Any C^*-algebra is isometrically isomorphic to a subalgebra of linear bounded operators in an appropriate Hilbert space [15]. This will allow us in the future to use the typical apparatus of Hilbert space, in which the superposition property arises in a natural way.

Remark. The standard approach to quantum mechanics usually assumes that all self-adjoint bounded operators in a Hilbert space are observables. This assumption is not satisfied in models with super selection rules [25]. The algebraic approach (including the one considered here) dispenses with this assumption.

In the algebraic approach, a state is defined as a positive linear functional Ψ on the set of observables that satisfies the normalization condition $\Psi(\hat{I}) = 1$. In the standard approach to quantum mechanics, a state is usually given either by a vector in Hilbert space or, in more general case, by a density matrix. However, not every interesting (from physical viewpoint) state can be specified using the density matrix (see [6]). Therefore, the algebraic definition is more general. Often a state defined in this way is called an algebraic state. Since $\varphi(\hat{I}) = 1$ (see (S.7.b)), the functional $\Psi_{\varphi_\eta}(\hat{B})$ defined by Eq. (4) satisfies the normalization condition. Therefore, the quantum state introduced here is an algebraic state. Since a linear positive functional defined on the set of observables is uniquely

extended to the algebra of dynamical quantities, then further we will call a normalized linear positive functional, defined on the algebra $\mathfrak{A}$, as an algebraic state.

Definition 42. An algebraic state Ψ is called pure if the equality

$$\Psi = \lambda\Psi_1 + (1-\lambda)\Psi_2 \quad 0 < \lambda < 1, \tag{15}$$

where Ψ_1 and Ψ_2 are two states, is possible only in the case $\Psi_1 = \Psi_2$.

It is easy to verify that the quantum state Ψ_{φ_ξ}, which we introduced in definition (D.40) is an algebraic pure state. Indeed, suppose that the functional Ψ_{φ_ξ} can be represented in the form (15). We narrow down Eq. (15) to the subalgebra $\mathfrak{O}_\xi$. On this subalgebra, i.e., for all, $\hat{B} \in \mathfrak{O}_\xi$, it is true that $\varphi_\xi(\hat{B}) = \Psi_{\varphi_\xi}(\hat{B})$. However, a functional $\varphi_\xi(\cdot)$ is a character of the subalgebra $\mathfrak{O}_\xi$. Every character of a commutative algebra is a pure state (see [14]). Therefore, it follows from the equality $\Psi_{\varphi_\xi}(\hat{B}) = \lambda\Psi_1(\hat{B}) + (1-\lambda)\Psi_2(\hat{B})$ that for $\hat{B} \in \mathfrak{O}_\xi$, $\Psi_1(\hat{B}) = \Psi_2(\hat{B}) = \Psi_{\varphi_\xi}(\hat{B}) = \varphi_\xi(\hat{B})$ is true and, in particular,

$$\Psi_1([\hat{B} - \varphi_\xi(\hat{B})]^2) = \varphi_\xi([\hat{B} - \varphi_\xi(\hat{B})]^2) = 0.$$

It follows from this that when $\hat{B} \in \mathfrak{O}_\xi$,

$$\int\limits_{\varphi\in\Psi_1} P_{\hat{B}}(d\varphi)\varphi([\hat{B} - \varphi_\xi(\hat{B})]^2) \equiv \Psi_1([\hat{B} - \varphi_\xi(\hat{B})]^2) = 0.$$

Therefore, if $\varphi \in \Psi_1$, then almost definitely $\varphi(\hat{B}) = \varphi_\xi(\hat{B})$ for $\hat{B} \in \mathfrak{O}_\xi$. This means that almost definitely the elementary states $\varphi \in \Psi_1$ form a class of equivalence $[\varphi]_{\varphi_\xi}$. From this, it follows that

$$\Psi_1(\hat{B}) = \int\limits_{\varphi\in\Psi_1} P_{\hat{B}}(d\varphi)\varphi(\hat{B})$$
$$= \int\limits_{\varphi\in\Psi_{\varphi_\xi}} P_{\hat{B}}(d\varphi)\varphi(\hat{B}) = \Psi_{\varphi_\xi}(\hat{B})$$

for all $\hat{B}$. It is similar for $\Psi_2(\hat{B})$, i.e., $\Psi_1(\hat{B}) = \Psi_2(\hat{B})$.

The procedure that implements a relation between a C^*-algebra and a Hilbert space is the so-called canonical Gelfand–Naimark–Segal (GNS) construction (see, e.g., [6, 13]). Briefly, it is as follows.

Let there be some C^*-algebra $\mathfrak{A}$ and a linear positive functional Ψ_0 on this algebra. We will consider two elements $\hat{U}, \hat{U}' \in \mathfrak{A}$ equivalent if for any $\hat{W} \in \mathfrak{A}$, the equality $\Psi_0\left(\hat{W}^*(\hat{U} - \hat{U}')\right) = 0$ is valid. Denote by $\Phi(\hat{U})$ the equivalence class of an element $\hat{U}$ and consider the set $\mathfrak{N}(\Psi_0)$ of all classes of equivalence in $\mathfrak{A}$. Turn the set $\mathfrak{N}(\Psi_0)$ into a linear space by setting $a\Phi(\hat{U}) + b\Phi(\hat{V}) = \Phi(a\hat{U} + b\hat{V})$. The scalar product in $\mathfrak{N}(\Psi_0)$ is defined by the formula

$$\left(\Phi(\hat{U}), \Phi(\hat{V})\right) = \Psi_0(\hat{U}^*\hat{V}). \tag{16}$$

This scalar product generates the norm $\|\Phi(\hat{U})\| = [\Psi_0(\hat{U}^*\hat{U})]^{1/2}$ in the algebra $\mathfrak{N}(\Psi_0)$. The completion in this norm turns $\mathfrak{N}(\Psi_0)$ into a Hilbert space. Each element $\hat{V}$ of the algebra $\mathfrak{A}$ is uniquely represented in this space by a linear operator $\tilde{V}$ acting according to the rule

$$\tilde{V}\Phi(\hat{U}) = \Phi(\hat{V}\hat{U}). \tag{17}$$

Thus, the GNS construction allows constructing a representation of any C^*-algebra. Let us consider a GNS construction in which the functional $\Psi_{\varphi_\eta}(\cdot)$, appearing in Eq. (4), acts as the functional that generates the representation. Let $\Phi(\hat{I})$ be the equivalence class of the identity element $\hat{I}$, then according to Eqs. (16) and (17), we have

$$\Phi(\hat{I}),\ \tilde{V}\Phi(\hat{I}) = \Psi(\hat{V}). \tag{18}$$

This is nothing but the Born postulate [26]. Thus, the GNS construction allows the standard mathematical apparatus of quantum mechanics to be reproduced.

At the same time, at this point there is a significant difference between the proposed approach to quantum mechanics and the standard one. Born's postulate is *sufficient* for quantum mechanical calculations, but its *necessity* is not clear. In contrast, in our case, Eq. (18) is a consequence of phenomenologically necessary postulates.

We will return to the Born postulate, but for now let us recall what representations there are.

The representation may be exact or approximate. In the exact representation, different elements of the algebra are associated with different operators in the Hilbert space.

Definition 43. A representation $\hat{V} \to \tilde{V}$ is called exact, if from $\tilde{V} = 0$, it follows $\hat{V} = 0$.

A representation may be null.

Definition 44. A representation $\hat{V} \to \tilde{V}$ is called null if $\tilde{V} = 0$ for any $\hat{V}$.

Definition 45. A representation $\hat{V} \to \tilde{V} \equiv \Pi(\hat{V})$ is a direct orthogonal sum $\Pi(\hat{V}) = \Pi_1(\hat{V}) \oplus \Pi_2(\hat{V})$ of two (or more) representations if operators $\Pi(\hat{V})$ act in a Hilbert space $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ according to the rule $\Pi(\hat{V})\hat{U} = \Pi_1(\hat{V})\hat{U}_1 + \Pi_2(\hat{V})\hat{U}_2$. Here $\hat{U} = \hat{U}_1 + \hat{U}_2$, $\hat{U}_1 \in \mathfrak{H}_1, \hat{U}_2 \in \mathfrak{H}_2$, while $\Pi_1(\hat{V})$ and $\Pi_2(\hat{V})$ are the representation operators in the spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively.

Definition 46. A representation $\hat{V} \to \Pi(\hat{V})$ is called degenerate if it can be expressed as a direct orthogonal sum of representations, among which at least one is zero.

Definition 47. A representation $\hat{V} \to \Pi(\hat{V})$ is called irreducible if it cannot be expressed as a direct orthogonal sum of two other representations.

Definition 48. A representation $\hat{V} \to \Pi(\hat{V})$, acting in a Hilbert space $\mathfrak{H}$, is called cyclic if in $\mathfrak{H}$, there exists a vector Φ (called cyclic) such that the set of vectors $\Pi(\hat{V})\Phi$ is everywhere dense in $\mathfrak{H}$.

It is equivalent to the fact that, using the $\Pi(\hat{V})\Phi$ construction, it is possible to construct a basis in $\mathfrak{H}$.

It is obvious that the representation constructed using the GNS construction is cyclic and nondegenerate. It can be shown that this representation is irreducible if and only if the state Ψ_0 is pure. In the general case, this representation is not exact. However, there exists a so-called universal representation $\hat{V} \to \Pi_u(\hat{V})$. This representation is the direct sum $\Pi_u(\hat{V}) = \oplus_i \Pi_i(\hat{V})$ of the representations. Each representation $\hat{V} \to \Pi_i(\hat{V})$ is built according to the GNS construction with the state Ψ_i. The summation is carried out over all algebraic states Ψ_i.

Any nondegenerate representation of the C^*-algebra is isomorphic to some subrepresentation of the universal representation. The universal representation is exact. This means that the algebra of elements $\hat{V}$ is isomorphic to the algebra of operators $\Pi_u(\hat{V})$. In other words, the C^*-algebra $\mathfrak{A}$ is isomorphic to some subalgebra of bounded linear operators in a Hilbert space $\mathfrak{H}_u$. To establish any algebraic relation between the $\mathfrak{A}$, it suffices to establish the corresponding relations between the operators that implement any exact representation of it. The existence of a universal representation guarantees that at least one this representation exists.

It has already been said earlier that the quantum state introduced here is a pure algebraic state. Now we show how functionals can be constructed that have the required properties. Consider first the case where the commutative algebra $\mathfrak{O}_\xi$ that defines the quantum state contains a one-dimensional projector $\hat{p}_0$. Most clearly, a one-dimensional projector can be defined as an element of an algebra, which in any exact representation corresponds to the projector operator onto a one-dimensional Hilbert subspace.

Remark. In the standard approach to quantum mechanics, it is typically assumed that any bounded self-adjoint operator corresponds to some observable. In this case, any maximal commutative subalgebra has one-dimensional projectors. Conversely, any one-dimensional projector belongs to some commutative subalgebra. In this situation, the case under consideration is a general one.

So let $\hat{p}_0 \in \mathfrak{O}_\xi$. Consider an exact representation of the algebra $\mathfrak{A}$. In the Hilbert space of this representation, there exists a vector $|\Phi_0\rangle$ such that $\hat{p}_0|\Phi_0\rangle = |\Phi_0\rangle$, $\langle\Phi_0|\Phi_0\rangle = 1$, $\hat{p}_0 = |\Phi_0\rangle\langle\Phi_0|$. Consider the expression $\hat{p}_0\hat{C}\hat{p}_0 = |\Phi_0\rangle\langle\Phi_0|\hat{C}|\Phi_0\rangle\langle\Phi_0| \equiv \vartheta(\hat{C})|\Phi_0\rangle\langle\Phi_0|$, i.e.,

$$\hat{p}_0\hat{C}\hat{p}_0 = \vartheta(\hat{C})\hat{p}_0 \tag{19}$$

for any $\hat{C} \in \mathfrak{A}$.

Equation (19) is a relation between elements of the algebra $\mathfrak{A}$. Therefore, it is determined only by the structure of the algebra $\mathfrak{A}$ and does not depend on its specific representation. In particular, the functional $\vartheta(\hat{C})$ does not depend on this representation. It is easy to verify that $\vartheta(\hat{C})$ is an algebraic state on the algebra $\mathfrak{A}$. The linearity of this functional follows from the relation

$$\vartheta_0(\hat{C} + \hat{D})\hat{p}_0 = \hat{p}_0(\hat{C} + \hat{D})\hat{p}_0 = [\vartheta_0(\hat{C}) + \vartheta_0(\hat{D})]\hat{p}_0.$$

The positivity follows from the relation $\hat{p}_0\hat{C}^*\hat{C}\hat{p}_0 = \vartheta_0(\hat{C}^*\hat{C})\hat{p}_0$. Since the operators $\hat{p}_0\hat{C}^*\hat{C}\hat{p}_0$ and $\hat{p}_0$ are positive, then $\vartheta_0(\hat{C}^*\hat{C}) \geq 0$. Finally, the normalization follows from the relation $\vartheta_0(\hat{I})\hat{p}_0 = \hat{p}_0\hat{I}\hat{p}_0 = \hat{p}_0$. Moreover, a restriction of a functional $\vartheta_0(\cdot)$ to a subalgebra $\mathfrak{O}_\xi$is a character of this subalgebra. Indeed, let $\hat{B} \in \mathfrak{O}_\xi$ and $\hat{C} \in \mathfrak{O}_\xi$, then

$$\vartheta_0(\hat{B}\hat{C})\hat{p}_0 = \hat{p}_0\hat{B}\hat{C}\hat{p}_0$$
$$= \hat{p}_0\hat{B}\hat{p}_0\hat{p}_0\hat{C}\hat{p}_0 = \vartheta_0(\hat{B})\vartheta_0(\hat{C})\hat{p}_0.$$

Thus, the functional $\vartheta_0(\cdot)$ has a property of linearity and multiplicativity. In addition, $\vartheta_0(\cdot)$ is positive and satisfies the normalization condition. These are exactly the conditions that must be satisfied for a functional describing a quantum state. Equality (19) is purely algebraic. Therefore, a value of the functional $\vartheta_0(\hat{C})$ depends only on $\hat{p}_0$ (the quantum state) and on $\hat{C}$ as an element of the algebra $\mathfrak{A}$, but not on any particular commutative subalgebra ($\hat{C}$ may belong to several such subalgebras).

Let us now show that the inverse statement is true. If a quantum state Ψ^0_ξsuch that $\varphi_\xi(\hat{p}_0) = 1$, corresponds to the functional $\Psi^0_\xi(\cdot)$, then $\Psi^0_\xi(\cdot) = \vartheta_0(\cdot)$. Indeed, from Eq. (4) it follows

$$\Psi^0_\xi(\hat{p}_0) = \Psi^0_\xi(\hat{I}) = 1. \tag{20}$$

From the Cauchy–Bunyakovsky–Schwarz inequality (see Eq. (12)) we obtain

$$\left|\Psi^0_\xi\left(\hat{C}(\hat{I}-\hat{p}_0)\right)\right|^2 \le \Psi^0_\xi C^*\hat{C})\Psi^0_\xi(\hat{I}-\hat{p}_0).$$

Whence, making use of Eq. (20), we have

$$\Psi^0_\xi(\hat{C}) = \Psi^0_\xi(\hat{C}\hat{p}_0) = \Psi^0_\xi(\hat{p}_0\hat{C}). \tag{21}$$

Making the change $\hat{C} \to (\hat{I}-\hat{p}_0)\hat{C}$ in (21), we obtain

$$\Psi^0_\xi(\hat{C}) = \Psi^0_\xi(\hat{p}_0\hat{C}\hat{p}_0). \tag{22}$$

Using Eq. (19) on the right-hand side of (22), we arrive at the equality

$$\Psi^0_\xi(\hat{C}) = \Psi^0_\xi\left(\vartheta_0(\hat{C})\hat{p}_0\right) = \vartheta_0(\hat{C}). \tag{23}$$

Let us now pass to the case when the subalgebra $\mathfrak{O}_\xi$ does not contain one-dimensional projectors. In this situation, we should consider an exact representation of the algebra $\mathfrak{A}$. Let $\mathfrak{H}$ be the Hilbert space of this representation, while $\mathfrak{B}(\mathfrak{H})$ be the set of all bounded linear operators in $\mathfrak{H}$. We can assume that $\mathfrak{A}$ and $\mathfrak{O}_\xi$ are subalgebras of the algebra $\mathfrak{B}(\mathfrak{H})$.

Let $\mathfrak{O}'_\xi$ be the maximum real commutative subalgebra of the algebra $\mathfrak{B}(\mathfrak{H})$ such that $\mathfrak{O}'_\xi \supset \mathfrak{O}_\xi$. Consider the set of all projectors belonging to $\mathfrak{O}'_\xi$. These projectors are mutually commuting self-adjoint operators in $\mathfrak{H}$ with discrete spectra. In the space $\mathfrak{H}$, there exists an orthonormal basis consisting of the eigenvectors of these operators. Let $\{\hat{p}\}$ be the set of projectors onto such basis vectors. All these projectors are one-dimensional, they belong to $\mathfrak{B}(\mathfrak{H})$, but in the case under consideration they do not belong to $\mathfrak{O}_\xi$. Each of the projectors $\hat{p}_i \in \{\hat{p}\}$ defines a linear functional $\vartheta_i(\cdot)$ on $\mathfrak{B}(\mathfrak{H})$: $\hat{p}_i\hat{B}\hat{p}_i = \vartheta_i(\hat{B})\hat{p}_i$. The restriction of this functional to the algebra $\mathfrak{A}$ has all the properties necessary to describe the corresponding pure quantum state.

6. PROBLEM OF PHYSICAL REALITY

In the famous Einstein–Podolsky–Rosen (EPR) work [28], the authors have formulated the principles which a complete physical theory should satisfy: (a) every element of physical reality should have a copy in a complete physical theory; (b) if, without any perturbation of the system, we can predict the value of a physical quantity with certainty (i.e., with probability one), then it means that there is an element of reality corresponding to this quantity.

The standard quantum mechanics did not accept this thesis. A single experiment does not have an adequate copy in the mathematical apparatus of standard quantum mechanics. Moreover, the opinion is firmly established that such a copy cannot exist and there is no objective physical reality that determines the result of a single experiment.

Quite weighty arguments are presented in favor of this opinion. Perhaps the best-known argument is based on Bell's inequality [29, 30]. Bell derived his inequality by reasoning in terms of the EPR thesis. After Bell, many variants of similar inequalities were proposed. Here, we consider the variant proposed in [31]. This variant is usually abbreviated as CHSH.

Let a particle with spin 0 split into two particles A and B with spins 1/2. These particles scatter over a long distance and are recorded by devices D_a and D_b, respectively. For a particle A, the device D_a measures the spin projection on the direction a, while for the particle B, the device D_b measures the spin projection on the direction b. The corresponding observables will be denoted by $\hat{A}_a$ and $\hat{B}_b$, while the measurement results, by A_a and B_b.

Let us assume that a state of the initial particle is characterized by some physical reality, which can be tagged with the parameter ν. We will use the same parameter for describing the physical realities that characterize the decay products. Accordingly, the results of measuring the observables $\hat{B}_a$, $\hat{C}_b$ can be considered as functions of $B_a(\nu)$, $C_b(\nu)$ of the parameter ν. Let the distribution of events with respect to the parameter ν be characterized by a probability measure $P(\nu)$ satisfying the standard conditions

$$\int P(d\nu) = 1, \quad 0 \le P(\nu) \le 1.$$

We introduce the correlation function $E(a,b)$:

$$E(a,b) = \int P(d\nu)A_a(\nu)B_b(\nu) \tag{24}$$

and consider the combination

$$\begin{aligned} N &= \left|E(a,b) - E(a,b')\right| + \left|E(a',b) + E(a',b')\right| \\ &= \left|\int P(d\nu)A_a(\nu)[B_b(\nu) - B_{b'}(\nu)]\right| \\ &+ \left|\int P(d\nu)A_{a'}(\nu)[B_b(\nu) + B_{b'}(\nu)]\right|. \end{aligned} \tag{25}$$

For any directions a and b

$$A_a(\nu) = \pm 1/2, \quad B_b(\nu) = \pm 1/2. \tag{26}$$

Therefore

$$\begin{aligned} N &\le \int P(d\nu)\left[\left|A_a(\nu)\right|\left|B_b(\nu) - B_{b'}(\nu)\right|\right. \\ &+ \left.A_{a'}(\nu)\left|B_b(\nu) + B_{b'}(\nu)\right|\right] = 1/2\int P(d\nu) \\ &\times \left[B_b(\nu) - B_{b'}(\nu) + B_b(\nu) + B_{b'}(\nu)\right]. \end{aligned} \tag{27}$$

Due to Eqs. (26) for any ν, one of the expressions

$$\left|B_b(\nu) - B_{b'}(\nu)\right|, \quad \left|B_b(\nu) + B_{b'}(\nu)\right| \tag{28}$$

is equal to zero and the other to one. Note that the same value ν appears in both expressions.

Considering the property of expressions (28), inequality (27) yields the Bell inequality (CHSH)

$$N \le 1/2 \int P(d\nu) = 1/2. \quad (29)$$

In standard quantum mechanics, the correlation function is easily calculated, the result is

$$E(a,b) = -1/4 \cos\theta_{ab},$$

where θ_{ab} is the angle between the directions a and b. For directions $a = 0$, $b = \pi/8$, $a' = \pi/4$, $b' = 3\pi/8$, we obtain

$$N = 1/\sqrt{2},$$

which contradicts inequality (29).

The experimental results are consistent with quantum mechanical calculations and do not confirm Bell's inequality. Typically, these results are considered as evidence that for a quantum mechanical system there is no physical reality that would predetermine the results of the measurement.

However, from the point of view of current probability theory, the above derivation of Bell's inequality is too naive. This derivation assumes that there is a probability distribution with respect to the parameter ν. In its meaning, this parameter marks an elementary event. And as noted earlier (see Section 4), it is far from always possible to attribute any probability to an elementary event. Before speaking about probability, it is necessary to equip the considered set of elementary events with the structure of a measurable space. In this regard, we will try to repeat the derivation of Bell's inequality, using the elementary state φ as a parameter ν.

According to the condition of the problem, the initial particle is in a definite quantum state. This means that we should consider the equivalence class $\{\varphi\}_{\varphi_\eta}$ as the space $\Omega(\varphi_\eta)$ of elementary events. In other words, if the observable $\hat{A} \in \mathfrak{O}_\eta$, then for all $\varphi \in \{\varphi\}_{\varphi_\eta}$ the value of this observable will be the same. The elementary states $\varphi \in \{\varphi\}_{\varphi_\eta}$ differ from each other due to values of the observables $\hat{B} \notin \mathfrak{O}_\eta$. It is easy to make sure that, due to this difference, the set $\varphi \in \{\varphi\}_{\varphi_\eta}$ will have the cardinality of the continuum. To verify this, let us consider some subalgebra $\mathfrak{O}_\xi \ne \mathfrak{O}_\eta$. Since the subalgebras $\mathfrak{O}_\xi$ and $\mathfrak{O}_\eta$ are maximal, then there exists at least one observable $\hat{B}$ such that $\hat{B} \in \mathfrak{O}_\xi$ and $\hat{B} \notin \mathfrak{O}_\eta$. The spectrum of this observable cannot consist of a single point. If this point λ is only one, then a spectral radius of the element $\hat{B} - \lambda\hat{I}$ is equal to zero: $r(\hat{B} - \lambda\hat{I}) = 0$. However, for the C^*-algebra, $\|\hat{B} - \lambda\hat{I}\| = r(\hat{B} - \lambda\hat{I})$. It means that $\hat{B} = \lambda\hat{I} \in \mathfrak{O}_\eta$. This implies that there are at least two elementary states $\varphi \in \{\varphi\}_{\varphi_\eta}$ with different values of the observable $\hat{B}$. The same reasoning can be repeated for another subalgebra $\mathfrak{O}_\xi \ne \mathfrak{O}_\eta$. Since the set of these subalgebras $\mathfrak{O}_\xi$ has the cardinality of the continuum, then the set of subalgebras distinct from each other $\varphi \in \{\varphi\}_{\varphi_\eta}$ will also have the cardinality of the continuum.

Let us now turn to Eq. (24) for the correlation function. We need correlation functions for four combinations of observables: $\hat{A}_a\hat{B}_b$, $\hat{A}_a\hat{B}_{b'}$, $\hat{A}_{a'}\hat{B}_b$, and $\hat{A}_{a'}\hat{B}_{b'}$. Of interest is the variant when the directions a, a', b, and b' in pairs are not parallel to each other. In this case, the listed observables in pairs are incompatible with each other. Therefore, in order to experimentally find the correlation functions, we must conduct four separate series of experiments. In the real case, each of these series consists of a finite number of experiments, in the ideal case, of a countable number.

Thus, in the experiment we are dealing not with a single space $\Omega(\varphi_{\varphi_\eta})$ of elementary events, but with four separate random samples from it. We denote them Ω_{ab}, $\Omega_{ab'}$, $\Omega_{a'b}$, and $\Omega_{a'b'}$. Since even in the ideal case these samples are countable, and the space $\Omega(\varphi_{\varphi_\eta})$ has the cardinality of the continuum, then the probability that there are common elements in these samples is zero. In addition, for these selections to become measurable spaces, it is necessary to choose the corresponding σ-algebras $\mathcal{F}_{ab}$, $\mathcal{F}_{ab'}$, $\mathcal{F}_{a'b}$, and $\mathcal{F}_{a'b'}$. These subalgebras are not only different, but, as explained in Section 4, they cannot be subalgebras of the same σ-algebra to which some probability measure corresponds. In other words, each sample should have its own probability measure: P_{ab}, $P_{a'b}$, $P_{ab'}$, and $P_{a'b'}$.

Thus, Eq. (24) will now look as follows

$$E(a,b) = \int\limits_{\Omega_{ab}} P_{ab}(d\varphi)\varphi(\hat{A}_a\hat{B}_b),$$

while Eq. (25), as

$$N = \left| \int\limits_{\Omega_{ab}} P_{ab}(d\varphi)\varphi\left(\hat{A}_a\hat{B}_b\right) - \int\limits_{\Omega_{ab'}} P_{ab'}(d\varphi)\varphi\left(\hat{A}_a\hat{B}_{b'}\right) \right| + \left| \int\limits_{\Omega_{a'b}} P_{\hat{A}_{a'}\hat{B}_b}(d\varphi)\varphi\left(\hat{A}_{a'}\hat{B}_b\right) + \int\limits_{\Omega_{a'b'}} P_{a'b'}(d\varphi)\varphi\left(\hat{A}_{a'}\hat{B}_{b'}\right) \right|. \quad (30)$$

Although the same symbol $d\varphi$ is used in all four terms on the right-hand side of Eq. (30), it must be borne in mind that the sets of elementary states corresponding to $d\varphi$ will be different. They are elements of different σ-algebras. Moreover, the probability that they have common elements is equal to zero. Therefore, first, it is impossible in Eq. (30), as it is done on the right side of Eq. (25), to combine the integrals under the modulus sign into one integral. Second, it is impossible to form pairs similar to those involved in

Eq. (28). In turn, this does not allow inequality (29) to be proven. Thus, if we associate a concept of physical reality with an elementary state, then the violation of Bell's inequality in no way proves the inconsistency of this concept.

Another argument against the use of the concept of physical reality in quantum physics is the so-called Kochen–Specker forbidding theorem [32]. The meaning of this theorem is as follows. Let us consider as a physical system a particle with a spin of unity. Let the directions x, y, z be mutually orthogonal. Then the observables $\hat{S}_x^2, \hat{S}_y^2, \hat{S}_z^2$ that describe squares of spin projections on the corresponding directions commute with each other. Therefore, they are compatible and can be measured simultaneously. Let us assume that there is some physical reality that uniquely predetermines the result of measurement in any direction. When measuring in one of the directions, one should get zero, while in the other two directions, unity must be obtained. We fix one of the latter directions and consider two directions (different from the previous ones) which are normal to it and to each other. In one of these directions, the measurement result should be zero, in the other direction, it will be unity. We fix the first direction and repeat the whole procedure from the beginning. After a finite number of such steps, we can arrive at the previously considered direction. In this case, it turns out that if initially in this direction a value of the spin projection squared was equal to zero, then, when returning to this direction, the same square should be equal to unity.

From this contradiction, it is concluded that there cannot be a physical reality that predetermines the measurement result. However, this reasoning completely ignores the problem of measurability. Meanwhile, here we have to deal with two triplets of directions: x, y, z and x, y', z'. Within each of the triples, all directions are mutually orthogonal, but in different triples, there are nonorthogonal directions. Therefore, the observables $\hat{S}_x^2, \hat{S}_y^2, \hat{S}_z^2$ and $\hat{S}_x^2, \hat{S}_{y'}^2, \hat{S}_{z'}^2$ belong to different commutative subalgebras of the algebra $\mathfrak{A}$. Correspondingly, the devices that perform measurements that are compatible within each of the triples belong to different types. These instruments do not necessarily have to yield the same result when measuring the observable $\hat{S}_x^2$. This was tacitly assumed in the proof of the theorem. We recall that the elementary state does not uniquely fix the values of all observables. It unambiguously fixes the readings of instruments of *a definite type*. For different types, these readings may be different.

Thus, within the proposed approach, the Kochen–Specker theorem does not exclude the possibility of existence of an objective physical reality associated with an elementary state.

7. PARADOXES

Critics of the standard formalism of quantum mechanics have pointed to a large number of situations in which quantum mechanical reasoning leads to paradoxical results. In this section, we will discuss only two, perhaps the most frequently mentioned, paradoxes. These are the Einstein–Podolsky–Rosen (EPR) paradox and the Schrödinger cat paradox. It must be said at once that the most orthodox supporters of the standard formalism assert that there are no paradoxes. It is only necessary to correctly use the formulas of quantum mechanics. Therefore, before discussing specific paradoxes, let us fix our own position. It is as follows.

The formulas of the standard formalism of quantum mechanics are definitely valid in the case of considering quantum ensembles. They correctly describe the average values of the observed quantities and the probabilities of events also in those physical models that are proposed by the authors of the paradoxes. Therefore, it is of interest to discuss only single phenomena. There are two possible positions here. See review [33] on this matter. First, we can assume that single phenomena lie outside the competence of quantum mechanics. In this case, the subject of the dispute disappears. However, single phenomena definitely exist. Therefore, the question arises about the completeness of the quantum mechanical description. Second, we can assume that for single phenomena, quantum mechanics should predict only probabilities, and the completeness of the description is limited to the prediction of the corresponding probabilities. In this case, it is necessary to consider that the probability is some self-dependent essence of this single phenomenon.

This is not the case in current mathematical probability theory. Recall that before introducing the concept of probability measure, the concept of the space of elementary events is introduced. Accordingly, a single phenomenon (elementary event) is considered as an element of the definite set (ensemble). In this case, the same single phenomenon can be considered as an element of different sets. Depending on this set, different probabilities will correspond to the same phenomenon, or no probability will correspond to it.

The orthodox supporters of the standard formalism reject this viewpoint and prefer to consider a probability as the fundamental undefined essence of a single phenomenon, which in the mathematical apparatus of quantum mechanics is matched by either a Hilbert space vector or a density matrix. Formally, paradoxes can be avoided in this way, but the physical essence of the phenomena remains beyond the scope of discussion.

After these preliminary comments, we proceed directly to the discussion of paradoxes. Let us start with the EPR paradox. In the original work [28], this paradox was considered using the example of measuring the position and momentum. A simpler physical

model was proposed by Bohm [34]. There, the same problem is discussed using the example of measurements of spin projections on different directions. Here we will focus on the variant proposed by Bohm. In this case, the same physical system is considered as in the discussion of Bell's inequality.

Let a particle with spin 0 decay into two identical particles B and C with spins 1/2, which fly apart over a large distance. The spin state of this system, according to formulas of the standard approach, is described by the state vector

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big[\Big|B_z^{(+)}\Big\rangle\Big|C_z^{(-)}\Big\rangle - \Big|B_z^{(-)}\Big\rangle\Big|C_z^{(+)}\Big\rangle\Big], \qquad (31)$$

where $\Big|B_z^{(\pm)}\Big\rangle$ and $\Big|C_z^{(\pm)}\Big\rangle$ are the eigenvectors of operators of spin projection on the z axis with eigenvalues $+1/2$ and $-1/2$. This is a so-called entangled state. In this state, neither the particle B nor C has a definite value of spin projection onto the z axis. The spin state of each of the particles can be described by the density matrix. For example, for a particle B, the density matrix is written as follows:

$$\rho(B) = \frac{1}{2}\Big[\Big|B_z^{(+)}\Big\rangle\Big\langle B_z^{(+)}\Big| + \Big|B_z^{(-)}\Big\rangle\Big\langle B_z^{(-)}\Big|\Big].$$

This matrix means that with the probability 1/2 the particle has a spin projection $+1/2$ and with the same probability, a projection $-1/2$.

At the moment when the particles B and C are in space-like regions, we measure the spin projection onto the z axis for the particle C. Let the result be $+1/2$. Then, according to the postulate about the quantum state collapse (the projection principle), the state $|\Psi\rangle$ is instantly replaced by the state

$$|\tilde{\Psi}\rangle = \hat{p}_+|\Psi\rangle\Big/\sqrt{\langle\langle\Psi|\hat{p}_+|\Psi\rangle}, \qquad (32)$$

where $\hat{p}_+$ is the projector of the form

$$\hat{p}_+ = \hat{I}_B \otimes \Big|C_z^{(+)}\Big\rangle\Big\langle C_z^{(+)}\Big|. \qquad (33)$$

Here $\hat{I}_B$ is the identity operator in the state space of the particle B.

Substituting (33) into (33), we obtain $|\tilde{\Psi}\rangle = -\Big|B_z^{(-)}\Big\rangle\Big|C_z^{(+)}\Big\rangle$. The particle density matrix corresponding to this state has the form $\tilde{\rho}(B) = \Big|B_z^{(-)}\Big\rangle\Big|B_z^{(-)}\Big\rangle$. This means that in the subsequent measurement of the spin projection onto the z axis for the particle B, we will obtain the value $-1/2$ with the probability one. This perfectly describes the experimental situation. Thus, as a recipe for getting the correct result, the projection principle works very well. However, we would like to understand what *physical* process ensures the effectiveness of this recipe.

Two variants of the process immediately arise. The first one is as follows. At the decay instant, the particles acquired definite spin projections on the z axis (of the opposite sign), but before measuring the projection of the particle C, we do not know which projections these are. By measuring the projection of the particle C, we automatically came to know the projection of the particle B. However, this explanation is not consistent with the general concept of the standard approach to quantum mechanics.

The point is that the same quantum state $|\Psi\rangle$ can be presented in the form

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big[\Big|B_x^{(+)}\Big\rangle\Big|C_x^{(-)}\Big\rangle - \Big|B_x^{(-)}\Big\rangle\Big|C_x^{(+)}\Big\rangle\Big],$$

where the designations are the same as in Eq. (31), only instead of projections onto the z axis, projections onto the x axis appear. Now we can repeat all the arguments given after Eq. (31), replacing the z axis in them with the x axis. As a result, we obtain that at the moment of decay, the particles must acquire definite values of the spin projections on the x axis. However, the observables corresponding to the spin projections on the z and x axes are mutually incompatible and, according to the standard approach, cannot simultaneously have definite values.

The second variant of the process looks as follows. After the decay, particles B and C did not acquire definite values of spin projections on any axis. As a result of measuring the projection on the definite axis, they acquired these values of projections on this axis. It is not difficult to believe that this process is possible for a particle C that has interacted with the measuring instrument. However, how this measurement could affect a particle B located in a space-like region relative to the measuring instrument cannot be imagined without violating the principles of the theory of relativity. Thus, both versions of the explanation of the physics involved turn out to be untenable. Herein lies the paradox.

Objecting to the presence of a paradox, Bohr wrote in [35] that when discussing a system in which there are correlations, it cannot be considered as consisting of two independent parts. Therefore, any measurement over one part of this system should be considered as a measurement over the entire system. This explanation does not seem particularly convincing. The fact is that there are two types of correlations that can be rationally explained. The first type includes correlations, which owe their origin to the interaction between parts of the system. In the case of the EPR paradox, this interaction would have to be transmitted at superluminal speed. The second type includes correlations due to some relation between the initial conditions for the considered particles. In the case of the EPR paradox, this relation exists since the particles B and C emerged as a result of the decay of a single primary particle. However, the presence of this relation is not sufficient for the unambiguous correlation of these particles in the future. For this purpose, it is also nec-

essary that the initial conditions should unambiguously fix the subsequent temporal evolution of these particles. And this means that immediately after the production, even before the moment of measurement, the particles B and C must have had a definite property that would unambiguously determine the result of the measurement. This contradicts the general concept of standard quantum mechanics.

Of course, it can be assumed that there is some special quantum type of correlations that cannot be rationally interpreted. However, this explanation is, from the viewpoint of science, the worst of all possible, since the main task of scientific theory is to reduce the number of truths that cannot be rationally interpreted.

Fock's arguments [36] seem to be more successful. Fock believed that in the quantum case, an objective meaning should not be assigned to the concept of "state." Rather, it should be understood as "status information." With this interpretation, the paradox can indeed be avoided. But the question arises: Is there something objective about which we receive this information?

Within the approach proposed in this paper, this "something" exists. It is an elementary state. The elementary state is an objective characteristic of a physical system. It does not depend on any knowledge of the system. In contrast, a quantum state, i.e., some equivalence class of elementary states, is not a completely objective characteristic of a physical system. This concept is an objective characteristic of an *ensemble* of physical systems. We can consider the definite system of interest to us as an element of different ensembles (freedom of choice). Accordingly, it will be characterized by different quantum states. Therefore, in a quantum state there is a subjective factor.

Turning directly to the EPR paradox, we can give it the following interpretation. Both before and after the decay of the initial particle, the physical system is characterized by stable (zero) values of observables $\hat{S}_{\mathbf{n}}$ (projections of the total spin onto the direction $\mathbf{n}$). After the decay, the values of the observables $\hat{B}_{\mathbf{n}}$ and $\hat{C}_{\mathbf{n}}$ (projections of spins onto the direction $\mathbf{n}$ for particles B and C, respectively) satisfy the relation

$$B_{\mathbf{n}} + C_{\mathbf{n}} = S_{\mathbf{n}} = 0. \tag{34}$$

In principle, each of the observables $\hat{B}_{\mathbf{n}}$ and $\hat{C}_{\mathbf{n}}$ could be unstable. However, in the case of a two-level system, which indeed a particle with spin 1/2 is, these observables will be stable. In the elementary state, the incompatible observables can simultaneously have definite values. Only these values cannot be simultaneously measured using a classical instrument. In a specific experiment, we can measure the observable $\hat{C}_{\mathbf{n}}$ for any, but only for one, direction $\mathbf{n}$, since for different directions $\mathbf{n}$ and $\mathbf{n}'$, the observables $\hat{C}_{\mathbf{n}}$ and $\hat{C}_{\mathbf{n}'}$ are incompatible. Thanks to Eq. (34), with this measurement, we automatically measure a value of the observable $\hat{B}_{\mathbf{n}}$. This is the so-called indirect measurement. Thus, in this approach, the EPR paradox is resolved trivially.

Using this physical system as an example, we can give a completely rational interpretation of the phenomenon such as the quantum state collapse. In the context of the standard approach, this phenomenon looks mystical.

Before measuring the spin projection of a particle C, we know that our physical system is in some elementary state, which belongs to the equivalence class characterized by zero values of observables $\hat{S}_{\mathbf{n}}$. In other words, we know that the system is in a singlet quantum state, but we do not know what specific elementary state it is in. After measuring the observable $\hat{C}_{\mathbf{n}}$, thanks to Eq. (34), we acquire knowledge not only about the value of this observable, but also about the value of the observable $\hat{B}_{\mathbf{n}}$. Therefore, now we know that after the measurement, the system under consideration will be in the elementary state, which belongs to the equivalence class characterized by the values $B_{\mathbf{n}} = -C_{\mathbf{n}}$ (the value $C_{\mathbf{n}}$ is known) of the observables $\hat{B}_{\mathbf{n}}$ and $\hat{C}_{\mathbf{n}}$. Here we assumed that the measurement of the observable $\hat{C}_{\mathbf{n}}$ was reproducible. Now, again, we do not know in which particular elementary state the physical system turned out to be, but we know that it is in a definite quantum state (of the type $|\tilde{\Psi}\rangle$, Eq. (32)).

Due to the interaction with the measuring instrument, the value of the observables $\hat{C}_{\mathbf{n}'}$ for directions $\mathbf{n}' \neq \mathbf{n}$ varies in an uncontrollable manner. Therefore, for these directions, Eq. (34) is violated. This corresponds to the fact that the considered system ceases to belong to the singlet state. Thus, all signs of the quantum state collapse are reproduced. Note that before measuring we could describe the quantum state of particle B with the help of the density matrix

$$\rho(B) = \frac{1}{2}\left[\left|B_{\mathbf{n}}^{(+)}\right\rangle\left\langle B_{\mathbf{n}}^{(+)}\right| + \left|B_{\mathbf{n}}^{(-)}\right\rangle\left\langle B_{\mathbf{n}}^{(-)}\right|\right], \tag{35}$$

while after measuring with the help of the density matrix

$$\tilde{\rho}(B) = \left|-C_{\mathbf{n}}\right\rangle\left\langle -C_{\mathbf{n}}\right|. \tag{36}$$

Although quantum state (35) is mixed, and state (36) is pure, this does not mean that some change happened in the elementary state of particle B. Simply we acquired additional information about this elementary state.

Equality (34) can be given another useful interpretation. In the decay of the primary particle, each of the secondary particles produces a "measurement" of the *elementary state* of its partner in the sense that the elementary state of one of the particles is a negative copy of the elementary state of the other particle. The devel-

opment of this copy can be called a measurement using a *quantum* instrument. One particle is a quantum measuring instrument for another. In contrast to the measurement with a classical instrument, this measurement can unambiguously fix the elementary state of the particle under measurement. However, in order for the result of this "measurement" to become available to us, it is necessary using a classical instrument to carry out a direct measurement for this quantum instrument. As a result of this secondary measurement, we obtain knowledge only about the equivalence class to which the elementary state belongs.

The scenario of the second paradox, which we will discuss here, was proposed by Schrödinger [37] (on this subject, see also [38]). The scenario looks as follows. A cat and a radioactive source of very low intensity are placed in the box. When an atom decays in the source, the Geiger counter is triggered. The impulse from the counter is fed through an amplifying device to the automation, which breaks the ampoule of poison. The cat dies from the poison. The observer does not know whether the decay has occurred or not. Therefore, according to the standard rules of quantum mechanics, it must describe the state of a complex system (a cat plus a radioactive source) by a state vector that is a superposition of two quantum states: an undecayed atom and a living cat plus a decayed atom and a dead cat. The superposition of a live and a dead cat looks strange to say the least.

There is a statement that there will be no paradox if we go from describing the state using the Hilbert space vector to describing it using the density matrix. However, here we must clearly agree on what game we are playing. If we consider that the density matrix describes an ensemble of physical systems, then there will be no paradox. However, in this case, we will describe the state of not the cat alone, but state of an ensemble of cats, in which a part of the cats is alive, the other part is dead. In this case, each of the cats is either definitely alive or dead, but which of the cats we have to deal with is determined by the probability theory. However, in the scenario of the Schrödinger paradox, it is understood that we are dealing with one cat. In this case, the mentioned interpretation of the density matrix is not suitable. If we consider that the density matrix describes the state of one cat, then it is no easier to imagine a mixed state of an alive and a dead cat than a superposition of these cats.

Definitely, there will be no paradox if Fock's interpretation is accepted. In other words, it is considered that the term "state" in quantum mechanics actually implies our knowledge of the objective state of a physical object. However, firstly, standard quantum mechanics does not accept this point of view. Secondly, the question remains, whether this objective state exists.

Within the concept of an elementary state, the paradox is again resolved trivially. The pair under study (a cat plus a radioactive atom) is in a definite elementary state. At a given time moment in this state, the cat is either definitely alive or definitely dead. There is no mixed elementary state of a living and dead cat. The quantum state describes the equivalence class of these elementary states. Among these elementary states, there are those that correspond to an alive cat at a given moment in time, while there are those that correspond to a dead cat at the same time moment.

When we place a cat in a box, we only have the information about the equivalence class, but not about the individual elementary state. The equivalence class is fixed by classically recorded conditions: at the time moment of preparation of the system under study, the cat was alive, and the atom did not decay. On the other hand, the unambiguous evolution of a particular physical system is determined precisely by its elementary state. Using classical observations, this state cannot be unambiguously fixed.

8. ELEMENTARY STATE AND GENERAL THEORY OF RELATIVITY

The 20th century gave mankind two remarkable physical theories: quantum mechanics and the theory of relativity. However, the relationship between these theories was not cloudless. The special theory of relativity, though not without problems, has been combined with quantum mechanics. From this union a viable child was born: the quantum field theory. However, the general theory of relativity (GR) has not yet been able in any way to be combined with quantum mechanics in a harmonious union. The contradictions turned out to be too serious.

The whole of GR is permeated with the spirit of locality (see, e.g., [39]). Formally, in quantum field theory, the axioms of locality and causality are also among the fundamental ones. However, at the same time, the so-called projection principle [1] is widely used, which fits very poorly with the locality property. There are numerous "proofs" that locality and causality are incompatible with the mathematical apparatus used in quantum mechanics.

The most striking inconsistency between the mathematical apparatus of GR and quantum mechanics is manifested in the basic equation of GR. This equation relates the curvature of the four-dimensional spacetime $\mathfrak{M}$ with the energy-momentum (4-momentum) tensor of matter. The curvature of space $\mathfrak{M}$ is one of the basic concepts of general relativity and is described in terms of c-numerical functions. On the other hand, the 4-momentum tensor of matter is an important concept in quantum theory and is described in terms of operators in the Hilbert space, i.e., with the help of q-numerical functions. Therefore, when trying to directly unify general relativity with quantum mechanics, we would be forced to equate c-numerical and q-numerical functions. This indicates that it is

impossible to do without the modernization of general relativity or quantum theory, or without both at the same time.

At present, the gravitational field quantization has been chosen as the main direction of harmonization of general relativity and quantum theory. A very beautiful mathematical scheme has already been built: so-called supergravity. However, this scheme was not without significant drawbacks. For example, there is no way to construct a renormalizable theory of the gravitational field. The most important is that supergravity, like any supersymmetric theory, assumes the existence of superpartners of ordinary particles. Nothing similar has been found experimentally.

The approach used here proposes to go in a different direction. In the previous sections, we abandoned the literal identification of the "observable" and "operator in a Hilbert space" concepts accepted in the standard approach to quantum mechanics. For classical observables, this identification is obviously unsuitable. Instead, it was proposed to use the algebraic approach, in which the observables are considered as elements of some algebra without any reference to their representation in the form of operators. This approach is suitable for both quantum and classical observables.

In the previous sections, the relationships between quantum and classical systems were described without considering the general theory of relativity. To take GR into account, the main provisions should be somewhat modified and supplemented. Let us supplement the postulates of the previous sections with one more.

Postulate 11. The metric tensor $g_{\mu\nu}(x)$ of a four-dimensional manifold $\mathfrak{M}$ is the value of the classical observable $\hat{g}_{\mu\nu}(x)$.

The manifold $\mathfrak{M}$ is not an $\mathfrak{M}$ linear space. Following Kartan [40], it is convenient to represent a four-dimensional nonlinear space $\mathfrak{M}$ as a smooth four-dimensional surface in a linear space $\mathfrak{R}$ of higher dimension. However, we will not ascribe any physical meaning to the space $\mathfrak{R}$.

The observables $\hat{g}_{\mu\nu}(x)$ are components of the covariant tensor. This tensor can be associated with a contravariant tensor with components $\breve{g}^{\mu\nu}(x)$. No fundamental difference is often made between tensors $\hat{g}^{\mu\nu}(x)$ and $\breve{g}_{\mu\nu}(x)$, and the components of one tensor are expressed in terms of the components of another. Though, from the viewpoint of mathematics, they are elements of different (mutually conjugate) spaces. In our case, these tensors will play essentially different roles. While the covariant tensor $\hat{g}_{\mu\nu}(x)$ describes a gravitational observable, the contravariant tensor $\breve{g}^{\mu\nu}(x)$ does not describe any observable. It defines some linear mapping. Further, as observables, respectively, elements of the C^*-algebra, we will consider the covariant components of tensor quantities. Contravariant components will be used to construct linear mappings; components of mixed tensors will be avoided. The operations of integration and differentiation with respect to x are also used to construct linear mappings.

We assume that the mapping $\breve{g}^{\mu\nu}(x)$ acts on the observable $\hat{g}_{\mu\nu}(x)$ as follows

$$\breve{g}^{\mu\lambda}(x)\hat{g}_{\lambda\nu}(x) = \delta^{\mu}_{\nu}\hat{I}. \tag{37}$$

Here δ^{μ}_{ν} is the Kronecker symbol, and $\hat{I}$ is the identity element of the algebra of observables.

Along with the observables $\hat{B}(x)$, their derivatives often have to be used

$$\frac{\partial\hat{B}(x)}{\partial x^{\sigma}} = \partial_{\sigma}\hat{B}(x) \equiv \hat{B}_{,\sigma}(x),$$
$$\frac{\partial^2\hat{B}(x)}{\partial x^{\sigma}\partial x^{\tau}} = \partial_{\sigma}\partial_{\tau}\hat{B}(x) \equiv \hat{B}_{,\sigma\tau}(x).$$

In GR, generally, observables $\hat{B}_{,\sigma}(x)$ and $\hat{B}_{,\sigma\tau}(x)$ are not components of tensors, i.e., with the coordinate transformation, they are not transformed, as it should be for tensors. On the other hand, with respect to the mapping $\breve{g}^{\mu\nu}(x)$, they behave like covariant components of tensors. Until now, there is no well-established term in GR for these quantities. We will use the term "paratensor" (alleged tensor). Genuine tensors in GR are the so-called covariant derivatives $\hat{B}_{;\sigma}(x)$, $\hat{B}_{;\sigma\tau}(x)$.

Differentiating both parts of Eq. (37) with respect to x^{σ}, we obtain

$$\left(\breve{g}^{\mu\lambda}_{,\sigma}(x)\right)\hat{g}_{\lambda\nu}(x) = -\breve{g}^{\mu\lambda}(x)\left(\hat{g}_{\lambda\nu,\sigma}(x)\right).$$

Multiplying both parts of this equality by $\breve{g}^{\nu\gamma}(x)$ and using Eq. (37), we obtain

$$\breve{g}^{\mu\nu}_{,\sigma}(x) = -\breve{g}^{\mu\lambda}(x)\hat{g}_{\lambda\kappa,\sigma}(x)\breve{g}^{\kappa\nu}(x).$$

The performed calculation is similar to the corresponding calculation in the book [41]. We have only consistently paid attention to the difference between observables and linear mappings of observables.

In the general theory of relativity, much attention is paid to the so-called coordinate-free description of a physical system. With this description, the relations are established between the observed quantities that do not depend on the particular choice of the reference frame in the four-dimensional spacetime $\mathfrak{M}$.

In our approach, observables are the elements of an algebra. In other words, observables are the quantities that satisfy the definite algebraic relations. These rela-

tions, naturally, do not depend in any way on the choice of coordinates in the space $\mathfrak{M}$.

Another important characteristic of a physical system is the elementary state. The elementary state assigns to each observable its value recorded by the measuring instrument. For a physical system that is in a definite elementary state, different measuring instruments for the same observable can record different values. This is analogous to the fact that in GR the values of observables can depend on the frame of reference. Therefore, in our approach, we will identify the reference system with some system of classical measuring instruments.

As pointed out in Section 3, an elementary state in the quantum case is an infinite set of functionals on the set of quantum observables. The former set fixes the measurement results obtained by the instrument of any type. In this sense, the elementary state does not depend on the choice of measuring instrument. In Section 3, we dealt with observables of matter in the absence of a gravitational field. Now let us see what changes should be made when the gravitational field is considered.

These changes will be of two kinds. First, the gravitational observables will appear. Therefore, the functionals characterizing elementary states of the physical system (or subsystem) under consideration must also be defined on gravitational observables. Second, the collection of instruments that can be used to measure both gravitational observables and observables of matter will be significantly extended. Now we must consider that the measuring device can move arbitrarily relative to the physical system under study. Each such device (system of devices) should be associated with its own functional in the elementary state. Thus, the set of functionals that make up each individual elementary state should be significantly extended. After the inclusion of such functionals in the elementary state, the latter will describe the results of measurements for any system of devices. This means that such an extended elementary state does not depend on the system of devices. Therefore, we can assume that the elementary state is an objective physical reality determined only by the system under study.

Another thing is the recorded values of observables. They depend both on the system under study and on the system of measuring instruments. Therefore, they are not an objective characteristic of the system under study. In this regard, a commonly used statement of this type seems unfortunate. An object moving at a high constant speed experiences the Lorentz contraction. This statement gives rise to the illusion that some objective changes are taking place in the moving object. In reality, no change occurs in the object itself, it merely interacts differently with measuring instruments that move relative to it at different speeds.

In other words, the dimensions of the moving object are not its objective characteristic. In this regard, we can recall the nearly forgotten term *mass of motion*, which is also not an objective characteristic. By contrast to this, *mass* (*rest mass*) is such a characteristic. In the terms used here, we can say that the mass is a stable observable for physically realizable elementary states of the system under study.

A stable observable, as defined in Section 3, is an observable on which the various functionals that enter into a given elementary state take the same value. As applied to GR, the concept of a stable observable can be extended. It can include all observables that are tensors. Of course, the result of measuring each component of a tensor observable may depend on the system of measuring instruments used (on the frame of reference). However, knowing the values of all components found by one instrument, we can unambiguously predict the result of measuring these components if we use another instrument associated with the first instrument by the corresponding coordinate transformation. In this case, we can assume that the tensor observable, considered as a whole, is stable. Accordingly, it can be assumed that the value of this observable (the totality of the values of all its components) is an objective reality.

Observables that have a structure of paratensors should not be considered stable. The values of their components in the same elementary state, found using one system of measuring instruments, cannot be unambiguously recalculated into values that can be found using another system of instruments. To be more precise, this recalculation is possible. However, for its implementation, a complete history of the transition from one instrument system to another is required. We can synchronize the instrument readings at different points of the space $\mathfrak{M}$. However, the result of this synchronization may depend on the path connecting these points.

9. TENSORS OF THE 4-MOMENTUM

Using the standard procedure (see, e.g., [41, 42]), from a metric tensor and its derivatives other basic gravitational observables can be constructed:

Christoffel symbols

$$\hat{K}_{\mu\nu\sigma} = 1/2\,(\hat{g}_{\mu\nu,\sigma} + \hat{g}_{\mu\sigma,\nu} - \hat{g}_{\nu\sigma,\mu}),$$

curvature tensor

$$\hat{R}_{\mu\nu\rho\sigma} = 1/2\,(\hat{g}_{\mu\sigma,\rho,\nu} - \hat{g}_{\nu\sigma,\rho\nu} - \hat{g}_{\mu\rho,\sigma\nu} + \hat{g}_{\nu\rho,\sigma\mu}) + \breve{g}^{\alpha\beta}(\hat{K}_{\alpha\mu\sigma}\hat{K}_{\beta\nu\rho} - \hat{K}_{\alpha\mu\rho}\hat{K}_{\beta\nu\sigma}),$$

Ricci tensor

$$\hat{R}_{\nu\rho} = \breve{g}^{\alpha\beta}\hat{R}_{\alpha\nu\rho\beta},$$

scalar curvature

$$\hat{R} = \breve{g}^{\alpha\beta}\hat{R}_{\alpha\beta},$$

Einstein–Hilbert tensor

$$\hat{G}_{\mu\nu} = -\frac{1}{8\pi}\left(\hat{R}_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu}\hat{R}\right),$$

covariant derivative of the vector

$$\hat{B}_{\rho;\sigma} = \hat{B}_{\rho,\sigma} + \breve{g}_{\mu\tau}\hat{K}_{\rho\sigma\mu}\hat{B}_{\tau}.$$

According to the Bianchi identity, for this tensor, the following equality is true:

$$\hat{G}^{;\nu}_{\mu\nu}(x) = 0. \tag{38}$$

By virtue of the definition of a physical system that we have adopted, we can consider a domain $\mathbb{O}$ in space $\mathfrak{M}$, together with observables of matter determined on it, gravitational observables $\hat{g}_{\mu\nu}(x)$ and their first two derivatives, as a certain physical system. From this system, we can select two subsystems. The elements of the first subsystem are the points of the domain $\mathbb{O}$ together with the algebra of gravitational observables. This algebra is commutative and belongs to the center of the algebra containing all observables. Accordingly, this subsystem will be classical.

The elements of the second subsystem are the points of the domain $\mathbb{O}$ together with the algebra $\mathfrak{A}(\mathbb{O})$ generated by observables of matter which are defined on the points $x \in \mathbb{O}$. In the general case, the algebra $\mathfrak{A}(\mathbb{O})$ is noncommutative. Accordingly, the second subsystem will be a quantum one. For the terminology not to become more difficult, we will call observables belonging to algebras $\mathfrak{A}(\mathbb{O})$ as quantum observables. Although, among them there may be observables belonging to the center of algebra $\mathfrak{A}$.

The first subsystem will be called a gravitational field. It is natural to call the second subsystem the matter. These two subsystems are mutually open and interact with each other. The basic equation of general relativity is precisely intended to describe this interaction. Outwardly superficially, this equation looks very simple:

$$\hat{G}_{\mu\nu}(x) = \hat{T}_{\mu\nu}(x). \tag{39}$$

Here, $\hat{T}_{\mu\nu}(x)$ is the 4-momentum tensor of matter.

The basic equation of GR, written in the form (39), is an equation that relates the observables of two subsystems that are localized in the same domain $\mathbb{O}$. If both subsystems are described in a classical way, then Eq. (39) is not internally inconsistent. However, it turns out to be contradictory if we want to describe the matter in terms of quantized fields, and the curvature of spacetime in classical terms.

In quantum field theory, the 4-momentum tensor of matter is described in terms of an operator-valued generalized function. To take this fact into account, Eq. (39) could be rewritten as

$$\int dx f(x)\hat{G}_{\mu\nu}(x) = \int dx f(x)\hat{T}_{\mu\nu}(x),$$

where $f(x)$ is the arbitrary readonably good (basic) function. However, this is not enough, since in this equality the operator is equated with a number.

As already mentioned in Section 8, at present, the main hopes for resolving the contradiction that has arisen are associated with attempts to quantize the spacetime. However, this requires a new revolution in physics with a very unclear outcome. Except that, one can try to do without such radical changes.

Recall that modern physics already has an experience in describing the interaction between the quantum and classical systems. The first thing that comes to mind is the motion (scattering) of quantum particles in classical fields. However, this description does not consider the counteraction of the quantum particles on classical fields. Meanwhile, there is a process in which this impact is the main goal of the study. This process is the interaction of quantum particles with a classical measuring instrument. It can be attempted to use the here-gained experience to remove the contradiction that arises in the direct use of Eq. (39) in the case when the matter is described in a quantum way.

We will consider the gravitational field associated with the domain $\mathbb{O}$ as a classical measuring instrument designed to measure the value of the 4-momentum tensor of matter that is localized in this domain. Accordingly, the value of tensor $G_{\mu\nu}$, as the reading of this instrument. In this case, Eq. (39) can be rewritten in the form

$$\int dx f(x)G_{\mu\nu}(x) = \varphi\left(\int dx f(x)\hat{T}_{\mu\nu}(x)\right). \tag{40}$$

On the left-hand side of Eq. (40), $G_{\mu\nu}(x)$ should be considered not as a classical observable (Einstein–Hilbert tensor), but as a value of this observable at the point x. On the right-hand side of Eq. (40), there is a value of the quantum observable (4-momentum tensor) in the elementary state φ. This is the elementary state in which the subsystem under consideration is located.

Equation (40) should not be confused with the equation

$$G_{\mu\nu}(x) = \langle\hat{T}_{\mu\nu}(x)\rangle, \tag{41}$$

which is widely used in the so-called semiclassical theory (see, e.g., [43]).

In this theory, the gravitational field is considered classical, and matter fields are treated in a quantum way. In this case, the Einstein–Hilbert tensor is associated with the average value $\hat{T}_{\mu\nu}(x)$ of the energy-momentum tensor over a quantum state. Equation (41) can be obtained using the variational principle from a

definite *effective action*. In some cases, this effective action can be useful.

Equation (41) is suitable for describing the interaction of a gravitational field with a material object, which can be considered as a quantum ensemble of some more elementary objects. To describe the latter, the quantum theory can already be used. However, it is desirable to have a theory in which the matter fields are consistently described in a quantum way from the very beginning.

In contrast to Eq. (41), Eq. (40) is suitable for describing an *individual* event of the interaction of a quantum object with a gravitational field. An example of this event is the birth of our Universe. We can hardly hope to collect statistics on these events.

There is one more significant difference in Eqs. (40) and (41). An individual event can occur here and now. Therefore, with the help of Eq. (41), it is possible to describe local events. For GR, this is very important. In the case of Eq. (41), it is about average values. To get average values, one has to collect statistics. To this end, the measurements can be simultaneously made in different places, the measurements can be made in one place, but at different moments of time. In any case, we are deprived of the possibility of a local description of the physical system.

Unlike Eq. (39), Eq. (40) is not an equation of motion for observables. In Eq. (38), the observables $\hat{G}_{\mu\nu}(x)$ and $\hat{T}_{\mu\nu}(x)$ can be considered independent. Equation (40) is an equation for an elementary state. Of course, it defines the elementary state far from uniquely. This equation is satisfied by a whole class of equivalence in the set of elementary states. The situation is typical for quantum systems.

In the standard approach to quantum mechanics, it is typically assumed that if, as a result of the interaction of a quantum system with a classical measuring instrument, the definite result is recorded for some observable, then after this measurement, the quantum system goes into a quantum state in which this observable has a recorded value. This statement is even fixed in the projection principle [1]. Strictly speaking, this is not necessarily the case. The measuring instrument is not necessarily at the same time the device that prepares the quantum state. However, in the case under consideration, this is what happens most likely.

We will assume that the classical subsystem (gravitational field) does not simply measure a value of the quantum observable $\hat{T}_{\mu\nu}(x)$, but simultaneously prepares a quantum state with the definite value of this observable. Then the reverse action of the classical subsystem on the quantum one will lead to the fact that the quantum subsystem will turn out in the elementary state $\varphi = [\varphi_\xi]$ that is stable on the observable $\hat{T}_{\mu\nu}(x)$. In this case, on the right-hand side of Eq. (40), as $\varphi(\cdot)$ the functional $\varphi_\xi(\cdot) \in \varphi$ can be taken, which is the character of any of the commutative subalgebras $\mathfrak{O}_\xi$ containing the observable $\hat{T}_{\mu\nu}(x)$. Then the results of the measurement of the classical observable $\hat{G}_{\mu\nu}(x)$ and the quantum observable $\hat{T}_{\mu\nu}(x)$ will coincide when using any suitable measuring instruments. This corresponds better to the meaning of the basic GR equation.

If the assumption made is correct, then this means that the observable $\hat{T}_{\mu\nu}(x)$ is preferred with respect to all other observables. Its value is a physical reality inherent in a quantum object and does not depend on the instrument used to measure it.

If indeed the value of the 4-momentum tensor of matter is a physical reality inherent in a quantum object, then it is possible to go from writing the basic GR equation in the form of (40) (in terms of generalized functions) to the form

$$G_{\mu\nu}(x) = T_{\mu\nu}(x), \tag{42}$$

which is used commonly in GR. It is only necessary to keep in mind that now this equation relates not two observed quantities, but physically realizable numerical values of these quantities.

One should consider as well that in the quantum case we cannot experimentally fix in which elementary state the quantum system under consideration is located. At the most, we can find out what quantum state it is in. If this quantum state is an eigenstate of the observable $\hat{T}_{\mu\nu}(x)$, then the value $T_{\mu\nu}(x)$ is uniquely fixed. It is with this value, Eq. (42) should be solved. If the fixed quantum state is not an eigenstate for the observable $\hat{T}_{\mu\nu}(x)$, then before using Eq. (42), we must find the probability of each of the values of the observable $\hat{T}_{\mu\nu}(x)$. After that, we can solve Eq. (42) for various values. The resulting solutions will describe the physical reality with the found probabilities.

If the matter is described classically, then the elementary state is determined by a single functional. Therefore, in the proposed approach, Eq. (39) remains valid. In this case, we can assume that Eq. (39) relates two observables. However, this will be true only when we associate each of the observables with a function whose values coincide with the physically realizable values of this observable. In other words, Eq. (39) remains valid only in the preferred representation of observables. This is usually taken for granted, but this cannot be automatically transferred to the quantum case.

It is instructive to approach the derivation of the basic GR equation from the other side, using the principle of stationary action. We start from the following formulation.

The actual evolution of a physical system occurs in such a way that a *value* of the variation of action is equal to zero.

Let us take a look at the word *value*. It is often omitted. It will play a key role for us. In a standard way (see, e.g., [41]), we assume that the action $\hat{S}_m$ of a physical system consists of two parts: the matter part of the action and the gravitational part $\hat{S}_g$:

$$\hat{S}_m = \int \hat{L}_m(x)\sqrt{-\hat{g}(x)}dx,$$

where $\hat{g}(x)$ is the determinant of a metric tensor, and $\hat{L}_m(x)$ is the scalar function of the quantum observables and the mapping $\breve{g}^{\mu\nu}$.

$$\hat{S}_g = \frac{1}{16\pi}\int \hat{L}(x)\sqrt{-\hat{g}(x)}dx.$$

Denote by $\hat{q}$ the quantum observable, which plays the role of a generalized coordinate. The gravitational action does not depend on $\hat{q}$, therefore, with its variation, only $\hat{S}_m$ will change. Accordingly, to find a value of the variation of action, it suffices to consider the quantum subsystem. Then, from the principle of the stationary action, we have

$$\varphi_\xi(\delta_q\hat{S}) = \varphi_\xi\left(\int \frac{\delta\hat{L}_m(x)\sqrt{-\hat{g}(x)}}{\delta\hat{q}(x)}\delta\hat{q}(x)dx\right) = 0. \quad (43)$$

We assume that $\delta\hat{q}(x) \neq 0$ only in a small region around a point y in the space $\mathfrak{M}$ and belongs to the center of the algebra of observables. Considering that $\varphi_\xi(\cdot)$ is a linear multiplicative functional, Eq. (43) can be rewritten as

$$\varphi_\xi\left[\frac{\partial\hat{L}_m(x)\sqrt{-\hat{g}(x)}}{\partial\hat{q}} - \left(\frac{\partial\hat{L}_m(x)\sqrt{-\hat{g}(x)}}{\partial\hat{q}_{,\mu}}\right)_{,\mu}\right] = 0.$$

If the point y is not fixed, then to describe the physically realizable elementary states of the quantum subsystem, all characters $\varphi_\xi(\cdot)$, restricted to the subalgebra of quantum observables, are admissible. By virtue of Postulate 10, from here, it follows

$$\frac{\partial\hat{L}_m(x)\sqrt{-\hat{g}(x)}}{\partial\hat{q}} - \left(\frac{\partial\hat{L}_m(x)\sqrt{-\hat{g}(x)}}{\partial\hat{q}_{,\mu}}\right)_{,\mu} = 0. \quad (44)$$

Thus, from the principle of the stationary action for quantum observables, equations of motion are obtained. In these equations, gravity acts as an external classical field, while gravitational observables are represented in the form of their values in the considered elementary state.

Consider the variation $\delta\hat{S}_m$ with the transformation of coordinates $x^\mu \to x'^\mu = x^\mu + \zeta^\mu$, where ζ^μ are small quantities. Since $\hat{S}_m$ is a scalar, the variation must be zero. Repeating the calculations of [41], we obtain

$$\hat{T}^{;\nu}_{\mu\nu}(x) = 0, \quad (45)$$

where

$$\hat{T}_{\mu\nu} = \frac{2}{\sqrt{-\hat{g}}}\left[\left(\frac{\partial\hat{L}_m\sqrt{-\hat{g}}}{\partial\breve{g}^{\mu\nu,\tau}}\right)_{,\tau} - \frac{\partial\hat{L}_m\sqrt{-\hat{g}}}{\partial\breve{g}^{\mu\nu}}\right]$$

is the 4-momentum tensor of matter.

A similar calculation can be done for the variation $\delta(\hat{S}_m + \hat{S}_g)$ and it can be obtained that the total 4-momentum tensor $\hat{\Theta}_{\mu\nu} = \hat{T}_{\mu\nu} - \hat{G}_{\mu\nu}$ satisfies the equation

$$\Theta^{;\nu}_{\mu\nu}(x) = 0. \quad (46)$$

Note that our Eq. (46) is not a consequence of Eq. (40) and the Bianchi identity (38). In particular, this is due to the fact that, for noncommuting observables, the elementary state does not have the linearity property.

Let us consider the variation $\delta(\hat{S}_m + \hat{S}_g)$ in varying the gravitational observable $\hat{g}_{\mu\nu}$. Now we must use the principle of stationary action, i.e., must consider that

$$\varphi(\delta(\hat{S}_m + \hat{S}_g)) = 0 \quad (47)$$

for a physically realizable elementary state φ.

In contrast to q- and c-numbers, the operation of addition is defined for all elements of the C^*-algebra. Therefore, we can again use the calculations of [41]. It is necessary to consider only Eq. (47) and the fact that the tensor $\hat{\Theta}_{\mu\nu}(x)$ is a generalized function. As a result, we arrive at the equation

$$\varphi\left(\int dx f(x)\hat{\Theta}_{\mu\nu}(x)\right) = 0, \quad (48)$$

which coincides with Eq. (40). It follows from this equation that the total 4-momentum tensor of each domain of the space is equal to zero. This is the law of conservation of energy and momentum in our approach. The law is satisfied not only globally, but also locally.

On the other hand, if the total 4-momentum tensor is defined as the sum of the tensor $\hat{T}_{\mu\nu}(x)$and the 4-momentum paratensor $\hat{t}_{\mu\nu}(x)$, then the usual proof of conservation of the total 4-momentum fails when $\hat{T}_{\mu\nu}(x)$ is a quantum observable. Instead of the conservation of this tensor, conservation law of another paratensor can be proven.

Let us introduce the paratensor

$$\hat{Q}_{\mu\nu}(x) = -\hat{t}_{\mu\nu}(x) - \hat{G}_{\mu\nu}(x).$$

This tensor consists only of gravitational observables. Therefore, it can be treated as a classical observable. Then, repeating the calculations from [41], in which the tensor $\hat{T}_{\mu\nu}(x)$ is replaced by the tensor $\hat{G}_{\mu\nu}(x)$, we obtain

$$\left(\hat{g}(x)\hat{Q}_{\mu\nu}(x)\right)^{,\nu} = 0. \quad (49)$$

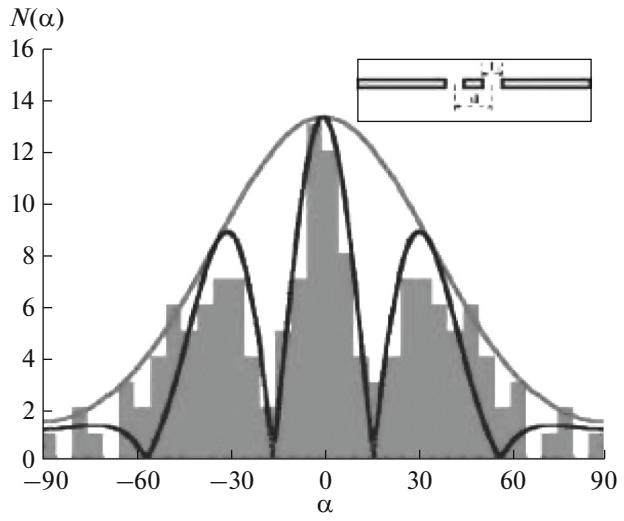

Fig. 1. Histogram describing the statistics of the crossing of bouncing oil droplets through a baffle with two slits.

This equality holds identically. Therefore, it is valid for any elementary state. On the other hand, since $\hat{g}(x)\hat{Q}_{\mu\nu}(x)$ is a paratensor, then its value is not uniquely determined by the elementary state. In our terminology, this observable is not stable on a physically realizable elementary state. The result of its measurement depends on the measuring instrument used. It is noteworthy that each fixed instrument does not notice the evolution of this observable in time. In this sense, this observable is hidden.

The division of the 4-momentum tensor $-\hat{G}_{\mu\nu}(x)$ of the gravitational field into two parts $\hat{Q}_{\mu\nu}(x)$ and $\hat{t}_{\mu\nu}(x)$ is ambiguous. The measurement result of each of these parts is not an objective reality but depends on the type of measuring instrument. This is very reminiscent of the division of internal energy in thermodynamics into heat and work. In our case, the role of heat is played by $\hat{Q}_{\mu\nu}(x)$, while the role of work, by $\hat{t}_{\mu\nu}(x)$.

10. EXPERIMENT WITH A WANDERING DROP

The introduction of a concept of "elementary state" makes it possible to consider gravitational and quantum phenomena from unified positions. However, before discussing quantum effects, it is useful to consider a very curious classical experiment that models many of the specific features of quantum experiments. We mean the experiment with a wandering droplet carried out by a team of French researchers [44, 45].

The experimental set up included a flat tank filled with silicone oil, which was subjected to vibration in the vertical direction with a characteristic acceleration greater than the acceleration due to gravity. On the surface of the silicone oil, a droplet of the same oil, about one millimeter in diameter, fell. Due to vibration, the droplet bounced off the surface without sticking to the oil bulk. In this jumping state, the droplet could exist indefinitely.

With each impact on the surface, the droplet generated a surface wave. The conditions were selected in such a way that, during each fall, the drop fell on the outer falling side of the hump of the surface wave that was generated by the previous fall of the drop. When rebounding from an inclined surface, the drop acquired a speed both in the vertical and horizontal directions. As a result, a complex system was formed, which consisted of a surface wave and a bouncing drop that was coherent with it and moved in a horizontal direction. The authors called this system a walker.

Several types of experiments were carried out with the walker: the reflection of a walker from the wall, the interaction of two walkers, the diffraction of a walker on one slit, the interference of a walker on two slits. Perhaps the most curious was the experiment with interference.

This experiment was organized as follows. The oil tank was partitioned by a metal strip lying at the bottom of the tank. The thickness of the oil layer above the strip was so small that the surface wave above it was attenuated. As a result, the movement of the bouncing drop over the strip in the horizontal direction halted. Having made two transverse passages in the strip, 8 mm wide and 15 mm apart, the experimenters organized a two-slit experiment with a walker.

In the experiment, the high-speed video recording was carried out to constantly monitor the drop and the surface wave. With the help of this survey, it was established that the droplet passed through one of the slits, while the surface wave, through both. Behind the barrier, both parts of the wave interfered with each other, while the wandering drop interacted already with the resulting wave and, when moving in the horizontal direction, deviated from the perpendicular to the partition by an angle α. This angle was measured, after which the experiment was repeated from the very beginning with another drop.

Under the same initial conditions, the deflection angles α of different droplets vary strongly and, it would seem, chaotically. However, the most surprising was the result of statistical processing of the data obtained. This result is shown in Fig. 1.

In this figure, it is easy to recognize a typical interference pattern. It can be fitted with good accuracy by scattering a plane wave on two slits. Though, of course, the droplets could not form any physical wave since they existed at different time intervals. In other words, in this case, the interference pattern is a purely statistical effect. It can be said that jumping droplets form a wave of probabilities.

The probability distribution is most significantly determined by the type of ensemble of events under

consideration. In this case, the experimenters selected the events in which the droplets initially moved perpendicular to the partition. In addition, the experimenters tried to equalize the density of the number of droplets crossing the slits along the partition.

In the droplet experiment, a decisive factor in the emergence of a wave of probabilities was the coherent nature of the interaction of each drop with its accompanying surface wave. As a result, a strongly coupled system—"walker"—emerged. The strong coupling manifested itself by the fact that precisely the droplet–surface wave interaction determined the droplet trajectory in the horizontal direction.

The results of this experiment are strikingly similar to results of corresponding experiments in the microcosm, in particular, to the electron scattering by two slits. This suggests the idea: to try to find an analog of a walker in experiments with an electron. Using this analogy, we will try to modify the current model of the atom.

11. SOFT AND SUPERSOFT PHOTONS

Rutherford's planetary model is the precursor of the modern model of the atom. According to this model, electrons, like planets, revolve around the nucleus, being kept in orbit due to the Coulomb force of attraction to the nucleus. Rutherford's model is very visual. However, according to the laws of classical electrodynamics, an electron by moving along a curvilinear orbit must constantly radiate energy and fall onto the nucleus very quickly. This shortcoming of Rutherford's model was overcome by Bohr, who supplemented the model with two postulates.

First, an electron can only revolve around the nucleus in definite stationary orbits. In this case, it does not radiate energy. Second, under the action of an external perturbation, an electron can jump from one orbit to another. In this case, it emits or absorbs the strictly defined portions of energy. In the context of modified planetary model, Bohr managed with a good accuracy to calculate the allowable energy levels for the hydrogen atom. However, rather a high price had to be paid for this: the model largely lost its visualization, since Bohr gave no physically visual justification for his postulates.

In the Bohr model, the electron was considered as a point particle moving in definite orbits. However, this point-like nature of the electron and the determination of orbits have become an insurmountable obstacle to the further development of the atomic model. Therefore, at the next stage of model modification, it was decided to replace the concept of "particle" with "wave function," and "orbit" with "orbital." In this way, it was possible to achieve a good agreement between the results of calculations and a large amount of experimental data. However, the model has almost completely lost visibility. In it, the description of the electron motion is replaced by the postulation of a definite equation for the evolution of vectors in a Hilbert space. The model has actually turned from a physical one into a mathematical one.

Here we will try to return to a more physical method of building new models. Namely, it is proposed to give an overview of various physical phenomena, to try to find common features in them. After that, it is proposed to transfer the experimentally established regularities in one of the compared phenomena to another phenomenon. At the same time, it is proposed to choose a mathematical apparatus that would be suitable for describing comparable physical phenomena.

Specifically, we will compare the structure of the atom and the structure of the Universe. This comparison can be very useful. We can experimentally study the Universe from the inside, and the atom from the outside. In addition, we can experimentally investigate intermediate objects—macroscopic bodies—and study them "from the side." The data obtained can significantly complement each other.

We will proceed from the idea that the fundamental properties of the matter do not depend on the scale of the object under study. Based on this idea, we will try to modify the Rutherford–Bohr model.

According to current concepts, the larger part of the mass in the Universe is concentrated in the dark matter and dark energy. However, what dark matter and dark energy consist of is not currently established. The various (and often highly exotic) candidates for these roles are offered. In practice; it can only be considered as the firmly established fact that neither one nor the other is recorded by modern measuring instruments.

At the same time, objects with suitable properties are well known in modern physics. These objects are soft and supersoft photons. The presence of undetectable soft photons is necessary for a consistent description of the well-studied process of elastic scattering of an electron by a nucleus. When calculating the cross section of this process within perturbation theory, the infrared divergences appear. A very efficient method has been developed in quantum field theory to overcome this difficulty.

The method is as follows (see, e.g., [46]). Experimentally, the process of elastic scattering (Fig. 2a) cannot be separated from the process of electron scattering with the emission of soft bremsstrahlung photons (Fig. 2b), if the energy of these photons is below the sensitivity threshold of the instruments used. When this process is considered, infrared divergences are compensated. With an increase in the order of perturbation theory, the number of bremsstrahlung photons taken into consideration should be increased.

Thus, although it seems to us that one electron emerges from the scattering region, in reality, an electron, surrounded by a cloud of soft photons, flies out.

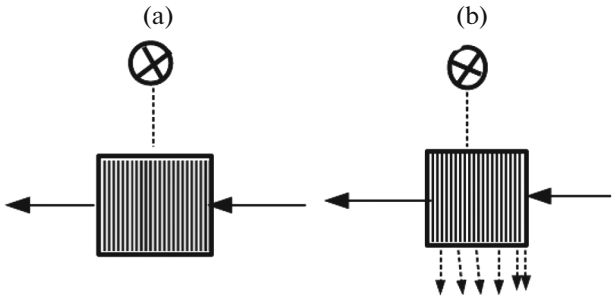

Fig. 2. Feynman diagrams describing the (a) elastic electron scattering and (b) bremsstrahlung. Solid arrows represent electrons, dashed arrows represent photons.

These photons are not virtual, but quite real. A similar situation also takes place in other processes. With each act of scattering, part of the energy of scattering particles is spent on the production of bremsstrahlung photons. As a result, more and more photons are produced, and the energy of each of them, on average, is decreasing. Under normal conditions, processes of this type are dominant. They are in good agreement with the second law of thermodynamics.

As a result of such processes, the soft and supersoft photons should be produced in great numbers and they are quite suitable for the roles of dark matter and dark energy. In principle, the presence of other components in the dark energy and matter is also possible, but we will focus on the contribution of soft and supersoft photons. Further, we will call these photons dark. The specific values of the required frequencies of mentioned photons are, of course, conventional and may depend on the problem under consideration. As the upper limit, $\omega = 10^{-3}$ Hz can be taken (conditional lower limit of the observed radio frequencies).

Dark photons can also act as a tool for thermalization of the system under consideration. Let us explain the latter with a simple classical example. Let the thermodynamic properties of a gas in a vessel be studied. This gas can often be considered as ideal, and it can be assumed that the gas molecules do not interact with each other and are elastically reflected from the vessel walls. However, to explain a reason for the emergence of thermodynamic equilibrium in the gas, we must assume that the molecules of this ideal gas actually interact (secretly for us) with each other and with the vessel walls.

In each specific process, many bremsstrahlung photons are produced, but they carry away a very small portion of the energy. Therefore, it should be expected that they must be well described by classical electrodynamics. However, in classical electrodynamics, the energy is carried by electromagnetic waves. On the other hand, any wave is a relatively stable and ordered object. Therefore, if we want to describe this ensemble of photons within the quantum theory, then we should consider that they are in a coherent state.

At present, the theory of quantum coherent states is well developed (see Glauber works [47, 48]). In the Glauber formalism, the vector potential of the electromagnetic field $\hat{A}$ is represented by an operator in Hilbert space, which is expanded in terms of mode functions $\mathbf{u}_k$:

$$\hat{A}(rt) = \sum_k \sqrt{\frac{\hbar}{2\omega_k}}\left[b_k^- \mathbf{u}_k(r)e^{-i\omega_k t} + b_k^+ \mathbf{u}_k^*(r)e^{+i\omega_k t}\right].$$

Here, the index k is a multi-index determined by both the projectors of wave vector $\mathbf{k}$ and the wave polarization $\mathbf{u}_k$. Since it is much more convenient to work with discrete indices than with continuous ones, then it is considered that the system occupies a large but finite volume. The functions $\mathbf{u}_k$ are solutions to the wave equation

$$\nabla^2 \mathbf{u}_k + \omega_k^2 \mathbf{u}_k = 0$$

with the corresponding boundary conditions, form a complete orthonormal set and satisfy the transversality requirement.

The quantum mechanical properties of $\hat{A}$ are completely fixed, by postulating that the operators $b_k^\pm$ satisfy the canonical permutation relations

$$[b_k^-, b_{k'}^+] = \delta_{kk'}, \quad [b_k^\pm, b_{k'}^\pm] = 0.$$

After this, each mode can be considered separately.

The state of the electromagnetic field in the Glauber formalism is described in terms of coherent states $|\beta_k\rangle$ that satisfy the relation

$$b_k^- |\beta_k\rangle = \beta_k |\beta_k\rangle,$$

where β_k is a complex number $\beta_k = |\beta_k|\exp(i\phi_k)$.

A state $|\beta_k\rangle$ can be considered as a Hilbert space vector representing a tagged wave, for which $|\beta_k|$ is a parameter characterizing the wave amplitude, while ϕ_k is a parameter characterizing the position of some tag that fixes the wave phase.

A simple illustrative example of a tagged wave is a wave on the surface of water in which a float is swimming. This float can be considered as a tag that marks a definite phase of the wave at each moment of time. A characteristic property of a float is that it moves coherently with the wave.

Returning to the considered process of electron scattering on the nucleus, accompanied by the emission of soft photons, we can notice a very strong similarity with this example. Especially if we assume that the electron moves coherently with the wave of soft photons. Since the scattered electron and the soft bremsstrahlung photons are produced in a single physical process, this assumption looks quite natural.

If the introduced vector $\mathbf{k}$ is related to the electron momentum ($\mathbf{k}\hbar$ is the momentum of the electron),

then it is easy to recognize the de Broglie wave in the tagged wave. It must be said that de Broglie himself did not state that a quantum particle is a wave. He asserted (see, e.g., [49]) that a particle is *associated* with a wave with frequency ω_k. By the way, this statement is physically much more natural than the assertion that an electron is both a wave and a particle simultaneously. Of course, all the reasoning can be repeated by considering the scattering of the observed photon on the nucleus. In this case, we will arrive at a tagged wave, in which the observed photon will serve as a tag.

We want to find common features in the structure of an atom and the Universe. It is known in advance that electromagnetic forces and quantum phenomena play an important role in the structure of the atom. Electromagnetic forces also play an important role in the structure of the Universe. In addition, gravity plays a huge role. Therefore, it is highly desirable to consider electromagnetic forces, quantum phenomena and gravity from a unified standpoint.

As noted earlier, the algebraic approach allows us to provide this consideration. Recall that the Born rule automatically arises in it. It is possible to look upon this rule also from this point of view. Born's rule reflects the experimental situation in quantum processes. When quantum systems interact with classical instruments that perform reproducible measurements of quantum observables, the quantum system automatically passes into the corresponding quantum state, in which the distribution of values of quantum observables obeys the Born rule.

Here the situation is very similar to that which takes place in thermodynamics. The zeroth law of thermodynamics states (see, e.g., [50]): "For each thermodynamic system, there is a state of thermodynamic equilibrium, which it spontaneously reaches under fixed external conditions over time." It suffices to replace the definition "thermodynamic" with "quantum" and the words "fixed external conditions" with "reproducible measurement of quantum observables."

This means that quantum states have a preferred property—quantum equilibrium (it can be said: stability). Therefore, we can consider that the Postulate (QM) (see Section 5) reflects the stability property of the ensembles corresponding to the equivalence classes. Otherwise, we can say that the average value of the observable $\hat{B}$, which satisfies the relation

$$\Psi_{\varphi_\eta}(\hat{B})\tilde{p}_{\varphi_\eta} = \tilde{p}_{\varphi_\eta}\tilde{B}\tilde{p}_{\varphi_\eta}, \qquad (50)$$

corresponds to a stable quantum distribution.

In (50), $\tilde{p}_{\varphi_\eta}$ is a certain one-dimensional projector in the Hilbert space, in which the exact representation of the algebra is implemented, while $\tilde{B}$ is the operator in this representation which corresponds to $\hat{B}$.

The presence of incompatible observables in quantum systems significantly affects the procedure for describing the dynamics of these systems. The dynamics of a classical system in the ideal case can be described by Newton's equations. This means that for describing this system, it is sufficient to know all the forces that act in the system and the full set of boundary conditions. The initial conditions can be considered as an integral part of this set. Definitely, for real systems this complete set is impossible to know. There will always be uncontrollable perturbations, and, strictly speaking, it is necessary to investigate the stability of motion with respect to these perturbations.

For quantum systems, the situation is aggravated. Even in the ideal case, it is impossible to establish a complete set of the boundary (initial) conditions, since any set of measuring instruments can fix only a set of compatible observables. Therefore, the issue of motion stability becomes fundamentally important. Actually, when formulating the laws of dynamics of quantum systems, the Founders of quantum mechanics did not especially deal with the motion stability problem, but they successfully went along the path of using not Newtonian mechanics but applying Hamiltonian formalism. In this formalism, the Hamiltonian and the action play a central role. In terms of the algebraic approach, this means that these quantities must be included in the algebra of observables.

It should be noted that the Newtonian formalism is by no means always applicable to classical systems. The simplest example is a gas in a thermal bath. It is clear that it is completely hopeless to try to describe the dynamics of this system in terms of the forces of interaction between the gas molecules and the bath walls and among themselves. On the other hand, this system is quite successfully described in terms of temperature and pressure, provided that an equilibrium distribution in the gas is established. In this case, temperature and pressure should be considered as observables. At the same time, with a very small number of molecules, the Newtonian formalism, in which forces are considered as observables, is quite applicable. However, in this case, the condition of equilibrium distribution will not be realistic.

Thus, the question of which quantities should be included in the set of observables does not always have an unambiguous answer. This set may depend on the way of the system description, first of all on the approximation that is used. The algebraic approach is convenient for its flexibility. It allows us to choose the observables to be considered in various ways and to group them in various ways into the ensembles under study.

12. PHOTON BOSE CONDENSATE AND THE PROTON MODEL

In the Rutherford–Bohr model, the nucleus of an atom was considered as a structureless force center; it is now firmly established that the nucleus has a com-

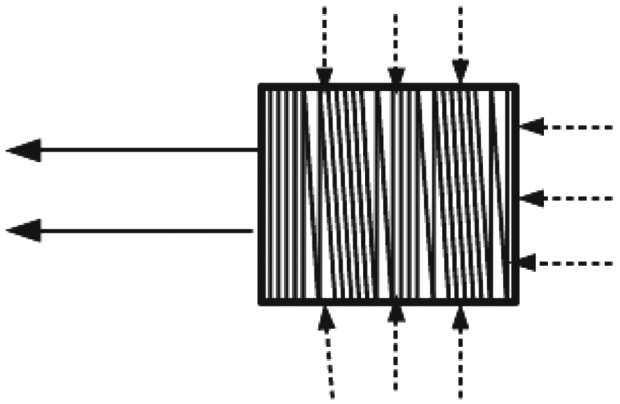

Fig. 3. Feynman diagrams describing the process of energy concentration.

plex internal structure. Therefore, before building a model of an atom, let us try to build a model of the simplest nucleus—of a proton.

According to the now accepted Standard Model, a proton consists of valence and sea quarks interacting by exchanging gluons. We will not distinguish between valence and sea quarks and combine them into a single group. These quarks are not virtual, but one must be very careful when talking about their mass, since they are involved in the strong interaction, and a concept of mass is strictly defined only for free particles.

By the way, we note that virtual particles are not physical, but mathematical objects, convenient for visualizing a series of the perturbation theory. Not more than that, therefore, it is necessary to endow them with some physical properties with even greater caution.

Quarks are fermions and cannot form a condensate of arbitrarily high density. This limitation is absent for photons. Photons, in principle, can form a structure of any density. In particular, they can form a Bose condensate.

Considering these facts, it is possible to offer a quite plausible planetary model of the proton. "Planetary" one means not in the sense of Rutherford, but in the sense of the structure of an individual planet. A proton, like a planet, consists of several spherical layers, the density of which, in accordance with the law of Archimedes, decreases as it moves from the center to the periphery.

In the proton center, there is a nucleus, in which the main mass of proton is concentrated. The nucleus appears as a Bose condensate of dark photons. Next, the mantle comes, which consists of uncondensed dark photons. Then the core comes, which involves a suspension of quarks and gluons in a sea of photons. At the very periphery, there is an atmosphere consisting of dark photons, but with a density lower than that of the core.

Numerous unsuccessful attempts were made to observe the Bose condensate of photons. Finally, one of them was crowned with success (see [51, 52]).

To facilitate the localization of photons in the microcavity, the authors of the experiment used slightly curved mirrors as cavity walls. Due to the finite curvature of the cavity walls, the photons were concentrated in the geometric center of the cavity. Photons interact with each other very weakly. Therefore, in order to obtain a thermodynamically balanced ensemble of photons, the authors filled the cavity with liquid organic paint. The equilibrium was obtained due to the balance between the processes of absorption and reemission of photons by paint molecules. Laser pumping was used to increase the number of photons in the cavity. When a definite critical (for a given temperature) density was reached, a phase transition to the Bose condensate state occurred in the ensemble of photons. This transition was observed visually: the blurred photon "cloud" was transformed into a bright narrow spot. In other words, a distinctive feature of this phase transition was a sharp change in the reflective properties of many photons concentrated in the cavity center.

Further, we will be interested in the case when the considered microcavity is filled with a very large number of very soft photons. In this case, it is easy to determine a physical cause of this phase transition.

At a very high concentration of soft photons, the process displayed in Fig. 3 may take place. This process, in a sense, is the reverse of the process shown in Fig. 2b. In the process of Fig. 3, a large number of soft photons collide and a certain number of particles are created (in the figure they are shown by solid arrows), among which there are quite energetic ones.

Inside the cavity, photons will interact with each other. Let ω be the characteristic frequency of photons and $\hbar\omega \ll mc^2$, where m is the electron mass. Then, the calculation of a cross section for the photon–photon scattering yields (see [53])

$$\sigma = 0.031\alpha^2 r_e^2 \left(\frac{\hbar\omega}{mc^2}\right)^6, \qquad (51)$$

where α is the fine structure constant, and r_e is the so-called classical electron radius:

$$r_e = 2.8\times10^{-13}\ \text{cm}.$$

Thanks to the factor $(\hbar\omega/mc^2)^6$ in Eq. (51), if a photon with a frequency greater than ω enters the cavity, then the process shown in Fig. 3 becomes more probable. In this process, photons with a frequency higher than ω will be created again. In other words, we will get a probabilistic process with positive feedback, which is typical for a phase transition.

Of course, at $\hbar\omega \ll mc^2$, the cross section given by Eq. (51) will be very small, but at a very high concen-

tration of photons, the described process can lead to the development of a chain reaction, the consequence of which will be the radiation observed in the experiment with a photon Bose condensate. We call this radiation as condensational.

For the start of a chain reaction, the initial perturbation must be strong enough. This explains why thermalization of the photon ensemble was required in the observed case. Photons of sufficiently high energy appear in this ensemble.

The details of the described phase transition are not of interest to us at the moment. Therefore, we confine ourselves to the most general statement. This transition proceeds in compliance with the energy conservation law. In other words, the first law of thermodynamics holds for this transition. With the second law of thermodynamics, this transition is not consistent since the energy concentration rather than its dissipation occurs in it.

But the second law of thermodynamics does not follow from the fundamental laws of physics. It is a conclusion made on the basis of the observation of physical systems with a not very high concentration of matter. Experimental data at a very high concentration of matter have simply not been available so far.

Now let us return to our planetary proton model. It is easy to see that in this model, the proton structure is very similar to the cavity structure in the experiment described in [51, 52]. The described cavity is bounded by curved mirrors, which are designed to concentrate photons in the cavity center. In the proton, quarks, which are concentrated in the crust, can act as a spherical mirror. The mirror is not absolutely perfect, since scattering occurs not only towards the proton center, but also towards the periphery. However, even in the experiment with the Bose condensate, the mirror was also not perfect. Therefore, a replenishment with laser photons was required.

In the proton, such replenishment is implemented by a strong gravitational field, due to which the photons located in the mantle and crust are concentrated in the region near the proton core. The gravitational field also acts on quarks. However, quarks are fermions and their concentration in a bounded cavity cannot be too high. In addition, quarks have an electric charge. Therefore, due to electromagnetic interaction, they interact quite strongly with condensational radiation, which will drive them away from the center. As a result, quarks and gluons strongly interacting with them cannot concentrate near the proton core but form a crust. Thus, it is the photons that are concentrated by gravity in the center, the density of which can become extremely high. In other words, prerequisites for the emergence of a black microhole are created in the vicinity of the core.

Thus, in the case of the proton core, we are just dealing with a very high concentration. Accordingly, the validity of the second law of thermodynamics in this case becomes unclear. In a proton, after being reflected in a quark mirror, a photon may lose energy, but may also gain it. In other words, the mirror thermalizes photons in the mantle and plays the role of paint molecules.

In other words, all the processes mentioned in [51, 52] also operate in the proton. Therefore, the result should be expected to be the same. In other words, the proton core will be a source of condensational radiation. The intensity of this radiation will be the greater, the more photons are fed to the proton core from the environment. In this case, we get a probabilistic process with negative feedback. These processes lead to a stable dynamic equilibrium.

Simultaneously, the condensational radiation makes it possible to avoid a great deal of trouble in the physical interpretation of the processes under consideration. Instead of the disappearance of matter at the so-called singular point (in a black microhole), we will deal with the transformation of the form of matter-energy. The energy moves from a less concentrated form to a more concentrated one.

It should be said that the very concept of "singular point" is not physical, it is just a mathematical abstraction. However, in the case under consideration, this mathematical concept can be completely replaced by a physical one: the "photon Bose condensate."

Now let us try to make some numerical estimates. We will consider a proton as a small Universe and use the Friedmann equations with the Λ-term (see, e.g., [54]):

$$\frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 = \frac{4\pi G}{3}(\rho + \rho_\Lambda) - \frac{kc^2}{2a^2}, \tag{52}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2} - 2\rho_\Lambda\right). \tag{53}$$

Here a is the dimensional scale factor (in our case, a distance from the center of the proton core), G is the gravitational constant, ρ and P are the density and pressure of the observed matter, c is the speed of light, k is the curvature parameter (in our case, $k = 1$), and ρ_Λ is the Λ-term. We will not associate the latter with zero oscillations of vacuum, but we will link it to the buoyancy force of Archimedes and assume that ρ_Λ is the density of the dark field. Note that in the system under consideration, in addition to the observed matter, there are dark field photons. They are not recorded by instruments, but they can influence the observed matter according to Archimedes' law.

We will be interested at what values of the parameters characterizing the system under study, it can be in equilibrium. This means that we need to find the values of these parameters for which Eqs. (52) and (53) admit solution $\ddot{a} = 0$ and $\dot{a} = 0$. Since we only need the dynamic equilibrium, it is sufficient that this solu-

tion existed for the values of the parameters averaged over a small region in the space $\mathfrak{M}$.

Let us first consider the region of the core and mantle of the proton. In this area $3P = \rho c^2$. Therefore, in the equilibrium region, the following relation should hold:

$$\rho = \rho_\Lambda = \frac{3c^2}{16\pi G a^2}. \quad (54)$$

This relation can be rewritten as

$$M_a = \frac{4\pi}{3} a^3 (\rho + \rho_\Lambda) = M \frac{a}{R_G}, \quad (55)$$

where M is the proton mass, M_a is the mass of the central proton region of radius a, $R_G = 2GM/c^2$ is the so-called gravitational radius of the proton, $R_G = 2.5\times10^{-52}$ cm. The most natural physical interpretation of Eq. (55) is as follows. The region of radius a is the proton core, this region is dynamically balanced, it discards the excess mass (entering it due to the gravity) in the form of condensational radiation. For parameters, it holds: $M_a \sim M$, $(M_a < M)$, and $a \sim R_G$.

Now, consider the region of the proton core. In this region, in addition to photons, there are charged particles of the quark–gluon matter. We will again use Eqs. (52) and (53). However, since the available experimental data, relating to this region, deal with the electric charge distribution in this region, then we will consider a density of quark–gluon matter as a density of observables ρ in this region. Dark photons, as well as photons of condensational radiation, will be included in the Λ-term. Since in this region, the electromagnetic interaction is much stronger than the gravitational one, then the external influence on the observed quark–gluon matter will be determined by the electromagnetic interaction with the condensational radiation rather than by the buoyancy force of Archimedes.

Let in (53) the pressure be related to the density by the relation $P = b\rho c^2$. Since we will evaluate the quantities only by order of magnitude, the exact value of the coefficient b is not important, it is sufficient that it be less than 10. For simplicity, we set $b = 0$. Then it follows from Eq. (53) that $\ddot{a}$ will vanish at $\rho = 2\rho_\Lambda$. Substituting this value into Eq. (52) and setting $\dot{a}$ to zero, we obtain the condition of dynamic equilibrium in this region.

The experiment shows that the electric charge of the proton is concentrated in a narrow spherical layer of radius 1×10^{-13} cm. Substituting this radius as a value of a in Eq. (54), we obtain for the density of quark–gluon matter in a proton

$$\rho = \frac{c^2}{8\pi G a^2} = 6\times10^{52} \frac{g}{\text{cm}^3}. \quad (56)$$

This density can be compared with the electron density. If we assume that the electron has the same density, then its radius by order of magnitude should be equal to 10^{-27} cm. This does not contradict the current experimental data. The upper limit for the electron radius is 10^{-20} cm.

The proton stability in the vacuum is an experimentally established fact. In the proposed model, this stability is treated as a dynamic equilibrium of the physical system that simulates a proton. The key role in the proposed model is played by soft and supersoft undetectable photons and their gravitational and electromagnetic interactions. In this model, a proton has the properties of the black hole in which two oppositely directed processes take place. The first process is the concentration of supersoft photons in the central core. This concentration is provided by gravity. In turn, this concentration increases the probability of a process in which some of the soft photons are replaced by more energetic particles (condensational radiation).

The competition of these two processes takes place over the full proton size, including the proton boundary, where soft photons of the proton come into contact with soft photons of the surrounding vacuum. The vacuum in the model under consideration is not an empty space, but a space filled with soft and supersoft undetectable photons. These photons are not virtual, but quite real, only their energy is below the sensitivity threshold of the measuring instruments used. With a sufficiently high initial density of supersoft photons in the center of the core, this construction must necessarily have a point of dynamic equilibrium. The value of this equilibrium point determines the mass of the proton. Unfortunately, we cannot independently find the equilibrium point, because this value is determined by the details of the strong interaction of quarks. However, the very fact of the existence of equilibrium point does not depend on these details.

It is striking that the proposed conception of proton is very similar to the design of a system placed in a thermal bath. In this case, the role of the thermal bath is played by the "photon vacuum," which consists of photons of dark matter and dark energy. Note that Blokhintsev [55–58] was an ardent supporter of using the analogy with the behavior of a physical system in a thermal bath in quantum processes.

With minimal changes, all the arguments in this section can be used to build a model of an electron. It is only necessary to assume that there are some subelectron carriers of electric charge. In this case, just as for the proton, the gravitational and charge radii will arise by natural causes for an electron.

13. MODEL OF THE HYDROGEN ATOM

Having a model of the proton, it is relatively easy to construct a model of an atom, in any case, of hydrogen. It is necessary just to add an electron that interacts electromagnetically with a proton.

However, the locality problem immediately arises here. On the one hand, experiments on ultraprecise measurement of the magnetic moment of an electron [59] indicate that the electron is essentially a point particle. On the other hand, experiments with atoms show that the electron wave function is spread over a spherical layer, the radius of which is close to that of the atom. Contradictions of a similar type are encountered very often when the standard mathematical apparatus of quantum theory is used. To bypass them, even a special phenomenon was invented: the wave function collapse. However, any viable physical explanation for this phenomenon is usually not given. As already mentioned, the algebraic approach is much better suited for discussing local properties.

If there is an electron near a proton, then, contrary to Bohr's assumption, but in agreement with Newton, it will fall on the proton with acceleration. Again, contrary to Bohr, but in agreement with Maxwell, it will emit photons. These photons will break the dynamic balance that was earlier between the proton atmosphere and the environment. To restore the balance, a part of the photons from the atmosphere must go into the deeper layers of the proton breaking the dynamic balance in them. This way the perturbation will eventually reach the core of the proton.

Now let us follow the fate of an electron. It will fall at the quark–gluon proton shell. With some probability, it will be reflected from it, while with some probability it will break through it and rush to the proton core, breaking the dynamic balance in the deep layers of the proton. Ultimately, it can reach the Bose condensate of the proton. Thanks to the gravity and all the bremsstrahlung photons, which were generated by the electron on its way, the condensate will be overloaded with soft photons. Therefore, the process inside the proton, such as shown in Fig. 3, becomes more probable. The electromagnetic interaction between the condensate photons and the electron is much stronger than between the photons themselves. Therefore, the process of energy concentration will primarily affect namely the electron. As a result, the electron, together with condensational radiation, will leave the proton core, restoring its dynamic equilibrium. Further, like a stone thrown from the surface of the earth, the electron will move along some parabolic trajectory. Of course, on the way, it will again emit bremsstrahlung photons.

However, as was explained in the previous section, these additional photons will eventually enter the Bose condensate, by generating the condensational radiation in it. Thus, despite the emission of bremsstrahlung photons, the kinetic energy in the system will not tend to zero but will fluctuate within a definite finite interval. When the electron falls again on the condensate, it will again move along some other parabolic trajectory, and so on an infinite number of times.

Here, also as in thermodynamics, the "slow time" should be distinguished from "fast time" (see [50]). In thermodynamics, the fast time follows the particle motion, described by Newtonian dynamics. The slow time follows the change in mean values of the observables. Just as in thermodynamics, an infinitely small interval of slow time δt can be considered as infinitely large for the fast time. Therefore, it can be assumed that during this interval the electron will have time to make infinitely many dives.

When moving through the photon atmosphere of an atom, an electron will cause perturbations in it. So, waves may occur in this atmosphere. The waves can be of two types: radial and ring ones. The atom is a stable system. For its stability, it is necessary that the motions of the electron be consistent with oscillations in the waves.

Here we are faced with a situation similar to that, which arises in the previously mentioned macroscopic experiment with a wandering drop. In our model, a chain reaction in the condensate corresponds to the vibrations in this experiment, an electron corresponds to an oil droplet, soft photons correspond to the silicone oil. There is also some difference. In the macroscopic experiment, there is one vertical direction and one surface wave in the horizontal plane. In the proposed model of the atom, there are infinitely many vertical (radial) directions. Accordingly, there are also infinitely many horizontal planes, and in each of them a "surface" (ring) wave can form.

When studying the thermodynamics of a gas, it is almost impossible to follow the "Newtonian" motion of gas particles. Therefore, instead of Newtonian particles, "statistical" particles are monitored, in which the internal parameters are the same as those of Newtonian ones, but the trajectories are different. The values of kinematic parameters for statistical particles are equal to the values averaged over the measuring time of the corresponding parameters for Newtonian particles. The dynamics of statistical particles is determined not by Newtonian forces, but by thermodynamic potentials.

Like the situation in thermodynamics, it is impossible to follow the Newtonian motion of an electron. Therefore, instead of a "Newtonian" electron, the movement of a "statistical" electron can be followed. The latter has the same internal characteristics as the Newtonian electron. By that it differs from the virtual one, but its trajectory is different from the trajectory of the Newtonian electron. At the same time, at the points of a trajectory of the statistical electron, the values of its kinematic observables coincide with values of the corresponding Newtonian quantities averaged over the time interval δt (the time required to record

this value). Just as in thermodynamics, the dynamics of the statistical electron is determined not by Newtonian forces, but by potentials.

Due to the process of energy concentration (Fig. 3), all radiation losses during the accelerated motion of the Newtonian electron return back to the statistical electron. Therefore, we can assume that the statistical electron moves without these losses.

In a stable wave, oscillations must be consistent with the statistical electron momentum. In other words, a stable wave of soft photons must be coherent with a statistical electron. Thus, for the stability of an atom, it is necessary that these waves have the structure of tagged waves, as this concept was introduced in Section 11. The tag in this wave is a statistical electron with the wave vector $\mathbf{k}$, while the function $\mathbf{u}_k$, which determines the ring wave mode, must satisfy the periodicity condition. Accordingly, the length of the orbit must be equal to an integer number of wavelengths.

Under an external action on an atom, a statistical electron can move from a stable circular orbit to a quasi-stable one, by absorbing a definite portion of energy. In the quasi-stable state, the dives of the Newtonian electron will continue, causing oscillations in the photon atmosphere. As a result, a radial tagged wave will be formed, which will be emitted by the excited atom. This gives grounds to assume that the electron, like quarks, is never in a free state. It either enters into the composition of an atom, molecule, crystal, etc., or occurs as a marker in a tagged wave. Therefore, we constantly detect wave properties in the electron.

A tagged wave is a very interesting physical object. With the help of macroscopic instruments, this wave can be divided into two coherent parts: with a marker and without a marker. The part without a marker carries negligible energy but may contain significant information. In principle, this may open up new possibilities for the transmission of information.

As noted in Section 11, on the one hand, tagged waves can be considered as classical electromagnetic waves with a definite localization in three-dimensional space. On the other hand, these waves can be viewed as vectors $|\beta_k\rangle$ in some Hilbert space $\mathfrak{H}_k^{ph}$. Another space $\mathfrak{H}^{ph}$, which is a direct orthogonal sum of spaces $\mathfrak{H}_k^{ph}$ with different k, proves to be useful.

From the viewpoint of the algebraic approach, a tagged wave can be considered as a quantum ensemble consisting of a huge number of individual supersoft photons. Therefore, it is not surprising that this wave can be described by a vector in Hilbert space. However, from a physical point of view, it seems not very good to consider that an individual supersoft photon is in some elementary state. The point is that this elementary state is defined in Section 3 through the concept of "observable," while the distinguishing feature of a supersoft photon is the impossibility of its observation.

This inconsistency in mathematical and physical interpretation can be eliminated as follows. Recall that what we are actually following is a statistical electron (or an observed photon), a marker of this wave, rather than the supersoft photon waves. Let this electron (photon) have a wave vector k. During the time required to register the characteristics of this electron (photon), a Newtonian electron (photon) will pass through the recording area a huge number of times. Therefore, we will deal with an ensemble of these waves, whose markers have a wave vector $\mathbf{k}$, rather than with a separate tagged wave. The tags fixing a phase in these waves will be placed randomly. Let us identify a subensemble belonging to this ensemble, the elements of which will be the tags. Let us make a standard assumption in probability theory that for each position of the tag in the wave, the probability is the same. In this case, the spatial arrangements of tags will repeat the spatial arrangements of supersoft photons in one tagged wave. Therefore, the ensemble of these tags can be associated with a vector $|\beta_k\rangle_m$ in the Hilbert space $\mathfrak{H}_k^m$, which is isomorphic to the space $\mathfrak{H}_k^{ph}$. Accordingly, the vector $|\beta_k\rangle_m$ will be the image of the vector $|\beta_k\rangle$. Unlike the latter, the vector $|\beta_k\rangle_m$ cannot be represented as a classical wave in which particles interact with each other. This vector can only be interpreted as a probability wave. Instead, individual particles of this wave will be physically observable.

In the selected ensemble of tags, the transition from one element of the ensemble to another can be interpreted either as the movement of the tags along the orbit in the plane orthogonal to $\mathbf{k}$, or as the movement of the tag along the radial direction. Bohr's axioms will be valid for this motion. In other words, we arrive at a modified Rutherford–Bohr model. Accordingly, the results obtained by Bohr will be valid for the hydrogen atom.

At the same time, in the proposed model for a statistical electron, circular orbits are allowed, which correspond to any other direction of the vector $\mathbf{k}$. By virtue of spherical symmetry, all of them will be equally probable. Therefore, an adequate description of the states of an electron in an atom should be done not in terms of orbits, but in terms of orbitals, as is done in the standard modern model of the atom, i.e., in terms of vectors in a space $\mathfrak{H}^m$, isomorphic to $\mathfrak{H}^{ph}$.

In the proposed model of the atom, the soft and supersoft photons play a central role. A special role is assigned to their Bose condensate. This condensate disposes of all motion waste and recycles it into new motion. As a result, something similar to a perpetual-motion machine occurs.

The existence of this engine does not contradict the energy conservation law. On the contrary, it provides

the enforcement of this law. It allows us to abandon the assumption that there is a singularity point at which it is not clear what happens to the energy.

The proposed scheme can also be very useful in constructing a model for the evolution of black holes, since in this case, it is also possible to avoid the appearance of singularity points and the complete annihilation of a black hole.

In this section, a fairly physically descriptive model of the structure of an atom is proposed. The main provisions of the proposed scheme can be used to build a model of the evolution of the Universe. A possible variant of this model is considered in [60].

14. MODEL OF THE UNIVERSE WITH A PHOTON BOSE CONDENSATE

Cosmology has been developing intensively in recent years. Thanks to the development of new technologies, observational astronomy has achieved great success. In particular, the following interesting facts were established: the visible part of the Universe is expanding, and the expansion occurs with acceleration; on a large scale, the Universe is isotropic, homogeneous, and its curvature is close to zero. There is almost no antimatter in the Universe.

An analysis of the experimental data makes it possible to reconstruct in sufficient detail the history of the development of the Universe after the Big Bang, which occurred approximately 13.5 billion years ago. There are no experimental data on the period before the Big Bang. This, it seems, opens up a wide scope for speculations when building models for the development of the Universe during this period.

Therefore, before building a model of the Universe, it is desirable to agree on the rules that must be followed in such a construction. First of all, the model should not contradict any sufficiently firmly experimentally established fact. Only after that it is necessary to ensure that the largest possible number of experimentally established facts are reproduced in the model. Of course, one cannot demand that the model reproduce all the facts. However, it should not prohibit the reproduction of these facts.

Let us try to build a model of the Universe by using the experience gained in building a model of the proton. We start with building a chain of physical objects that can be considered as intermediate between the proton and the Universe.

An acceptable variant of this chain is the following: the nuclei of chemical elements; transuranic elements; planets (in particular, the Earth); stars (in particular, the Sun); galaxies; metagalaxies.

In this chain, we will first discuss the triton (the nucleus of tritium). In its internal structure, it is very similar to the proton. Therefore, to build its model, the arguments can be used that were applied for the proton. The only difference between a proton and a triton is a different set of quarks in the crust, but a specific form of this set for the proton has not been discussed.

However, in its behavior, the triton differs significantly from the proton. While the proton is a stable particle, the triton undergoes the beta decay. In other words, a microexplosion occurs in it.

Moving further along the links of the chosen chain, we will encounter both stable and unstable elements. Stability and instability depend on what the hadronic content of the element in question is. It does not make much sense to discuss the theory of this problem in detail. Here the strong interaction of quarks plays an essential role. Unfortunately, at present there is no theory of this process.

As long as we are at the microlevel in our consideration, we will encounter microexplosions. However, we have already learned how to artificially select the hadronic content in such a way that it is not microexplosions that occur, but explosions of very high power. This refers to nuclear and thermonuclear weapons. Moving further to the level of planets and stars, we will encounter explosions on the astronomical scale.

All this indicates that the Big Bang should be considered as an ordinary rather than a unique phenomenon. Let us pay special attention to the fact that among all explosions in elements of the chain there is not a single one in which something arose out of nothing (even remotely resembling "nothing"). Moreover, the latter phenomenon, in general, has never been observed in nature. This allows us to consider it experimentally firmly established that this can never be. Of course, one can believe in a miracle, but this is already beyond the scope of experimentally established facts.

Although there are no direct experimental data on the properties of the Universe before the Big Bang, an analysis of the links in the proposed chain makes it possible to describe these properties with great confidence.

Just before the Big Bang, the Universe was almost the same globally as immediately after the explosion. Precisely, before the explosion, the Universe was globally isotropic, homogeneous, and there were almost no antiparticles in it. The mentioned properties could change slightly as a result of the explosion, but against the general background, these changes should be very small. Observations show that this scenario of explosions occurs in all links of the chain.

Numerous attempts have been made to theoretically interpret the observed facts. The most popular of them is the so-called standard Big Bang scenario (see, e.g., [61]). However, there are annoying gaps in this scenario. What was before the Big Bang (or was there nothing)? What caused the Big Bang (or was there no cause)? The initial stage of the expansion of the Universe from a singular point to millimeter sizes (the stage of inflation) is somehow incredibly fleeting. In this case, the proposed process of inflation looks quite artificial, and the Universe in the form of a singularity

point is somehow visually, in general, impossible to imagine. The dark energy is proposed as a source of acceleration of the expansion of the Universe, but what it is, to put it mildly, is not entirely clear. It is not very clear why there is almost no antimatter in the Universe.

To avoid these complications, we can try to build the model of the Universe using the experience gained in building a model of the proton.

It is commonly said that the most abundant element in outer space is hydrogen. However, from the viewpoint of the physics of elementary particles, protons fundamentally differ little from photons, while the number of the latter is much greater. Thus, according to estimates (see, again, [61]), one cubic centimeter of cosmic space contains about 2×10^{-7} baryons and 500 photons. This refers to photons that can be recorded. Considering the bremsstrahlung photons, the number of photons is much larger.

In space, photons are distributed more or less uniformly, but, of course, fluctuations are possible, including very large ones. Too little time has passed since the Big Bang, and the probability of a large fluctuation is small. However, if world history did not begin with the Big Bang, but had an infinitely long prehistory, then a very large fluctuation is quite likely.

Due to density fluctuations in a definite region, conditions could arise for the beginning of the black hole formation. In this case, following the example of formation of stars from a dust cloud, a superstar, consisting of photons and baryonic matter (quarks and leptons), could form, but the latter will be much smaller. It can be expected that there will be approximately the same number of baryons and antibaryons, but, of course, the absolute equality is nearly impossible.

Following the example of the observed stars, the resulting superstar may be stable, or it may be unstable. We will talk about instability a little later, but for now we will assume that the stability period was quite long. Due to numerous chaotic rescatterings during the equilibrium time, the superstar should acquire the spherical symmetry (in its own coordinate system) and begin to rotate around an axis of its own.

If a superstar has dimensions numbered in gigaparsecs or more, then a centimeter-sized region located at a giga-parsec distance from the center will be almost homogeneous. If it is dominated by the baryonic matter, then due to rescattering, the baryonic antimatter will have time to fully annihilate.

To model the internal structure of a superstar, we make the same assumptions, as we did to model the internal structure of a proton, corrected for the astronomical dimensions of the superstar. In other words, we suppose that a superstar has a core, magma, crust, and atmosphere.

Again, let us assume that the core is a Bose condensate of supersoft photons, which has the structure of a black hole. Only now it is not a micro black hole, but, on the contrary, a gigantic black hole.

The superstar has a preferential direction towards the star center. However, if we allocate a relatively small area on the order of 1 cm^2 (a preimage of our Universe) in the superstar and associate the coordinate system with this area, then there will be no preferred direction in this area. In addition, in this area, the colossal force of attraction to the star center will not be felt, since this force will be balanced by the centrifugal force arising from the star rotation (weightlessness effect).

Just as in the case of the proton, we will assume that the collapse of a superstar into a singular point will be prevented by a chain reaction leading to the production of condensational radiation. No antigravity is needed here. However, depending on the baryon situation that has developed in the area where the chain reaction began, two options for the further development of events are possible. The first option is when the chain reaction gradually fades. This option ensures the stability of the superstar structure. The second option is when the chain reaction becomes a runaway one. In this case, a powerful explosion may occur, which will be accompanied by a powerful blast wave.

This explosion is quite suitable for a role of the Big Bang, which is believed to have begun the history of our universe. In this case, there may be two significant differences from the so-called standard scenario. Firstly, the blast wave does not have to propagate at a supersonic speed. Secondly, this wave arrives at an earlier-prepared section of the superstar. As mentioned above, this section was already homogeneous before the arrival of the blast wave and there was almost no antimatter in it. In other words, it had the properties that, in the standard scenario, should have occurred at the time of the explosion.

It is natural to assume that the scenario of subsequent events was usual for the case of the arrival of a powerful blast wave. In other words, at the area where the wave arrives, the temperature should increase significantly, the area should receive a large momentum and should be broken into many fragments, which, on average, will uniformly scatter in different directions. One fragment could later become our ancestral homeland.

The standard scenario can be used to describe our postexplosion Universe. In this case, it can be avoided to use the very controversial "preexplosive" part of the standard scenario. The assumption of the existence of a mysterious inflaton is also not necessary. Instead, one can make the following assumptions:

First, suppose that the period before the explosion was not very short, but very long.

Second, there was no singularity.

Third, the Universe was a very small part of the crust of a very large superstar.

Fourth, the fundamental laws of physics were the same as they are now.

After splitting into fragments in the zeroth approximation, the general structure of the Universe will remain the same. However, in the first approximation, the structure may change. Some violation of isotropy should be expected. The Universe will break up into many miniuniverses, the structure of which can be seen from Earth.

For the fragments (miniuniverses) that have not gone beyond the horizon of visibility, it becomes possible to observe their internal structure. These results should be expected. If there is a sufficiently powerful radiation source in the quark–gluon crust of the fragment, then it can be detected from the Earth. On the other hand, if the condensational radiation frequency for a fragment is sufficiently low, then the mantle region of the fragment and its core will look as a dark spot. Current astronomical observations suggest just this cellular picture.

At the same time, the situation is also of interest, in which the condensational radiation frequency is sufficiently high. In this case, this radiation can be observed from the Earth. Quasars may be quite suitable candidates for the role of these objects.

Now, repeating the technique of section 12, we will try to make some numerical estimates using the Friedman equations with a Λ-term (Eqs. (52), (53)):

$$\frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 = \frac{4\pi G}{3}(\rho + \rho_\Lambda) - \frac{kc^2}{2a^2}, \tag{57}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2} - 2\rho_\Lambda\right). \tag{58}$$

The Universe in which we live is located in the baryon layer of a superstar. The quantity a appearing in Eqs. (57) and (58) is the average radius of this layer. We recall that G is the gravitational constant, ρ and P are the density and pressure of the observed matter, c is the speed of light, k is the curvature parameter (in our case $k = 1$), and ρ_Λ is the Λ-term.

Let us consider the zeroth approximation first. In this approximation, the Universe is stable. Therefore, it should be assumed that $\dot{a} = \ddot{a} = 0$. We will also assume that $P = 0$. Then, by solving the equations, we find

$$\rho_{\hat{L}} = 0.5\rho, \quad a^2 = c^2/4\pi G\rho.$$

Assuming that $\rho \sim 10^{-30}$ g/cm^3, for the average radius we obtain $a \sim 10^{28}$ cm.

The first approximation regards that $\dot{a}$ and $\ddot{a}$ are different from 0. In this case, it is convenient to rewrite Eqs. (57), (58) in the form

$$\frac{c^2}{a^2} = 4\pi G\rho + \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2, \tag{59}$$

$$2\rho_{\hat{L}} = \rho + \frac{\ddot{a}}{a}\frac{3}{4\pi G}. \tag{60}$$

If we discard the physically meaningless solution $a < 0$, then for given ρ, $\dot{a}$, and $\ddot{a}$, Eqs. (59) and (60) uniquely determine a and $\rho_{\hat{L}}$. Existing experimental data do not fix ρ, $\dot{a}$, and $\ddot{a}$ absolutely unequivocally. However, they, at least, do not contradict the statement that Eqs. (59) and (60) have physically reasonable solutions.

Of course, the Universe is not just an enlarged copy of the proton. In particular, the proton is known to be nearly stable. On the contrary, it is known about the Universe that it is stable only in the zeroth approximation, and unstable in the next approximations.

For a proton, the mass, gravitational radius, and charge radius are known. For the Universe, all these quantities are free parameters. Our Universe can clearly be imagined in the form of a prominence that develops in a superstar. Thus, we live in a developing prominence.

15. MEASUREMENT PROCESS MODEL

In the quantum case, the elementary state $\varphi = [\varphi_\xi]$ of an individual physical system is a set of functionals $\varphi_\xi(\cdot)$, each of which is a character of the maximum real commutative subalgebra $\mathfrak{O}_\xi$ of the algebra $\mathfrak{A}$. The set Ξ ($\xi \in \Xi$) of such subalgebras has the cardinality of the continuum. Thus, the elementary state is a function-valued field over the set Ξ.

To set φ, it is necessary and sufficient to specify φ_ξ for each $\xi \in \Xi$. In turn, to specify φ_ξ, it is sufficient to specify a value φ_ξ on each generator of the subalgebra $\mathfrak{O}_\xi$. We can assume that each generator of the subalgebra $\mathfrak{O}_\xi$ is matched by a component of this function-valued field φ. The value of the functional φ_ξ on the generator can be considered as the value of the field component φ at the point ξ. Thus, φ is a real c-numeric multicomponent field over the set Ξ.

Accordingly, even a quantum system, which is typically considered as a system with a finite number of degrees of freedom (for example, a harmonic oscillator), in the quantum case is a field system, i.e., a system with an infinite number of degrees of freedom. From here it follows that in the elementary state of any quantum system, in principle, an infinite volume of information can be encrypted. However, this infinite volume is not available for real use. The fact is that for the information to be useful to us, we must be able to control it with the help of classical instruments. However, classical instruments cannot distinguish one elementary state from another, they only distinguish equivalence classes that correspond to quantum states. Therefore, the amount of controlled information turns out to be finite, but it can still be much larger than for classical physical systems. This is a physical prerequisite for the possibility of building quantum computers.

The elementary state of any physical system is the field over the space $\mathfrak{M}$ as well. The fact is that systems, which are traditionally considered in quantum mechanics as point systems, in reality are distributed in the space $\mathfrak{M}$.

Thus, the elementary state φ of any quantum system is described by a real c-number field over the set Ξ and the space $\mathfrak{M}$. This field has all the features of a real classical field. If we accept the classical paradigm, then it is permissible to assume that there is some matter field, the mathematical image of which is φ.

However, as was explained in Section 3, the dependence of this matter field on ξ cannot be found by any combination of measuring instruments. Therefore, we will call this field as hidden. An example of a hidden field is the field of dark photons.

The assumption of the material existence of a hidden field can help to solve one of the problems of quantum theory: the wave–particle duality. Here the term "corpuscular-field dualism" will be used for this concept. The field properties of a quantum system are naturally associated with a hidden field, i.e., with the elementary state of the system. By comparison, the corpuscular properties are related to observables of the system, more precisely, to local observables.

The corpuscular properties of a quantum system mean the following. A physical system has local observables, i.e., observables associated with a limited area in space $\mathfrak{M}$. These observables, or more specifically their complex combinations, form an algebra of local observables. There are stable sets of values of local observables, which we treat as quantum particles of the definite type: electrons, protons, nuclei, atoms, etc.

Measuring instruments perceive these observables as an undivided entity. In this, the corpuscular properties of quantum systems are manifested. The reaction of the measuring instrument is determined by the elementary state of the system (by a hidden field). In turn, the structure (value) of the hidden field is determined by the spectra of the corresponding observables. Note that the spectrum point is an indivisible entity. Thus, in a quantum system, the corpuscular and field properties are closely interwoven.

In the standard approach to quantum mechanics, a quantum state of a physical system is also associated with a c-number field—a wave function. However, the wave function is complex-valued. Therefore, it cannot directly correspond to the matter field. In the proposed interpretation, a wave function is related only to a probability, and rather indirectly. Namely, the average values of observables *can be represented* (but they are not) in the form of mathematical expectations of linear operators of some Hilbert space. In turn, the vectors of this Hilbert space *can be represented* as wave functions.

The hidden field can be considered as a physical storage media of the physical state of a quantum object. To be this storage media, it must be consistent (coherent) with the associated quantum object. This makes it possible to build a rather plausible model of the measurement process based on the hidden field. We recall that many inconsistencies that exist in the standard approach usually are attributed to the absence of this model.

Let us describe a similar model. A measuring instrument consists of an analyzer and a detector. Sometimes these components of a measuring instrument can be combined. An analyzer is a device with one input channel and multiple output channels. If the instrument is designed to measure the observable $\hat{B}$, then each output channel corresponds to a definite part of the spectrum of this observable, i.e., each output channel corresponds to a some equivalence class of elementary states.

The hidden field associated with the measured quantum object excites collective oscillations in the analyzer that are coherent to the field. The oscillations can be very weak, but due to coherence they interact with the quantum object in a resonant way. A microscopic description of this interaction is almost impossible. However, the result of this interaction can be described as a boundary condition. If the quantum state of the measured object describes an equivalence class that corresponds to one of the output channels, then the object uniquely found in this output channel. If the object under study is in a quantum state that does not correspond to any of the output channels, then the analyzer turns out to be a bifurcation region for this object. In this case, the resonant interaction of the object with the hidden field excited by the analyzer oscillations turns out to be the random force that directs the object to the definite output channel, namely, into the channel corresponding to the equivalence class to which the elementary state of the measured object belongs.

Here, the area of quantum object localization means the area of localization of its local observables, which can be recorded by classical measuring instruments. Further, this localization area will be called the kernel of a quantum object. An example of the kernel is the localization area of tagged waves associated with the physical object under study.

At the same time, as already was mentioned, a quantum object is accompanied by a field, which, on the one hand, is not recorded by measuring instruments, while, on the other hand, is an integral part of the hidden field. Therefore, the analyzer can become the branching region of the hidden field. In each output channel of the analyzer, the corresponding part of the hidden field will fall.

The analyzer is a classical object. The interaction of a hidden field with a classical object can be of two types. With the first type, the coherence of the hidden field with the radiating object is not violated; with the second type, it is violated. Since we assume that in the

analyzer, the hidden field excites oscillations, coherent to the field, and they interact resonantly with the quantum object, then it should be assumed that the interaction with the analyzer does not violate the field coherence. We will also assume that the impact of the quantum object on the analyzer is macroscopically undetectable. This recording takes place in the detector.

The detector is a classical system in a state of unstable equilibrium. The detector strongly interacts with the kernel of the quantum object. As a result of this interaction, the detector leaves the equilibrium state. A catastrophic, macroscopically recorded process develops in it. The detector(s) is located at one (several) output channel of the analyzer. As a result of the triggering of the detector, the output channel of the analyzer is fixed, into which the kernel of the quantum object has fallen. This is how the value of the observable of a quantum object is fixed. At the same time, the equivalence class, to which the elementary state of the measured object belongs, is fixed.

The reverse effect of the detector on the quantum object is also strong. In the case of a nonreproducible measurement, a complete change in the elementary state of a quantum object occurs. In a reproducible measurement, coherence is also violated, but the elementary state of the quantum object remains in the equivalence class, which corresponds to the output channel through which the kernel of the quantum object has passed. The kernel of a quantum object and the hidden field accompanying it cease to be coherent with the parts of the hidden field that have passed through other output channels of the analyzer.

If the detector is located at the output channel through which the quantum object kernel has not passed, then the detector experiences only a weak effect from the hidden field that has passed through this channel. The catastrophic process does not develop in the detector, and the macroscopic effect is not recorded. However, the reverse effect of the detector on the hidden field turns out to be significant. The field in this channel loses coherence with the quantum object kernel and the hidden field parts that have passed through other channels.

If detectors are not placed in any of the channels of the analyzer, then it is possible, in principle, to reconnect all parts of the hidden field that have passed through different channels. They will coherently add up, and the initial elementary state can be recreated. If there is a detector in some channel, then the corresponding part of the hidden field cannot take part in the coherent summation. Effectively, from the viewpoint of the elementary state of the quantum object, this part of the hidden field is lost. A measurement model similar to this one is described in [62].

Thus, a part of the hidden field that determines the elementary state of a quantum object can effectively "disappear" in two cases. Either the state of this part of the field changes (with violation of coherence), or nothing happens to this part of the field, but the state of the quantum object kernel changes. In both cases, the structure of the hidden field coherent to the kernel changes. It is this field that determines the elementary state of a quantum object. When the elementary state changes, the quantum state naturally changes. This change has all the markings of a quantum state collapse in the measurement.

The hidden field performs the functions that are usually attributed to hidden parameters. However, unlike the situation with hidden parameters, here an example of a hidden field is given. Therefore, there is no problem of the existence of this field. All the arguments that are often used against hidden parameters have no probative force in the case of a hidden field.

In contrast to hidden parameters, a hidden field is partially observable. It affects the behavior of the kernel of a quantum object in the case when this kernel is coherent to the hidden field. In turn, the detector of the classical measuring instrument reacts to the kernel. The classical instrument does not react in any way to a hidden field that has lost coherence with the kernel. However, this does not mean that the hidden field has disappeared. It can manifest itself in the form of dark matter or dark energy.

The idea that a quantum object consists of a kernel and a hidden field allows us to give a clear interpretation, consistent with the principles of locality and causality, to a large number of quantum effects in which these principles seem to be violated.

Let us start with the Aharonov–Bohm effect.

16. AHARONOV–BOHM EFFECT

The Aharonov–Bohm effect (AB effect) is one of the key effects of quantum physics. It is closely connected with a number of fundamental problems of quantum theory. These include the problem of quantum measurements, the relationship between quantum quantities and their classical counterparts, and the problem of locality and causality.

The effect was theoretically predicted in 1959 [63] and received its first experimental confirmation a year later [64]. However, many complaints have been made about this result and a number of subsequent experimental confirmations.

A very lively discussion developed, in which the existence of the effect was questioned. Much later, in 1986, Professor Tonomura's group [65] succeeded in finally experimentally proving that the AB effect really exists.

Already in [63], it was proposed to conclude from the effect presence that "...unlike classical mechanics, there exists an effect of potentials on a charged particle, even if in the area where it is located, all fields (and, consequently, forces acting on the particle) disappear," and therefore "further development of the

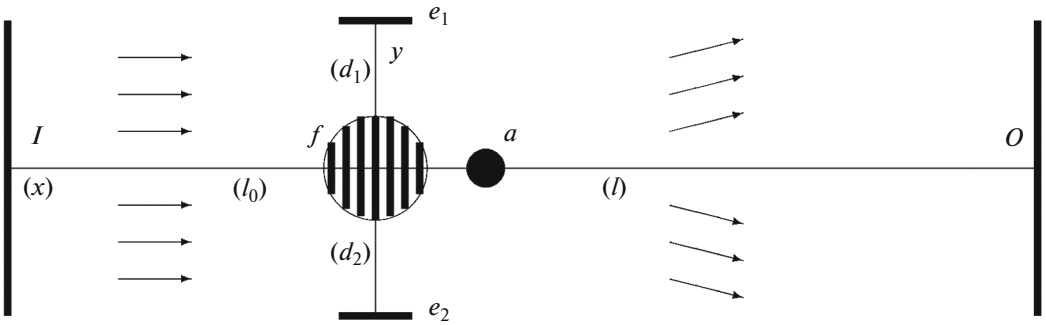

Fig. 4. Schematic diagram of the AB effect.

theory is necessary." Two directions of this development were proposed in the work. First, the locality condition can be abandoned, and second, it is possible to assign a special role to potentials in quantum theory. In a recently published article by Aharonov and his collaborators [66], the nonlocality is emphasized. Here we focus on the second possibility.

The schematic diagram of the mental experiment and the first one actually implemented is shown in Fig. 4.

The experiment is as follows. A beam of electrons with identical momenta falls on a biprism $[e_1 f e_2]$ from the left. The biprism consists of an aluminized quartz filament f with a diameter of 1.5 μm and two electrodes e_1 and e_2. A thin solenoid of infinite length (in the thought experiment) or a magnetized long rod 1 μm in diameter (in the real experiment) is placed in the shadow of the filament f. The filament and the rod are perpendicular to the plane of the figure. To simplify the mathematical calculations in the experiment shown in the figure, plate I, perpendicular to the plane of the figure, was taken as the electron source. In the real experiment, an electron microscope was used as the source, which rather would be well to depict as a dot. However, since the distances l_0 and l ($l_0 \sim 6.5$ cm, $l \sim 13.5$ cm) are much larger than the diameters of the filament and rod, this introduces an insignificant distortion in the final result.

The filament f divides the incident electron stream into two parts. One goes through the slit (d_1), the other through the slit (d_2). A small positive potential is applied to the filament, due to which, after bypassing the filament f and the rod a, these two parts are combined into a single flux of electrons scattered in the biprism. Falling on the screen O, these electrons form interference fringes.

Outside the rod a, the magnetic field strength is zero. Therefore, according to classical electrodynamics, the interference pattern on the screen should not depend on the magnitude of the magnetic induction flux passing through the rod a. On the other hand, the value A of the vector potential in the region of electron passage cannot be equal to zero everywhere, since, according to the Stokes theorem, the integral of A over any closed loop around the rod a must be equal to the flux of magnetic induction through this loop. The occurrence of the interference pattern dependence on the magnetic induction flux is called the AB effect.

In the quantum consideration, the picture of the process changes significantly. It is the vector potential $\hat{A}$ that enters into the Schrödinger equation for an electron, but not the magnetic field strength. Therefore, the vector potential affects the electron behavior even when the wave function of the electron is vanishingly small in the region where the magnetic field strength is different from zero. This gave grounds to Aharonov and Bohm to assume the special role of potentials in quantum theory. They even put this statement in the title of their article [63].

As already mentioned, the AB effect immediately provoked a lively discussion, the preliminary results of which were summarized in Feinberg's paper [67]. In his article, Feinberg substantiates the assertion that this role is not so special, and that something similar can be seen also in the classical consideration. Next, we will try to reveal the physical reasons for the influence of a magnetic rod on the behavior of scattered electrons.

Let us first describe the dynamics in physical, intuitively comprehensible terms. Then, we will try to put all this into a mathematical form. By assumption, the considered physical system is homogeneous along the axis perpendicular to the plane of the figure. Therefore, we will assume that all events occur in the plane of the figure. The consideration will start from the left edge of the figure.

The source I emits a stream of electrons. As explained in Section 14 dedicated to the atomic structure, this is not a set of free electrons, but a set of tagged waves $|\beta_k\rangle$. All these waves have the same wave vector k directed along the axis x. Each wave has one tag placed randomly. Further, each wave will be associated with a Hilbert space vector. Therefore, from the

very beginning, we will denote the waves by the same symbols as the corresponding Hilbert space vectors.

The filament splits the wave into two coherent parts: $|d_1\rangle$ and $|d_2\rangle$, which, respectively, will go to the slits d_1 and d_2. An electron (tag) will randomly go with one of the parts. When passing through the slits, the wavevectors of the waves will receive random small deviations in the direction of the axis y. In addition, Stokes' theorem states that the magnetized rod affects the structure of the electromagnetic field. In particular, it affects soft photons that form a tagged wave. As a result, the waves $|d_1\rangle$ and $|d_2\rangle$, remaining coherent, will acquire a phase difference. This can affect the magnitude of wavevector deviations. Due to the positive potential on the filament f, after passing through the slits, the waves $|d_1\rangle$ and $|d_2\rangle$ coherently add up. In the resulting wave, the wavevector will no longer be directed along the axis x. As a result, when soft photons hit the screen O, they could form an interference pattern on it. However, these photons have too small an energy to leave a recorded trace.

At the same time, in each tagged wave, in addition to soft photons, there is one electron. This electron has enough energy to leave a trace on the screen in the form of a dot. In reality, not a single tagged wave will hit the screen, but an ensemble of these waves, each of which contains one electron. These electrons will draw an already visible interference pattern on the screen. By virtue of what was said in Section 14, this pattern should be an observable copy of an invisible interference pattern formed by soft photons of one tagged wave.

For an electron, the magnetized rod a is in the shadow of the filament f. Therefore, the electron does not directly interact with the rod a. On the other hand, as already mentioned, any wave, including a tagged one, is a relatively stable structure. This means that between a set of soft photons and an electron included in the same tagged wave, the interaction must be significant. In other words, an electron interacts with a magnetic rod through an intermediary. If we forget about the mediator, then this interaction appears as a nonlocal one. But locality, as a general principle of the theory, is not violated.

Now let's proceed to the mathematical calculation of the interference pattern drawn by electrons. In the light of what was said above, in this case, it is possible to follow the soft photons of tagged waves rather than the electrons. Ensembles of these waves will be described by the corresponding Hilbert space vectors. In our calculations, we will focus on relation (19) in Section 5.

$$\hat{p}_0\hat{C}\hat{p}_0 = \vartheta(\hat{C})\hat{p}_0. \qquad (19)$$

There, a functional was constructed that describes the quantum average in the case when the commutative algebra defining the quantum state contains a one-dimensional projector. Let us start by constructing a suitable projection operator $\hat{p}_0$.

As already was noted, in the algebraic approach, it is necessary to choose a system of units so that all observables become dimensionless. In the case under consideration, it is convenient to choose a width of each of the slits ($|d_1| = |d_2| = d = 1$) as the length unit, and ω_k, as the unit of the wavevector.

The slit d_1 will select soft photons in the wave, which passed through it, the coordinate y of which lies in the interval $[r, r+1]$, where r is the radius of the filament f in units of d. Therefore, the state of this wave can be described by the vector

$$|d_1\rangle = 1 \quad \text{with} \quad y \in [r, r+1]$$
$$\text{and} \quad |d_1\rangle = 0 \ \text{if otherwise.}$$

Similarly

$$|d_2\rangle = 1 \ \text{with} \ -y \in [r, r+1]$$
$$\text{and} \ |d_2\rangle = 0 \ \text{if otherwise.}$$

As a result of the subsequent coherent addition of these parts, the state of the wave will be described by the vector

$$|s\rangle = 2^{(-1/2)}[|d_1\rangle + \exp(i2\pi\alpha)|d_2\rangle].$$

Here, the parameter α describes the shift of a phase of wave $|d_2\rangle$ relative to a phase of wave $|d_1\rangle$. The value of this parameter, according to Stokes' theorem, is determined by the magnetic flux through the rod a.

The main disputes between supporters and opponents of the AB effect focused precisely on this parameter. The fact is that Stokes' theorem includes the vector potential $\hat{\mathbf{A}}$ rather than the electromagnetic field strength. On the other hand, the value $\mathbf{A}$ of the vector potential is not uniquely determined but depends on the chosen gauge. Therefore, opponents argued, a vector potential cannot be an observable quantity and cannot generate a physically observable effect.

Within the framework of the algebraic approach used here, this reasoning is incompetent, since it is possible to require definite values only from the observables that are stable on the considered elementary states. Definiteness of values is not an obligatory property of observables. A necessary property of observables is that they belong to an algebra. It is quite possible that some observables are not stable, but their algebraic combinations are stable and have the definite values.

It is this situation that is realized in the AB effect. The change in the phases of each of the waves $|d_1\rangle$ and $|d_2\rangle$, caused by the vector potential, does not have a definite value, but the phase difference is a stable observable and has a definite value. Note that a similar situation is also characteristic of classical systems. The speed of any one classical object does not have a defi-

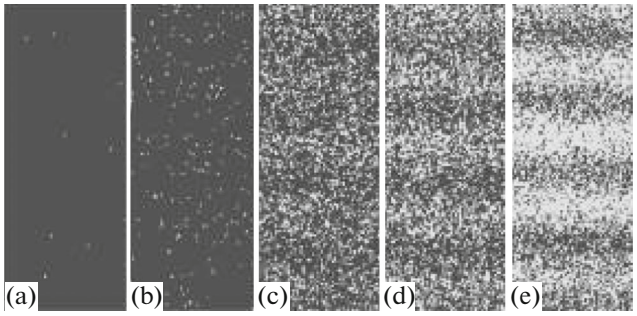

Fig. 5. Interference pattern in the scattering of electrons.

nite value (depends on the coordinate system), while the relative velocity of two objects has a definite value (it is a stable observable).

In Eq. (19), the projector operator $\hat{p}_0$ determines the structure of the ensemble over which the observable $\hat{C}$ is averaged. Accordingly, in our case, the operator $|s\rangle\langle s| = \hat{p}_0$ should be taken as this projector. As an observable $\hat{C}$, we take the projector onto the state $|k_y\rangle$, in which the projection of the wavevector onto the axis y has a definite value. In the coordinate representation, this state has the form $|k_y\rangle = \exp(iyk_y)$.

In these notations, Eq. (19) takes the form $\vartheta(\hat{C})(\hat{C})|s\rangle\langle s| = |s\rangle\langle s||k_y\rangle\langle k_y||s\rangle\langle s|$, which is equivalent to

$$\vartheta(\hat{C}) = |\langle s|k_y\rangle|^2. \quad (61)$$

To rewrite formula (61) in dimensionless terms, it is necessary to make the substitution $k_y \to k_y/\omega_k \equiv \theta$. Physically, θ characterizes an angle of scattering of a wave of soft photons. Accordingly, $\vartheta(\hat{C})$ has a meaning of a density $\rho(\theta)$ of the number of soft photons scattered through the angle θ. In Eq. (61),

$$\langle s|k_y\rangle = \frac{1}{\sqrt{2}}\int_r^{1+r} dy\left[e^{i\theta y} + e^{-i\theta y}e^{-i2\pi\alpha y}\right].$$

By calculating the integral, Eq. (61) can be rewritten as

$$\rho(\theta)) = \frac{8}{\theta^2}\cos^2\left(r\theta + \frac{\theta}{2} + \pi\alpha\right)\sin^2\left(\frac{\theta}{2}\right). \quad (62)$$

In the performed experiment, $r \sim 10^{-3}$. Therefore, we can put $r = 0$. The brightest part of the interference pattern is the central one ($\theta \sim 0$). The arrangement of the fringes in it is controlled by the cosine. It can be seen that with a change in the magnetic flux (with a change in α), the interference pattern shifts. It is this shift that is the AB effect.

Formula (62) directly describes the interference pattern that could be drawn by soft photons of one coherent wave. Of course, the energy of these photons is too small to leave a visible trace on the recording screen. However, as already was noted, when a large number of tagged waves fall on the screen, tags (electrons) statistically reproduce a visible copy of this pattern.

Thus, we see that in order to explain the AB effect, it is not necessary to invoke the assumption that the quantum interaction is nonlocal. However, this nonlocality is required for other purposes. Namely, it is required to justify the renormalization procedure (elimination of ultraviolet divergences). In this case, the nonlocality of the measurement procedure comes into play. Any measuring device has finite dimensions in four-dimensional space $\mathfrak{M}$ and cannot capture the details of the interaction at small distances. In the renormalization procedure, just the indeterminacy that arises in this case is used.

17. SCATTERING OF AN ELECTRON ON TWO SLITS

According to Feynman [69], this is "... such a phenomenon that it is impossible, clearly, absolutely impossible to explain in a classical way. In this phenomenon, the very essence of quantum mechanics is hidden."

Despite this categorical statement by Feynman, let us try to do it in a classical way. In other words, we avoid using not very clear "quantum logic" but will try to limit ourselves to the usual "classical logic."

Of course, the use of classical logic does not make scattering electrons to be classical particles. Their quantum nature can be considered by treating them not as free particles, but as tags in waves of unobservable coherent soft photons.

Let us discuss the results obtained by the previously mentioned group of Tonomura [70]. In these experiments, the scattering of an electron beam in a biprism, which is similar in its physical properties to a double-slit screen, was studied.

A schematic diagram of the biprism is given in Section 16. Only in this case there was no magnetized rod (a). In addition, the intensity of the beam was so low that, on average, there was less than one electron in the setup at the same time. This made it possible to neglect the influence of interaction of electrons with each other on the experimental results. In addition, in the experiment it was possible to fix a result of the passage of a small number of electrons.

The results of the experiment are shown in Fig. 5 borrowed from [70]. Individual photographs correspond to different exposure times. Photo (a) shows traces from 10 electrons; photo (b), from 200 electrons; photo (c), from 6000 electrons; photo (d), from 40000 electrons; photo (e), from 140000 electrons.

It can be seen that when a small number of electrons are recorded (photos (a) and (b)), no interference pattern is visible. This picture appears only in the

case of recording of a very large number of electrons (photos (d) and (e)).

This experiment speaks in favor of the fact that wave properties do not appear in a single electron. They appear only in an ensemble of electrons formed in a special way. In the case under consideration, all electrons had approximately the same momentum.

The tagged wave consists of a huge number of photons, each of which carries a very small portion of energy. In this situation, to a good approximation, this wave can be considered as a classical electromagnetic wave. According to the laws of classical optics, it must interfere on two slits.

The wave of soft bremsstrahlung photons itself does not manifest itself in any way on the recording screen, because these photons carry too little energy. A tag leaves a trace on the screen. The tag, like a float on a surface wave on water, can replace any photon of the wave. It is natural to assume that the probability of being replaced is the same for all wave photons. In this case, first, traces from a small number of tags will be randomly located on the screen. Second, traces from a large number of markers on the recording screen will be located with a density proportional to the probability of soft wave photons to hit a given place. Accordingly, an interference pattern will appear on the recording screen. This is the result that was obtained in Tonomura's experiments.

This result can be confirmed by a quantum-mechanical calculation performed within the algebraic approach that we used. However, a new calculation need not be carried out, but the result of the calculation performed in the previous section can be utilized.

Apart from the use of a source of electrons of very low intensity, the only difference of the setup used in the study of electron scattering on two slits, from the setup in the experiment to study the Aharonov–Bohm effect, is the absence of a magnetized rod. Therefore, we can repeat all the calculations in the previous section, while in the final result we can set α to zero.

As a result, for the density of the number of electrons scattered through the angle θ, we obtain the expression

$$\rho(\theta)) = \frac{8}{\theta^2}\cos^2\left(r\theta + \frac{\theta}{2}\right)\sin^2\left(\frac{\theta}{2}\right). \qquad (63)$$

Formula (63) itself describes the interference pattern that could be drawn by soft photons of one coherent wave. However, as already noted, the energy of these photons is too small to leave a visible trace on the recording screen. While a large number of tagged waves fall on the screen, the tags (electrons) statistically reproduce a visible copy of this pattern.

In each individual event, the electron left a point trace on the absorbing screen. This is in full agreement with our presented picture of electron scattering. In each individual event, we deal with one electron, which is a carrier of corpuscular properties and is recorded as a point on the screen. Its accompanying soft photons, which are carriers of wave properties, have too little energy to leave any trace on the screen. Thus, to interpret the result of the experiment, it is not necessary to involve the assumption of the quantum state collapse, which, from the viewpoint of classical physics, seems to be extremely unnatural and contradictory to the theory of relativity.

18. BEAM SPLITTERS

Most of the modern experiments that investigate the locality and causality properties are carried out with photons. Therefore, before the further consideration of specific experiments, we describe the principles of operation of the two most commonly used optical instruments.

The first optical device is a polarizing beam splitter PBS. It serves to separate the photon beam into two subbeams polarized in two mutually orthogonal directions. The device geometry determines three orthogonal directions (orthogonal polarization basis): **I** is the direction of the incident beam, **H** is the horizontal direction, **V** is the vertical direction. If the incident photon beam is polarized horizontally, then after passing through the PBS the photons propagate in the direction **H**, if polarized vertically, then in the direction **V**. If the incident beam is polarized at a definite angle ϑ ($\vartheta \neq \pi n/2$), then a part of the photons passing through the PBS propagates in the direction **H** and acquires the horizontal polarization, and the other part, in the direction **V** and acquires the vertical polarization.

As applied to an individual beam photon, this physical phenomenon has essentially different interpretations in the standard quantum mechanical approach and in the described algebraic approach. In the standard approach, it is preferable not to talk about the polarization of an individual photon at all. If a photon is known to belong to a polarized beam, then it is said to have the corresponding polarization. If there is no information about the prehistory of a photon, then it is said that it has no polarization. It is said that it acquired the definite polarization (horizontal or vertical) only after passing through a PBS. This process is random and is not caused by any physical reality.

In the proposed approach, the interpretation is essentially different. Each photon is in a definite elementary state. This elementary state describes the physical reality that predetermines a result of the interaction of a photon with a PBS for any orientation of the horizontal and vertical directions, i.e., in any polarization basis. Thus, it is predetermined in which of the two possible directions the photon will propagate after passing through the PBS for a given orientation of the polarization basis. However, if it is not known in advance that in this basis the photon has a

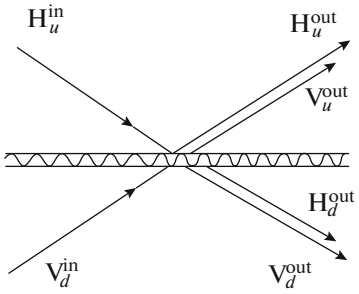

Fig. 6. A simple beam splitter.

definite polarization (belongs to a beam having vertical or horizontal polarization), then we cannot make a corresponding prediction.

As a result of observing a photon that has passed through the PBS, we acquire some information. Namely, we find what polarization the photon had in the polarization basis associated with the PBS. During the passage through the PBS, the polarization of a photon generally changes in an uncontrolled manner. However, if PBS implements a reproducible measurement, then the polarization does not change along the directions of the PBS polarization basis. It is changing in other directions. Thus, we can acquire information about the polarization of a photon in any direction, but only in one direction.

The second device is a (simple) beam splitter BS, which serves to split a coherent photon beam into two subbeams, or to mix two photon beams. Visually, this device can be represented as a translucent plate (see Fig. 6), on which beams of polarized photons are incident from two sides in the same plane perpendicular to the plate and at the same angles.

In this case, the device geometry also determines the polarization basis. We assume that the horizontal direction lies in the plane of the beams, while the vertical direction is perpendicular to this plane.

If photons from two beams fall on the plate not simultaneously, then each of them, without a change in polarization, either passes through the plate without a change in phase or is reflected with a change in phase by $\pi/2$. These two options are random and have the same probability. If two photons from different beams fall on the plate simultaneously, then they interfere according to the rule

$$\begin{aligned} |H,V\rangle_u^{in} &\to \frac{1}{\sqrt{2}}\Big[|H,V\rangle_u^{out} + |H,V\rangle_d^{out}\Big], \\ |H,V\rangle_d^{in} &\to \frac{1}{\sqrt{2}}\Big[|H,V\rangle_u^{out} - |H,V\rangle_d^{out}\Big]. \end{aligned} \tag{64}$$

In formulas (64), $|H,V\rangle$ means that a photon has either horizontal polarization, i.e., is in a quantum state $|H\rangle$, or vertical, i.e., is in a quantum state $|V\rangle$. In every line of Eq. (64), the polarization on the left and on the right is the same, in different lines the polarizations can be different. The index $u(d)$ denotes that the photon belongs to the up (down) beam, and the index *in* (*out*) means that the photon belongs to the incoming (outgoing) beam.

19. THE DELAYED CHOICE EXPERIMENT

The idea of tagged waves, in which sufficiently energetic photons act as tags, makes it possible to give a very illustrative interpretation of the so-called delayed choice experiment.

In the traditional approach, the dual behavior of a quantum system (wave or particle) is usually explained by the influence of the environment. So, it is said that when interacting with slits, the electron behaves like a wave, and when interacting with a recording screen, as a particle. To test this method of explanation, Wheeler [71] proposed a thought experiment forty years ago, which has recently been implemented in an almost ideal form [72].

The schematic diagram of the experimental setup is shown in Fig. 7. In the variant proposed by Wheeler, instead of beam splitters BS_{in} and BS_{out}, semitransparent mirrors appeared. The setup is a Mach–Zehnder interferometer with long arms. In a real experiment, they had a length of 48 m. At the classical level, the principle of operation of the installation looks very simple.

A beam of photons (P) is directed to the input translucent mirror (beam splitter BS_{in}). In this mirror, the beam splits into two coherent parts P_a and P_b, which follow the paths a and b, reflecting along the way from the mirrors M_a and M_b. There are two possible states of the setup. The first is when the output mirror (beam splitter BS_{out}) is absent. The interferometer is said to be open. In this case, each part is found in the corresponding detector (D_a or D_b). The second state is that the output mirror is present (closed interferometer). In this case, coherent addition of both parts of the beam occurs in the output mirror. The result of this addition is determined by the fact that when reflecting from the mirror, the phase changes by $\pi/2$, but when passing through the mirror, the phase does not change. Taking this into account, it is elementary to establish that after addition in the output mirror, the entire beam will arrive at the detector D_b.

At the quantum level, the picture is much more interesting. To get this picture in its pure form, the beam intensity is sharply reduced so that no more than one photon can be in the setup at the same time. Each of these photons can have both corpuscular and wave properties. We will assume that, depending on the state of the environment (on the state of the experimental setup), it exhibits either corpuscular or wave properties. If corpuscular, then as a result of interaction with the input mirror, the photon randomly

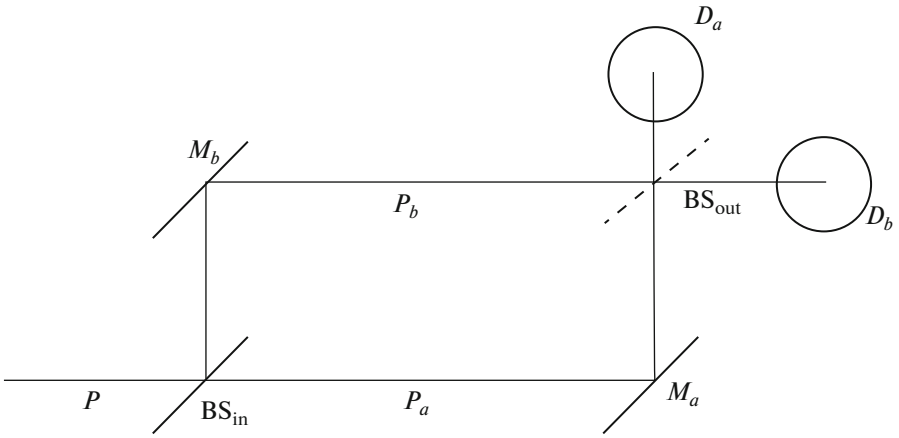

Fig. 7. Schematic diagram of the delayed choice experiment.

chooses one of the paths. If it has wave properties, then in the input mirror the wave is divided into two parts and propagates along both paths.

Let the state of the setup be such that there is no output mirror. Then, with the corpuscular behavior of the photon, one of the detectors will be triggered. By fixing which of the detectors fired, we can determine which path was chosen by the photon in the input mirror. With wave behavior, both detectors must work simultaneously. Now let the setup state be such that the output mirror is present. Then, with the corpuscular behavior of the photon, one of the detectors will again randomly fire. With the wave behavior the detector D_b will be always triggered.

Thus, in order for the quantum picture to match the classical one, in the absence of an output mirror, the photon must behave like a particle, i.e., must choose one of the paths. If the output mirror is present, then the photon should behave like a wave and propagate along both paths after the input mirror.

The choice of a separate path or both paths at once, it seems, must made by the photon, at the moment of passing the input mirror. To "disorientate" the photon, Wheeler suggested making the decision whether or not to insert the output mirror after the photon had passed the input mirror, and carried out that decision before the photon reaches the location of the output mirror. Thus, during the passage of the entrance mirror, "the environment will still not be clear" for the photon, However, in order to reproduce the classical picture, the photon has to make the right choice every time, i.e., it must guess in advance the whim of the experimenter.

It proved to be very difficult to actually carry out the manipulation of the mirror proposed by Wheeler. It was necessary to keep within 160 ns, which a photon spends on passing the interferometer base (48 m). The experimenters were able to carry out all the required manipulations in 40 ns. Of course, this cannot be done with a translucent mirror. Therefore, instead of a mirror, a beam splitter was used, which was switched on and off by an electrooptical modulator. In this case, the decision to turn the beam splitter on or off was made by a random number generator. The geometry of the setup was such that no signal propagating at a speed of less than the speed of light could transfer information about the decision made to the entrance beam splitter until the photon under study passed through it.

Despite all these precautions, the photon perfectly predicted the decisions of the random number generator. This means that in the time interval between the moments of passage of the input and output beam splitters, the photon has no alternative to be localized in one of the arms of the interferometer or in both at once. In some mysterious way, both of these possibilities are realized simultaneously. Formally, this does not contradict the standard mathematical apparatus of quantum mechanics. However, no clear physical picture of this phenomenon can be drawn up.

In contrast, in terms of elementary state, core, and hidden field, the physical picture of the phenomenon looks very simple. A photon entering the input beam splitter interacts with it. Depending on the elementary state of the photon, its core is either reflected in the beam splitter or passes through it. At the same time, as a result of the interaction, a hidden field is created that is coherent to the core. In this case, the hidden field is soft bremsstrahlung photons, and the core is a rather hard recorded photon. The hidden field is divided into two parts, one of which is distributed along one path, the other part, along the other path. Thus, in the time interval mentioned in the previous paragraph, the core of the incident photon and one of the hidden field parts are localized in one arm of the interferometer, and the second part of the hidden field is in the other arm. All parts of a photon retain coherence among themselves.

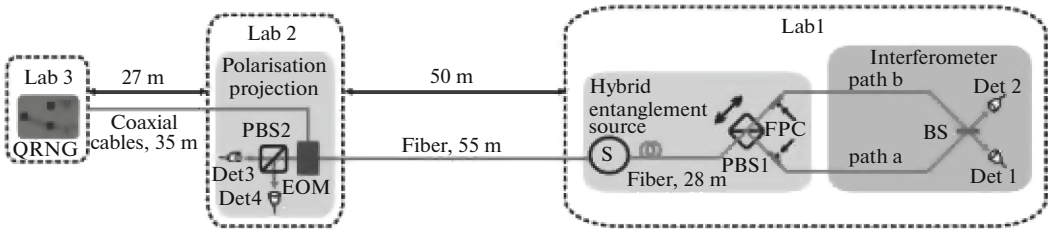

Fig. 8. Scheme of the Vienna experiment.

In the output beam splitter, both parts of the hidden field are coherently added, generating small secondary oscillations in the set of bremsstrahlung photons coherent to the core. These secondary oscillations resonantly interact with the core. Considering the phase shift between the hidden field parts, the resulting hidden field and the core after the output beam splitter propagate towards the detector D_b. When the core hits the detector, the latter registers this fact.

When the output beam splitter is off, the core proceeds on the path selected in the input beam splitter and enters one of the detectors, where it is recorded. The part of the hidden field propagating along a different path enters another detector. However, the detector does not react to the hidden field. In this case, the picture looks as if the photon has only corpuscular properties. Thus, the overall picture seems to be very clear and fully consistent with the principles of locality and causality.

20. ZEILINGER GROUP QUANTUM ERASER

The concepts of the core and hidden field make it possible to give a quite illustrative interpretation of the phenomenon, which is called the quantum eraser.

Let us first discuss the experiment of Zeilinger group [73]. Two versions of the experiment were implemented. The first one is Viennese with a base of 55 m. The second is Canaries with a base of 144 km. The results of both experiments were identical, and we will discuss only the first version.

The schematic diagram of the experiment is shown in Fig. 8, borrowed from [73]. The recording elements were located in three laboratories separated by distances which ensured that the experiments at different laboratories were separated from each other by spacelike intervals. Based on this, the authors of the experiment stated that their results contradict Einstein's locality condition. We will now see that contradictions can be avoided.

The experiment uses a pair of photons: a signal photon (s) and the surrounding photon (e). This pair is in an entangled state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}[|H\rangle_s|V\rangle_e + |V\rangle_s|H\rangle_e],$$

where H and V denote the horizontal and vertical polarizations.

The source (S) of these photons is a nonlinear crystal irradiated by an ultraviolet laser. Laser photons are scattered in the crystal. As a rule, with this scattering, one photon at the input gives one photon at the output. However, sometimes with much less probability one photon generates two photons whose total energy is equal to the energy of the incoming photon. This phenomenon is called the parametric frequency down-conversion. Depending on the properties of the crystal, different types of correlations between the polarizations of photons in the pair occur. If photons are produced with orthogonal polarizations, then it is said that this is a transformation of the second type.

One of the photons of the pair (photon (s)) is directed to laboratory 1. The other (photon (e)) is directed to laboratory 2. There it enters the electrooptical modulator EOM, which can be in two states: on, off. Turning on and off is done by a random number generator QRNG.

When the EOM is turned off, the photon (e) passes through it without changing its polarization. When the modulator is turned on, the cyclic polarization of the photon (e) changes to linear. After the EOM modulator, the photon (e) enters the polarization beam splitter PBS2. Depending on its linear polarization, the photon (e) is directed either to the detector Det3 or to the detector Det4.

The photon (s)—a partner of photon (e) is directed to the polarizing beam splitter PBS1. After this, depending on its polarization, it is directed either to the arm (a) or to the arm (b) of the interferometer. By shifting the beam splitter PBS1, it is possible to change the difference between the optical lengths of these arms. Next, the photon enters a simple beam splitter BS, after which it is directed either to the detector Det1 or to the detector Det2.

To collect statistics, the experiment is repeated many times in two modes: EOM is on, EOM is off. In both modes, the measurements are carried out for different shifts of the PBS1 beam splitter. The result of the data taking is the following. When the EOM modulator is turned off, the probability of a photon (s) to hit each of the detectors Det1 and Det2 is 1/2 and does not depend on the position of the beam splitter PBS1. When the EOM modulator is turned on, the probability depends on the position of the beam splitter PBS1. The graph of this dependence appears as a typical interference pattern.

Simultaneously, the following was established. With the EOM modulator switched off, the clicks of the detectors Det3 and Det4 unequivocally fix the arms (a) and (b), along which the photon (s) propagates. When the EOM modulator is on, these clicks do not provide this information.

Superficially, everything looks as if the random number generator QRNG affects the behavior of photons that reach the detectors Det1 and Det2. In this case, the following principle of complementarity is valid. With the EOM modulator turned off, it is possible (using detectors Det3 and Det4) to obtain information on which path the photon (s) propagates, but no interference pattern occurs. When the EOM modulator is on, information about the path of propagation of photon (s) is erased, but an interference pattern appears. It seems that the principle of causality is violated.

Now we will see that such a conclusion can be avoided. We will assume that the source (S) emits not individual photons (s) and (e) in some quantum states, but tagged waves, in which these photons are tags. As noted earlier, when collecting statistics, a definite coherent quantum state corresponds to the resulting ensemble of tags. The ensemble of soft photons, which forms a coherent de Broglie wave, corresponds to the same quantum state. Mathematically, a quantum state can be described by a Hilbert space vector, but Hilbert space is ill-suited for describing local properties.

In contrast, a soft photon wave is a physical object with well-defined local properties. Therefore, we will try to trace the relation between mathematical operations on Hilbert space vectors and physical operations on the corresponding wave of soft photons.

Let us first consider the case when the EOM modulator is turned off. After the EOM, the wave with the marker (e) enters the polarizing beam splitter PBS2. Depending on the polarization, it will go either to the detector Det3 or to Det4. For definiteness, we will assume that the wave with polarization H hits the detector Det4. In this case, a click of the detector Det4 will mean that the wave with the marker (e) emitted by the source (S) had a polarization H, while its partner, the wave with the marker (s), had a polarization V. Accordingly, after passing through the polarization beam splitter PBS1, it will go to the arm (b) of the interferometer. Let us denote the quantum state vector of this wave as $|b\rangle$.

Along the arm (b), this wave will reach a simple beam splitter BS. In this case, the phase of the wave will change by an angle ϕ, the value of which depends on the arm length. In the beam splitter BS, the wave is split into two parts. Half of the soft photons will pass through without changing the phase as a wave ($d1$) going to the detector Det1, the other half will be reflected with a phase change of $\pi/2$ and will go to the detector Det2 as a wave ($d2$). For each wave photon, the probability of hitting either of the two detectors is 1/2 and does not depend on the angle ϕ.

In terms of the Hilbert space, this process is as follows. Phase shift: $|b\rangle \to e^{i\phi}|b\rangle$. Passage of a beam splitter BS:

$$e^{i\phi}|b\rangle \to |d\rangle \equiv \frac{1}{\sqrt{2}}\left[e^{i\phi}|d_1\rangle + e^{i(\phi+\pi/2)}|d_2\rangle\right]. \qquad (65)$$

Here

$$\hat{d}_i|d_j\rangle = \delta_{ij}|d_j\rangle, \quad \langle d_i|d_j\rangle = \delta_{ij} \ (i,j=1,2),$$

where a click of the detector Det(i) is the value 1 of the observable $\hat{d}_i$. Obviously, $\langle d|\hat{d}_i|d\rangle = 1/2$.

Now consider the case when the modulator is on. For definiteness, we assume that the modulator transforms right-hand polarized photons into horizontally polarized photons, and left-hand polarized photons into vertically polarized ones. In this case, a click of the detector Det3 means that when a photon (e) was emitted from the source (S), it had a polarization R. Accordingly, its partner photon (s) must have polarization L. Similarly, a click of the detector Det4 indicates that the photon (s) has a polarization R.

When the modulator is on, an interference pattern is observed. However, it is visible only after collecting statistics, i.e., when we are dealing with ensembles of events rather than with individual events. In the case under consideration, these will be quantum ensembles for which Postulate 6 of Section 3 is valid.

Recall that a functional was constructed there, which describes the quantum average in the case when the commutative algebra defining the quantum state contains a one-dimensional projector.

For example, consider the case when the detector Det3 clicked. This means that the corresponding photon (s) has a polarization L. Dark (bremsstrahlung) photons coherent to it will also have polarization L. In our approach, this does not mean that they do not have any linear polarization. This fact means something else: in the set of these left-handed photons, half of the photons have a polarization V and the other half have a polarization H.

As a result, at the output of the beam splitter PBS1 we will have the following picture. The photon (s) will go along one of the paths, (a) or (b), while the wave of dark photons coherent with it will break into two parts. One of them will follow the path (a) and the other will follow the path (b). Immediately after the beam splitter PBS1, photons in different parts will have orthogonal polarizations. However, after passing through the polarization controls FPS, all photons will become equally polarized.

Incident on the beam splitter BS, both waves will add up coherently if the shift between phases is considered. This shift depends on the difference between the optical lengths of paths (a) and (b). Consequently, the number of wave photons that hit the detector Det1 and the detector Det2 will depend on the phase shift. Since, when collecting statistics, the probability of a marker to hit each of these detectors is equal to the probability of a wave photon to hit this detector, an interference pattern will be observed when the EOM modulator is on.

Again, let us see how this physical process can be represented in terms of the Hilbert space. At the input of the beam splitter PBS1, we have a soft photon wave, which is described by the vector

$$|L\rangle = \frac{1}{\sqrt{2}}[|H\rangle - i|V\rangle]. \quad (66)$$

Further, in a similar situation we will say that we have a wave (66). The beam splitter PBS1 converts it into the wave

$$\frac{1}{\sqrt{2}}[|a\rangle - i|b\rangle].$$

After the regulator PPC, the waves $|a\rangle$ and $|b\rangle$ will have the same polarization. Due to a shift of the beam splitter PBS1, at the input of the beam splitter BS, we will have the wave

$$\frac{1}{\sqrt{2}}[e^{-i\phi}|a\rangle - ie^{i\phi}|b\rangle].$$

According to Eq. (65), the beam splitter BS converts it into the wave

$$|d'\rangle = [-i\sin\phi|d_1\rangle + \cos\phi|d_2\rangle].$$

This implies

$$\langle d'|\hat{d}_1|d'\rangle = \sin^2\phi, \quad \langle d'|\hat{d}_2|d'\rangle = \cos^2\phi.$$

With the click of Det4, $\hat{d}_1$ and $\hat{d}_2$ change places.

The EOM modulator has no effect on the value of the observables $\hat{d}_1$ and $\hat{d}_2$ for each specific photon (s). It affects a set of events over which the values of these observables are averaged. Since data processing is carried out after the end of all stages of the experiment, it is completely indifferent how the individual stages are separated from each other.

Thus, there is no contradiction between Einstein's principle of causality and the results of this experiment.

21. KIM GROUP'S QUANTUM ERASER

Let us now consider an experiment carried out by Kim's group [74]. In this experiment, events that are correlated with each other are not separated by a spacelike interval. This experiment is an example of a delayed choice experiment in which the future seems to influence the past.

The schematic diagram of the experiment is shown in Fig. 9, borrowed from [74]. This experiment also uses entangled photon pairs obtained using the parametric frequency down-conversion of the second type. An ultraviolet laser irradiates a nonlinear BBO crystal through a double-slit grating in two regions A and B. In each individual event, one of these regions is a source of an entangled pair of photons: a signal photon (s) and the surrounding photon (e). The photon (s) is directed to the lens LS, in the focal plane of which the detector D_0 is located. This detector can move along the focal plane and record the number of photons (s) incident on different points of the plane.

The photon (e) is directed to the interferometer with arms of the same optical length. The interferometer consists of a prism, three simple 50 : 50 beam splitters (BSA, BSB, BS) and two mirrors M_A and M_B. Since the mirrors produce the same shifts of the phase, these shifts can further be ignored. After passing through the interferometer, the photon (e) enters one of the detectors D_1, D_2, D_3, D_4 (the detector D_4 is not shown in the figure). The signals from these detectors and from the detector D_0 are fed into a coincidence circuit, which makes it possible to determine for each photon (s), which of the detectors D_1, D_2, D_3, D_4 its partner, the photon (e), has hit. The overall dimensions of the installation are chosen such that the time of passage of a photon (e) to the detector D_0 is much less than the time of passage of its partner (e) to the recording detector

Let us follow an entangled pair of photons (s) and (e). Let the pair be created in region A. Photon (s), having passed the lens LS, enters the detector D_0. Its partner (e), reflected from the prism, hits the beam splitter BSA. After the beam splitter, it will hit the detector D_3 with a probability of 0.5, where it will be recorded. With the same probability, this photon will go to the mirror M_A, after which it will fall into the beam splitter BS. After this beam splitter, it will again fall into the detector D_2 with a probability of 0.5, where it will be recorded. With the same probability, it will go to the mirror M_B, after which he will hit the detector D_1.

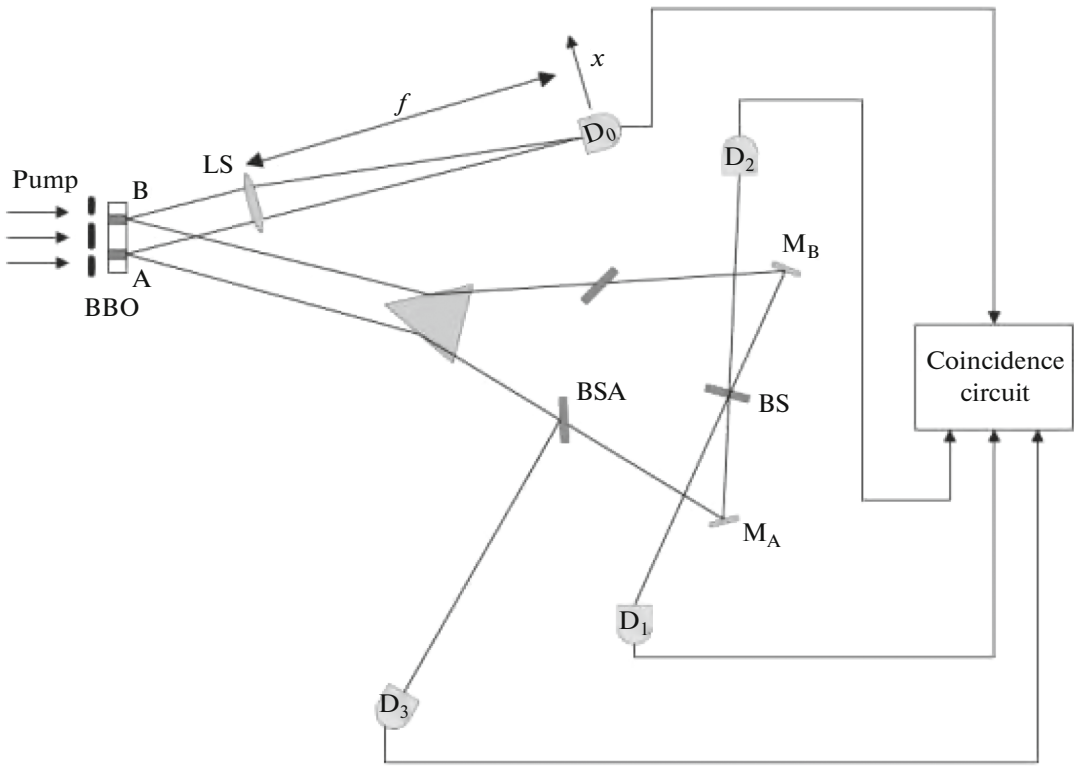

Fig. 9. Scheme of the experiment of Kim's group.

A pair created in region B behaves similarly. The only difference is that instead of a detector D_3, a photon (e) can enter detector D_4. Thus, a click of detector D_3 unambiguously indicates that the pair was created in area A, while a click of detector D_4 unambiguously indicates that the pair was created in region B. Clicks of detectors D_1 and D_2 give no information about the place where the pair was created.

An analysis of the detector operation results shows that if events with clicks of the detector D_3 or detector D_4 are selected, then the detector D_0 does not record any interference pattern. Let us pay attention to the fact that in these cases it is uniquely known in which region the photon pair was created.

If events with clicks of detector D_1 or detector D_2 are selected, then the detector D_0 reproduces the interference patterns. In these two cases, information about the place of the creation of the pair turns out to be erased.

Such an analysis of the experimental results allowed the authors to conclude that if information about the production place of a pair cannot be obtained from the results of the experiment, then an interference pattern takes place, and if this information can be obtained, then there is no interference pattern. Thus, it seems that the presence or absence of an interference pattern is determined by the results of operation of the detectors D_1, D_2, D_3, D_4, and on the other hand, the interference pattern itself is built from the detector readings that were obtained before the detectors were triggered. In other words, indeed, everything looks as if the future influences the past.

Now let us see how the experiment results can be interpreted in terms of the concept of tagged waves. We will assume that it is tagged waves, in which the observed photons are tags, rather than individual laser photons fall on the lattice along the axis z. Let us denote the width of each slit in the lattice as a, and the distance between the slits as r. Let us direct the axis x perpendicular to the slits in the lattice plane.

It is clear that only the waves whose tags passed either through one slit or through another one will take part in the formation of the interference pattern. The interference pattern itself is determined by a probabilistic distribution k_x of projections of the tag momenta onto the axis x. Thus, a lattice with two slits will form an ensemble of physical systems that will leave traces on the recording screen.

Each of these systems is described by a large number of observables related to both the tag and wave photons. In each this system, we single out a subsystem with observables: $\mathbf{k}_x$ and $\hat{x}$. Strictly speaking, these observables are not elements of the C^*-algebra,

but their spectral projectors $\hat{p}(k)$ and $\hat{p}(x)$ are the C^*-algebra elements. Recall that in the theory of self-adjoint operators in a Hilbert space, each section of the operator spectrum is associated with a definite projector operator.

A two-slit lattice selects subsystems in which tags pass through one or another slit. This means that when passing through the lattice, the coordinate of any of these tags will satisfy the condition

$$\frac{r}{2} < |x| < \frac{r}{2} + a, \tag{67}$$

and these tags will have the same value of the observable $\hat{p}(x)$—the projector corresponding to a region in the spectrum (67). So, the lattice will form a quantum ensemble (class of equivalence of elementary states). In this ensemble, the probability distribution must satisfy Postulate 6 from Section 3, and we can use the Eq. (19)

$$\hat{p}_0 C \hat{p}_0 = \vartheta(\hat{C})\hat{p}_0. \tag{19}$$

This formula involves operators of the exact representation of the C^*-algebra in the Hilbert space, therefore we can perform further mathematical calculations using the standard technique of quantum mechanics.

Since we are only interested in the general view of a picture at the screen, we somewhat idealize the experimental conditions. In particular, we will assume that when a pumping photon decays into photons (s) and (e), both energy and momentum are conserved, and they are distributed equally between these photons. In reality, this is true only on average. For more detailed calculations, it is possible to use the technique considered in [75].

Let us first follow the events when the detector D_1 is clicked. We will trace tagged waves, but, just as in the previous section, we will depict them as Hilbert space vectors describing the quantum state of these waves. Again, these vectors will be called waves. After passing through the lattice, we will deal with two waves

$$\frac{1}{\sqrt{2}}[|\chi_A(x)\rangle + |\chi_B(x)\rangle]. \tag{68}$$

Here we introduced two functions:

$$\chi_A(x) = \frac{1}{\sqrt{a}} \quad \text{for} \quad \frac{r}{2} < -x < \frac{r}{2} + a, = 0 \text{ otherwise}$$

and

$$\chi_B(x) = \frac{1}{\sqrt{a}} \quad \text{for} \quad \frac{r}{2} < x < \frac{r}{2} + a, = 0 \text{ otherwise}.$$

In (68), the first term corresponds to the wave with a tag passing through the slit A, and the second term corresponds to the wave with a tag passing through the slit B. Further, the functions $\chi_A(x)$ and $\chi_B(x)$ will play a role of wave functions of the corresponding quantum states in the x-representation. After the BBO crystal, each of the terms in (68) will describe two waves with tags (e) and (s) coming out of region A or region B.

Lest the formulas be cluttered up, we will further omit the terms corresponding to the waves going to the detectors D_3 and D_4. After passing the beam splitters BSA, BSB and BS, there will be another separation of the tagged waves into waves in which photons (e) propagate towards the detector D_1, or detector D_2. The result of this separation in terms of Hilbert space vectors can be represented as follows:

$$2^{-1}[|\chi_A(x)\rangle|d_2\rangle + i|\chi_A(x)\rangle|d_1\rangle + |\chi_B(x)\rangle|d_1\rangle + i|\chi_B(x)\rangle|d_2\rangle] \equiv |\chi, d\rangle,$$

where we considered a phase change of the wave tagged by the photon (e) upon reflection in the beam splitter. If we select the events in which the detector D_1 is clicked, then we get an ensemble of waves tagged by photons (s), which is described by the vector

$$\chi_1(x) \equiv [i|\chi_A(x)\rangle + |\chi_B(x)\rangle]/\sqrt{2} = \sqrt{2}\langle d_1|\chi, d\rangle. \tag{69}$$

In terms of probability theory (see, e.g., [4, 21]), this ensemble is described by the conditional probability

$$P(F_1|F_2) = \frac{P(F_1 \cap F)_2}{P(F_2)}.$$

In this formula $P(F_i)$ is the probability of the event F_i, $P(F_1 \cap F_2)$ is the probability of the simultaneous implementation of events F_1 and F_2. Although the term "simultaneous" is used here, it has nothing to do with time. It means that both events occurred, while in what sequence, or with what time interval they occurred, is completely indifferent. $P(F_1|F_2)$ is the probability of an event F_1 given that the event F_2 also occurred. In our case, $P(F_2) = 1/2$ is the probability of clicking of the detector D_1.

We can easily calculate $R_{01}(k_x)\delta k_x$—the conditional average of the number of photons (s) having the x-projection of momentum in the interval δk_x—provided that the photon (e) hits the detector D_1. To this end, we use Eq. (19), in which we substitute the operator $|\chi_1(x)\rangle\langle\chi_1(x)|$ as a projector $\hat{p}_0$, while $|k_x\rangle\langle k_x|\delta k_x$ we substitute as $\tilde{A}$. In this case, Eq. (19) can be rewritten as

$$R_{01}(k_x)\delta k_x = |\langle k_x||\chi_1(x)\rangle|^2 \delta k_x. \tag{70}$$

Here

$$k_x\chi_1(x) = \frac{1}{\sqrt{4\pi\hbar}}\left[\int dx \exp\left(\frac{-ixk_x}{\hbar}\right)(i\chi_A(x) + \chi_B(x))\right].$$

By integrating, we get

$$R_{01}(k_x)\delta k_x = \frac{\hbar}{\pi a k_x^2}\sin^2\left(\frac{ak_x}{2\hbar}\right)\cos^2\left(\frac{\pi}{4} + \frac{k_x(r+a)}{2\hbar}\right)\delta k_x.$$

Substituting $k_x \simeq 2\pi\hbar x(f\lambda)^{-1}$, this formula can be given the form

$$R_{01}(x)\delta x \simeq \frac{f\lambda}{2\pi^2 x^2 a}\sin^2\left(\frac{x\pi a}{f\lambda}\right) \times \cos^2\left(\frac{\pi}{4}+\frac{x(a+r)\pi}{f\lambda}\right)\delta x. \quad (71)$$

Here, $R_{01}(x)$ is the average density of the number of photons (s) that hit the point x of the recording screen, f is the focal length of the lens, $\lambda = 2\pi\hbar k^{-1}$.

Under the condition that the tag (e) is recorded by the detector D_2, we similarly obtain for the density of the number of photons (s) falling at the point x of the recording screen, we obtain the expression

$$R_{02}(x)\delta x \simeq \frac{f\lambda}{2\pi^2 x^2 a}\sin^2\left(\frac{x\pi a}{f\lambda}\right) \times \sin^2\left(\frac{\pi}{4}+\frac{x(a+r)\pi}{f\lambda}\right)\delta x.$$

Now consider the events when the detector D_3 is clicked. Again, omitting the waves with the tag (e), which go to the detectors D_1, D_2, and D_4, in this case, instead of Eq. (69), we will have

$$|\chi_3(x)\rangle = i|\chi_A(x)\rangle.$$

Accordingly, for the conditional average number of photons (s), for which the x-projection of the momentum falls into δk_x, provided that the photon (e) is recorded by the detector D_3, instead of Eq. (70), we obtain

$$R_{03}(k_x)\delta k_x = 1/2|\langle k_x|\chi_3(x)\rangle|^2\,\delta k_x. \quad (72)$$

The factor 1/2 on the right-hand side of this formula appeared due to the fact that in this case only half of the pumping photons contribute to the average number of coincidences: these are photons that have passed through the slit A. In the experiment, when calculating the number of coincidences, the number of pumping photons was taken as the initial quantity.

In Eq. (72),

$$\langle k_x|\chi_3(x)\rangle = \frac{i}{\sqrt{2\pi\hbar}}\int dx\exp\left(\frac{-ixk}{\hbar}\right)\chi_A(x) = \sqrt{\frac{2\hbar}{\pi a k_x^2}} \times \sin\left(\frac{ak_x}{2\hbar}\right)\left[i\cos\left(\frac{k_x(r+a)}{2\hbar}\right)-\sin\left(\frac{k_x(r+a)}{2\hbar}\right)\right].$$

Thus,

$$R_{03}(k_x)\delta k_x = \frac{\hbar}{\pi a k_x^2}\sin^2\left(\frac{ak_x}{2\hbar}\right)\delta k_x,$$

or

$$R_{03}(x)\delta x \simeq \frac{f\lambda}{2\pi^2 x^2 a}\sin^2\left(\frac{x\pi a}{f\lambda}\right)\delta x.$$

For events when the detector D_4 is clicked, the calculations and results are similar.

The presented calculation results are in good agreement with the experimental results given in [74]. However, the conclusions that follow from these results are quite different. The results of the operation of the detectors D_1, D_2, D_3 and D_4 in each specific elementary event do not affect the operation of the detector D_0 in any way. They influence the statistical processing of these results when we move from elementary events to random events. The latter most essentially depend on the fact of which elementary events are included in the random event of interest to us.

The probability of a random event may have nothing to do with the localization of individual elementary events. Thus, the results of the work of Kim's group in no way contradict Einstein's principle of causality.

22. ENTANGLED STATES, QUANTUM TELEPORTATION

When discussing the problems of locality, the most interesting and mysterious are the so-called entangled states. The term was once introduced by Schrödinger [76], and in the original it appeared as: "Verschrankung". The translation—"interwoven states"—would be more accurate, but the term "entangled states" has taken root in the Russian-language literature, though it sounds somewhat ambiguous. For a system consisting of two particles, each of which can be in two quantum states, orthogonal to each other, $|+\rangle$ and $|-\rangle$, the examples of typical entangled states are

$$\begin{aligned}
|\Psi^{(-)}\rangle_{12} &= \frac{1}{\sqrt{2}}[|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2],\\
|\Psi^{(+)}\rangle_{12} &= \frac{1}{\sqrt{2}}[|+\rangle_1|-\rangle_2 + |-\rangle_1|+\rangle_2],\\
|\Phi^{(-)}\rangle_{12} &= \frac{1}{\sqrt{2}}[|+\rangle_1|+\rangle_2 - |-\rangle_1|-\rangle_2],\\
|\Phi\rangle^{(+)}{}_{12} &= \frac{1}{\sqrt{2}}[|+\rangle_1|+\rangle_2 + |-\rangle_1|-\rangle_2].
\end{aligned} \quad (73)$$

Here $|\cdot\rangle_{12}$ depict the state vectors in the Hilbert space of a two-particle system, while $|\cdot\rangle_1$ and $|\cdot\rangle_2$ are the state vectors of the 1st and 2nd particles in the Hilbert space of one-particle systems, $|\cdot\rangle_1|\cdot\rangle_2$ is the direct product of the corresponding vectors. The quantum states depicted in formulas (73) are often called Bell states.

A distinctive feature of entangled states is that classical instruments can be used to prepare the corresponding pure states of a many-particle (in the case of formulas (73) two-particle) system. However, even after this it is impossible to say, in what pure quantum state each of the particles, which make up the system, is. On the other hand, if further measurements are

made on one particle, then it is possible to establish not only the pure state of this particle, but also the pure quantum state of its partner, which was not subjected to measurement.

For example, if it is known that a two-particle system is in a quantum state $|\Psi^{(-)}\rangle_{12}$, then nothing can be said in which of the two possible states, $|+\rangle$ or $|-\rangle$, each of the particles is. However, if as a result of a subsequent measurement on the 1st particle it is found that it is in the state $|+\rangle_1$, then with a probability of 1 it can be predicted that during the measurement on the 2nd particle we will find it in the state $|-\rangle_2$.

This state of affairs in the standard approach to quantum mechanics is fixed in the form of the so-called projection principle [1]. According to this principle, a measurement carried out on a part of the physical system under study leads to a change (reduction) in the quantum state of the entire system. In this case, the characteristics of not only the part of the system that was affected by the measuring device, but also the other part, which was not subjected to this impact, may change. Thus, in the case considered above, as a result of measuring the characteristics of the 1st particle, the state $|\Psi^{(-)}\rangle_{12}$ is reduced (collapses) into the state $|+\rangle_1|-\rangle_2$.

As a recipe for a mathematical description of the effect of a measuring instrument on a quantum object, the projection principle generally works very well. However, it is not possible to give this principle any visual physical interpretation consistent with the theory of relativity in the standard approach.

In his famous book [1], von Neumann introduces the notion of two types of influence on a physical system. As a result of the impact, which von Neumann attributed to the second type, the quantum state changes according to the Schrödinger equation. This change obeys the principle of causality and is unambiguously predictable. This is how the quantum state of the system changes when it interacts with another quantum system or an external classical field.

Von Neumann attributed an impact of a measuring device on a physical system to the first type. With this impact, the quantum state changes randomly and, according to von Neumann, is causeless. It looks very strange, since any measuring device can be considered either as some kind of quantum system, or as an external classical field. The only distinctive feature of the interaction of the measuring device with the physical system under study is that as a result of this interaction we obtain some information about the system. In this regard, von Neumann introduced the concept of psychophysical parallelism. According to this principle, in describing the impact of the first type, the inner self of the researcher plays a fundamental role. By this, von Neumann tried to explain the unusual properties of this type of impact.

Unlike other reasoning of von Neumann, this argument does not seem to be in any way convincing. Subsequently, numerous attempts were made to substantiate the projection principle, but all of them, to put it mildly, from a physical point of view, seem disputable.

Further, we will not discuss the projection principle in general (on this subject, see [10]), but will focus our attention on the problems that arise when applying this principle in one specific case, namely, in the so-called quantum teleportation.

Since the state is usually used when discussing the Einstein–Podolsky–Rosen paradox, in modern literature this quantum state is often called an EPR state, and the corresponding two-particle system is named an EPR pair.

The idea of the nonlocal nature of quantum measurements gave rise to great hopes for the possibility of a fundamentally new way of transmitting information (see, e.g., [78]). In the scientific literature, this method is called "quantum teleportation." Numerous experiments have already been carried out, which seem to confirm these hopes. At the same time, the element of mystery inherent in the "teleportation" concept is also preserved in the scientific literature. Next, we will try to remove this veil of mystery. See also [20] on this subject.

The essence of the teleportation phenomenon can be understood by considering a thought experiment, a scheme of which is shown in Fig. 10.

Here S is the source of the initial state; EPR is the source of EPR pairs; A is the Bell state analyzer (Alice); B is the unitary converter (Bob); $\{C\}$ is the classical communication channel; {1} is the carrier of the initial teleportable state; {2}–{3} is the EPR pair; {4} is the carrier of the final teleportable state.

In the standard form, the description of the teleportation phenomenon is as follows (see, e.g., [80]). The source S emits a particle {1} in the quantum state Ψ_1, which is matched by the Hilbert space vector $|\Psi\rangle_1 = \alpha|+\rangle + \beta|-\rangle$, where α and β are complex numbers satisfying the condition $|\alpha|^2 + |\beta|^2 = 1$. Particle {1} is heading towards Alice. The EPR source emits an EPR pair {2} and {3} in the state $\Psi_{23}^{(-)}$ (vector $|\Psi^{(-)}\rangle_{23}$, see Eq. (73)). One of the particles ({2}) of the pair goes to Alice, the other particle ({3}) goes to Bob.

According to the standard rules of quantum mechanics, the state of a three-particle system (particles {1}, {2}, {3}) is described by the quantum state vector $|\Psi\rangle_{123} = |\Psi\rangle_1|\Psi^{(-)}\rangle_{23}$. This vector can be repre-

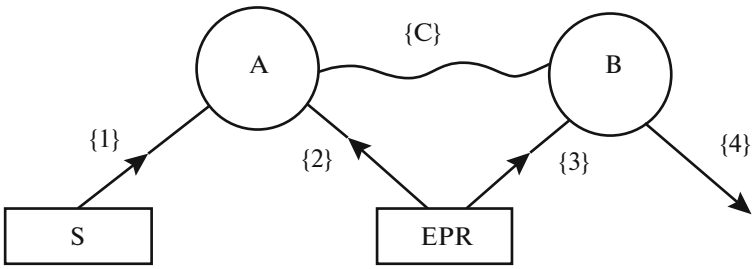

Fig. 10. Scheme of quantum teleportation.

sented as an expansion in terms of the Bell states of particles {1} and {2} (see Eq. (73)):

$$|\Psi\rangle_{123} = \frac{1}{2}\{|\Psi^{(-)}\rangle_{12}(-\alpha|+\rangle_3 - \beta|-\rangle_3) + |\Psi^{(+)}\rangle_{12}(-\alpha|+\rangle_3 + \beta|-\rangle_3) + |\Phi^{(-)}\rangle_{12}(\alpha|-\rangle_3 + \beta|+\rangle_3) + |\Phi^{(+)}\rangle_{12}(\alpha|-\rangle_3 - \beta|+\rangle_3)\}. \quad (74)$$

Alice, by using the analyzer A, determines, in what of the four possible Bell states, the particles {1} and {2} are, which have fallen to her. Let, e.g., them be in the state $|\Psi^{(-)}\rangle_{12}$. After this measurement, according to the projection principle of standard quantum mechanics, the three-particle state is reduced as follows

$$|\Psi\rangle_{123} \to |\Psi^{(-)}\rangle_{12}(-\alpha|+\rangle_3 - \beta|-\rangle_3). \quad (75)$$

Through the classical communication channel, Alice reports the result of her observation to Bob. Having received the message that Alice has recorded the state $|\Psi^{(-)}\rangle_{12}$, Bob, without doing anything, further transmits the particle {3}. This particle, according to the right-hand side of Eq. (96), will be in the state $|\Psi\rangle_4 = (-\alpha|+\rangle_3 - \beta|-\rangle_3)$.

The quantum states described by the vectors $|\Psi\rangle_1$ and $|\Psi\rangle_4$ coincide. Initially, there was no correlation between the quantum states of particles {1} and {3}. Alice manipulated only particles {1} and {2}. At the moment of these manipulations, the particle {3} could be in the region of space $\mathfrak{M}$, the space-like region of Alice's manipulations. However, particle {3} somehow mysteriously turned out to be in the quantum state that particle {1} was in. In this case, neither Alice nor Bob could know what quantum state the particle {1} was in.

If Alice gets a different result for {1} and {2} and reports it to Bob via the classical communication channel, then Bob will have to work a little. If the result is $|\Psi^{(+)}\rangle_{12}$, then Bob should perform a unitary operation on particle {3}, which reduces to the transformation $-|+\rangle \to |+\rangle$. If the result is $|\Phi^{(-)}\rangle_{12}$, then a transformation $|+\rangle \leftrightarrow |-\rangle$ is necessary. If the result is $|\Phi^{(+)}\rangle_{12}$, then the transformation $-|+\rangle \to |+\rangle \leftrightarrow |-\rangle$ is needed. After these operations, particle {3} will turn out in the state $|\Psi\rangle_4$ that coincides with the quantum state $|\Psi\rangle_1$. In these cases, due to the additional operations available, the paradoxical nature of the situation proves to be veiled. However, from the ordinary viewpoint, it looks as absurd as in the first case.

The standard words that are pronounced in this situation are something like this: "A lot of what is absurd from an ordinary viewpoint is the norm in the quantum world." However, one can do without this weak consolation.

For greater clarity, we will assume that the two-particle system under consideration consists of two particles, each of which has a spin equal to 1/2. In this case, $|+\rangle_1$ means the quantum state of the first particle with the spin projection on the selected axis (axis z) equal to $+1/2$, and $|-\rangle_1$, with the spin projection equal to $-1/2$. Similarly for the second particle.

Then, when using the concept of elementary state, the fact that the source S emits particles in a definite quantum state means the following. The source emits a beam of particles whose elementary states are different, but they all belong to one specific equivalence class. The beam need not be localized either in time or space. The equivalence class may not be known to anyone. The latter means that the numbers α and β appearing in the expansion of $|\Psi\rangle_1$, in terms of the basis $|+\rangle, |-\rangle$, may not be known, but they are the same for all beam particles (with an accuracy to a common phase factor). Therefore, there is such a coordinate system in three-dimensional space in which all particles of the beam will have a spin projection on the axis z equal to $+1/2$. In this coordinate system, $\alpha|+\rangle + \beta|-\rangle$ is matched by $S_z = +1/2$; $S_z = -1/2$ corresponds to the state vector $-\alpha|+\rangle + \beta|-\rangle$. The x axis can be directed so that the vector $\alpha|-\rangle + \beta|+\rangle$ was matched by $S_x = +1/2$, and the vector $\alpha|-\rangle - \beta|+\rangle$ was matched by $S_x = -1/2$.

Each particle of the beam emitted by the source S is analyzed by Alice together with the particle of the EPR pair emitted by the source EPR. Different EPR pairs are in different elementary states, but in each pair the elementary state of one particle is a negative copy of the elementary state of another particle. A physical

system consisting of an analyzer and a particle {1} can be considered as a complex measuring instrument. With this device, Alice sorts particles {2} into four groups. Each group includes particles {2}, which, together with the particle {1}, are in a two-particle elementary state belonging to one of the four quantum Bell states.

Since each of particles {2} has an EPR pair partner, this sorting can be considered as a splitting of the beam of particles {3} into four subbeams. According to Eq. (95), each such subbeam will contain particles {3} that have the definite value of the spin projection either onto the axis z or onto the axis x. This is because the elementary state of each particle {3} is strongly correlated with the elementary state of the partner-particle {2}.

Since Alice determines into which group each of particles {2} fell, she receives information about the group, into which the corresponding particle {3} fell. Alice communicates this information to Bob through the classical communication channel. This information is enough for him to choose the desired unitary transformation. With the help of this transformation, he changes the elementary state of particle {3} such that it turns out to be in a quantum state Ψ_1.

By her measurement, Alice did not affect the elementary state of particle {3} in any way, she only obtained some information about this elementary state with the help of indirect measurement, which she shared with Bob. After that, Bob performed some manipulations on particle {3}. As a result of these manipulations, the elementary state of particle {3} did not become an exact copy of the elementary state of particle {1}. Therefore, the term "teleportation" in this case does not seem particularly successful. Bob only succeeded in driving particle {3} into the same equivalence class as particle {1}.

A real experiment in which the quantum teleportation was observed (see [81]) was carried out with photons. A discussion of this experiment using the concept of an elementary state can be found in [20].

23. COMPUTER MODEL OF THE QUBIT

Recently, a new scientific direction that lies at the intersection of quantum physics and information theory, the physics of quantum information, has been intensively developed (see, e.g., [78]).

In classical information theory, a bit is accepted as an elementary unit. This is information available in any classical system, which can be in two mutually exclusive states: yes–no, 0–1, on–off, etc.

The elementary unit in quantum information theory is called a qubit. It is believed that the qubit carrier is a quantum system which has an observable that can take two values. For example, a particle with spin 1/2 can be in states in which the spin projection onto the selected axis is either $+1/2$, or $-1/2$. A photon can have a helicity equal to either $+1$, or -1, etc.

At the same time, it is believed that, unlike a classical system, a quantum system can be in a superposition of these states. Therefore, a qubit can potentially contain much more information than a bit. It is with this property of quantum systems that very high hopes are associated for their practical use in the field of storage, processing, and transmission of information.

From the viewpoint of information theory, the physical realization of the information carrier is completely irrelevant. Therefore, further, we will use the terms bit and qubit to refer not only to units of information, but also to carriers of the corresponding information. Potentially, a qubit seems to have great advantages over a bit. However, at the current level of technological development, it is practically very difficult to work with qubits. In particular, the problem of decoherence is very acute.

Here we will try to replace the qubit with the so-called soft qubit, or squbit for short. On the one hand, the squbit must carry the same observed information as the "hard" qubit, and on the other hand, the squbit must be capable of being implemented as a computer program. Unlike the state of a "hard" qubit, a computer program can be stored almost indefinitely and can be transmitted almost without distortion over any distance.

Technically, working with a squbit is incomparably easier than with a qubit. Therefore, the squbit is a much more convenient object for experiments. Of course, there is a danger that in the squbit we have not reproduced all the observable properties of the qubit. Therefore, the final word remains with qubit experiments. However, squbit experiments can be very good blueprints for qubit experiments. In addition, these experiments may cast doubt on many of the stereotypes that have developed in large numbers in quantum physics.

24. SINGLE SQUBIT

In the physics of quantum information, it is believed that the (pure) state of a qubit is described by a vector of a single-particle Hilbert space. However, this vector is related to the results of the experiment in a very indirect way. Namely, it is considered that the mathematical expectation of a definite self-adjoint linear operator on this vector is equal to the average value of the corresponding observable obtained in a series of experiments.

From the experimental data, this average value is calculated according to the laws of standard classical probability theory. This means that within the Kolmogorov approach, each result of a single experiment corresponds to an elementary event. However, it is believed that Kolmogorov's probability theory is not applicable in quantum physics. Indeed, the standard mathematical apparatus of quantum mechanics does

not contain an ingredient that has the properties of an elementary event. On the other hand, an elementary event is the basic concept of Kolmogorov's probability theory.

In the previous sections, a new approach to quantum mechanics was developed, in which a new concept of "elementary state" was introduced. On the one hand, this state is associated with a more complete description of a quantum system than a quantum state. On the other hand, it has the properties of an elementary event in Kolmogorov's probability theory.

Accordingly, we will assume that a single squbit should be characterized in more detail by an elementary state rather than by a quantum state. This will make it possible to work with squbits using Kolmogorov's theory of probability.

Just like a normal qubit, a squbit can be implemented in a variety of ways. We will use the method proposed in [11]. In this version, the squbit appears as a multilayered, gray-colored sphere of unit radius. Each layer is colored as follows. The positive pole is black (+1), the negative pole is white (–1). Intermediate regions have a gray color, the saturation of which with black changes according to the law

$$\rho_{\mathbf{R}}(\mathbf{r}) = (\mathbf{rR}),$$

where $\mathbf{r}$ is the unit radius-vector drawn to the current gray point, while $\mathbf{R}$ is the unit radius-vector drawn to the positive pole. Different layers may have different orientations. In addition, with each layer having a number k, a function $\varepsilon^{(k)}(\mathbf{r})$ is associated, which, for all r, firstly, satisfies the conditions

$$-1/2 < \varepsilon^{(k)}(r) < +1/2, \quad \varepsilon^{(k)}(-r) = -\varepsilon^{(k)}(r), \qquad (76)$$

secondly, one of the conditions

$$\left|\mathbf{R}^{(k)}\mathbf{r} + \varepsilon^{(k)}(\mathbf{r})\right| > 1/2, \qquad (77)$$

or

$$\left|\mathbf{R}^{(k)}\mathbf{r} + \varepsilon^{(k)}(\mathbf{r})\right| \le 1/2. \qquad (78)$$

The layer for which condition (77) is satisfied will be called active, while that for which condition (78) is satisfied, we will call passive.

When looking at the function $\rho_{\mathbf{R}}(\mathbf{r})$, an association immediately arises with the distribution function of the average value of the spin projection on the direction $\mathbf{r}$ for a particle with spin 1/2, which is in a fixed quantum state. This is quite natural. If we want the squbit to model the properties of the qubit, then those properties should somehow be embedded in the definition of the squbit. It is highly desirable that this definition (in particular, function $\rho_{\mathbf{R}}(\mathbf{r})$) be the same for all processes that we are going to describe using squbits. Otherwise, we will deal with the simulation of a separate quantum process (which seems to be a much less interesting task) rather than with the simulation of a qubit.

Further, for a squbit, the terminology is used, which is used for the qubit realized as a quantum particle with spin 1/2. The elementary state of a squbit is fixed uniquely if $\mathbf{R}^{(k)}$ and $\varepsilon^{(k)}(\mathbf{r})$ are given for all k and $\mathbf{r}$. This requires an infinite amount of information. Therefore, it may seem that the very concept of an elementary state is rather useless. In reality, this is not the case due to the specific nature of quantum measurements. As we will see below, in most cases, to obtain the numerical value of the observable under study, it is sufficient to know the characteristics of a small number of external layers. In addition, since in each individual measurement it is possible to measure the spin projection $S(\mathbf{r})$ in only one direction $\mathbf{r}$, then to fix the measurement result it is sufficient to fix values of the functions $\varepsilon^{(k)}(\mathbf{r})$ only for this direction $\mathbf{r}$.

The statistical nature of the measurements is easy to visualize, assuming the presence of a certain "host of an elementary event," which personifies all uncontrolled effects on the object under study. This host selects one particular event from all the allowed elementary events. In the case of a squbit, it chooses $\mathbf{R}^{(k)}$ and $\varepsilon^{(k)}(\mathbf{r})$, keeping r fixed. In a computer implementation of a squbit, a random number generator can act as this host.

We assume that the instrument measuring the spin projection $S(\mathbf{r})$ responds only to the active layer. If $\mathbf{R}^{(k')}\mathbf{r} > 0$, then $S(\mathbf{r}) = +1/2$, if $\mathbf{R}^{(k')}\mathbf{r} < 0$, then $S(r) = -1/2$. Thus, the result of a particular measurement depends on one controlled parameter $\mathbf{r}$ (controlled by the measuring instrument) and two uncontrolled parameters: the number of the active layer (k') and the pole of the active layer $\mathbf{R}^{(k')}$. These two parameters are determined by the parameters $\mathbf{R}^{(k)}$ and $\varepsilon^{(k)}(\mathbf{r})$, which are chosen by the host.

In Kolmogorov's theory of probability, no probability measure can be assigned to an elementary event (in our case, an elementary state) in the general case. These measures can be attributed only to some subsets of the set of elementary events. These subsets must form a σ-algebra (see Section 4). A distinctive feature of quantum systems is that it is impossible to construct a general σ-algebra for them, which would make it possible to describe the probabilities of events containing the definite values of incompatible observables (see [9, 20]). For each group of compatible observables, its own σ-algebra should be constructed. For squbits, as generators of these σ-algebras, the sets of elementary states can be taken (see [11]), for which $\mathbf{R}^{(k)}$ lie inside small solid angles $d\mathbf{R}^{(k)}$, while $\varepsilon^{(k)}(\mathbf{r})$, inside small intervals $d\varepsilon$. For each value $\mathbf{r}$, its own σ-algebra and, accordingly, its own probability measures should be constructed. In numerous "proofs" of the assertion that quantum distributions cannot be repro-

duced within the Kolmogorov probability theory, it is regularly forgotten, first, that the σ-algebra must be fixed, and second, that it cannot be fixed for incompatible observables. For example, in the proofs of various versions of Bell's inequalities, there is even no mention of σ-algebra, though σ-algebra is the basic concept of Kolmogorov's probability theory.

Let us now describe the effect on the squbit, which is implemented by a device that is analogous to a device that is called in optics a polarization beam splitter PBS. In optics, this device splits an unpolarized beam of light into two beams polarized in two orthogonal directions. In our case, we will assume that the geometry of the device prefers a definite direction $\mathbf{n}$ (a unit vector) and divides the squbit ensemble into two subensembles $\Psi^+_\mathbf{n}$ and $\Psi^-_\mathbf{n}$. For each individual squbit, the measurement result is determined by the actions of the host of the elementary event (random number generator). The host generates values $\mathbf{R}^{(1)}$ and $\varepsilon^{(1)}(\mathbf{n})$ for the first squbit layer. If $\mathbf{R}^{(1)}\mathbf{n} + \varepsilon^{(1)}(\mathbf{n}) > 1/2$, then PBS records the value $S(n) = +1/2$ and directs the squbit to the sub-ensemble $\Psi^+_\mathbf{n}$. If $\mathbf{R}^{(1)}\mathbf{n} + \varepsilon^{(1)}(\mathbf{n}) < -1/2$, then PBS records the value $S(\mathbf{n}) = -1/2$ and directs the squbit to the subensemble $\Psi^-_\mathbf{n}$. If $\left|\mathbf{R}^{(1)}\mathbf{n} + \varepsilon^{(1)}(\mathbf{n})\right| \le 1/2$, then the PBS does not record any value, but makes a second attempt, by going to the second layer and repeating the whole procedure. From a physical point of view, the host must first generate a large number of layers, and only then should these layers be checked for activity. However, from the computer's point of view, this sequence is associated with a waste of resources. Note that the parameters of the second layer $\mathbf{R}^{(2)}$ and $\varepsilon^{(2)}(\mathbf{n})$ do not depend on the values of the parameters of the first layer. The process continues until a definite value $S(\mathbf{n})$ is registered for the squbit under investigation. Theoretically, this may take an infinite number of attempts. However, in practice, the probability of an event with a large number of steps proves to be extremely small.

At first glance, it seems that subensembles $\Psi^+_\mathbf{n}$ and $\Psi^-_\mathbf{n}$ correspond to quantum states with the definite values $S(\mathbf{n}) = +1/2$ and $S(n) = -1/2$, respectively. However, the matter is more complicated. The fact is that, indeed, for each of the subensembles, a definite value $S(\mathbf{n})$ is recorded, but this is not sufficient. A quantum state is characterized by a well-defined probability distribution for the values of spin projections to other directions $\mathbf{r}$.

In addition, as a result of the interaction of a qubit (squbit) with a measuring instrument, its elementary state may change. Here three options are possible. First, the elementary state can change in an uncontrolled way. In this case, there is no need to talk about some kind of quantum state. Second, the elementary state may not change. This occurs in so-called indirect measurements, in which the measurement takes place without a physical contact of the measured particle with the measuring instrument. This is possible if it is known in advance that the elementary states of two particles, which are distant from each other, are correlated. Then, by measuring the characteristics of one of the particles, we obtain information about the characteristics of the other. In this case, the answer to the question about the quantum state depends on the ensemble characteristics before the measurement.

Finally, a third variant is possible, when the measuring instrument prepares the quantum state. This happens when the device changes the elementary state in a quite definite way. In our case, when, after measurements, $\mathbf{R}^{(k)}$ ($k = k'$) are randomly distributed over $\Re^+_\mathbf{n}$ the upper hemisphere with a central vector $\mathbf{n}$, and for passive layers, $\mathbf{R}^{(k)}$ ($k = k'$) are distributed over the entire sphere $\Re$. The functions $\varepsilon^{(k)}(\mathbf{r})$ for all k are randomly distributed over the interval $(-1/2, +1/2)$. For $S(\mathbf{n}) = -1/2$, the vector $\mathbf{R}^{(k')}$ must be randomly distributed over the hemisphere $\Re^-_\mathbf{n}$ with the central vector $-\mathbf{n}$.

Consider an event (a subset of elementary states) in which $\mathbf{r}$ and $d\varepsilon$ are fixed, $\mathbf{R}^{(k')} \in \Re^+_\mathbf{n}$, $\mathbf{R}^{(k)} \in \Re$ ($k = k'$), $\varepsilon^{(k)}(\mathbf{r}) \in d\varepsilon$ or $-\varepsilon^{(k)}(\mathbf{r}) \in d\varepsilon$.

Let us start with the first layer, i.e., let $k = 1$. The probability of realization of inequality (77) with an additional condition $\mathbf{R}^{(k)}\mathbf{r} > 0$ ($j = +1$), or with an additional condition $\mathbf{R}^{(k)}\mathbf{r} < 0$ ($j = -1$), is described by the expression

$$P^{(1)}_\mathbf{n}(\mathbf{r}, \varepsilon, j)d\varepsilon = d\varepsilon \frac{N}{2}\int d\mathbf{R}\hat{T}(\mathbf{Rn}) \times [\hat{T}[j(\mathbf{Rr} + \varepsilon) - 1/2] + \hat{T}[j(\mathbf{Rr} - \varepsilon) - 1/2]]. \quad (79)$$

Here N is the normalization factor, $d\mathbf{R} = d\varphi d\vartheta \sin\vartheta$, $\hat{T}(x)$ is the Heaviside threshold function. From (79) it follows

$$P^{(1)}_\mathbf{n}(\mathbf{r}, \varepsilon) \equiv \sum_{j=\pm 1} P^{(1)}_\mathbf{n}(\mathbf{r}, \varepsilon, j) = N\pi. \quad (80)$$

The probability of realization of inequality (78) is described by the expression

$$\tilde{P}^{(1)}_\mathbf{n}(\mathbf{r}, \varepsilon)d\varepsilon = d\varepsilon \frac{N}{2}\int d\mathbf{R}[\hat{T}[1/2 + \mathbf{Rr} + \varepsilon] \times \hat{T}[1/2 - \mathbf{Rr} - \varepsilon] + \hat{T}[1/2 + \mathbf{Rr} - \varepsilon] \times \hat{T}[1/2 - \mathbf{Rr} + \varepsilon]] = 2N\pi d\varepsilon. \quad (81)$$

From formulas (80) and (81) we get $N = (3\pi)^{-1}$ and

$$\tilde{P}_{\mathbf{n}}^{(1)}(\mathbf{r},\varepsilon) = 2/3. \quad (82)$$

For the second layer ($k = 2$), the same reasoning can be repeated. We only need to consider that, thanks to Eq. (82), we will have to deal with the second layer with a probability of 2/3. Continuing this process, we get that for fixed $\mathbf{r}$, $d\varepsilon$ and j, the probability for the active layer to have a number k' with an additional condition $j\mathbf{R}^{(k')}\mathbf{r} > 0$ is

$$P_{\mathbf{n}}^{(k')}(\mathbf{r},\varepsilon,j) = \frac{1}{2\pi}\left(\frac{2}{3}\right)^{k'} \int d\mathbf{R}\,\hat{T}(j\mathbf{Rn}) \times [\hat{T}[\mathbf{Rr} + \varepsilon - 1/2] + \hat{T}[\mathbf{Rr} - \varepsilon - 1/2]], \quad (83)$$

while the probability of detecting a number greater than k for the active layer is equal to

$$\tilde{P}_{\mathbf{n}}^{(k)}(\mathbf{r},\varepsilon) = \left(\frac{2}{3}\right)^{k}. \quad (84)$$

This probability decreases rapidly as k increases. Therefore, to obtain the final result, in practice, it will be necessary to generate the characteristics of only a small number of layers.

From Eq. (83), we obtain

$$P_{\mathbf{n}}(\mathbf{r}, j) = \sum_{k=1}^{\infty} \int_{-1/2}^{1/2} d\varepsilon P_{\mathbf{n}}^{(k)}(\mathbf{r},\varepsilon,j) = \frac{1}{\pi}\int d\mathbf{R}\,\hat{T}[j\mathbf{Rn}](\mathbf{Rr})\hat{T}[\mathbf{Rr}]. \quad (85)$$

Since

$$\sum_j P_{\mathbf{n}}(r, j) = 1, \quad \sum_j j P_{\mathbf{n}}(\mathbf{r}, j) = (\mathbf{rn}). \quad (86)$$

Then

$$P_{\mathbf{n}}(\mathbf{r}, j) = \frac{1}{2} B(1 + j(\mathbf{rn})). \quad (87)$$

This formula correctly describes the distribution of the values of the spin projections on the direction $\mathbf{r}$ in the quantum state in which $S(\mathbf{n}) = +1/2$. The same final result is obtained when squbits are included in the subensemble $\Psi_{\mathbf{n}}^{+}$, for which $\mathbf{R}^{(k)} \in \Re_{\mathbf{n}}^{+}$ for all layers. This means that this sample is quite representative of the quantum state with $S(\mathbf{n}) = +1/2$. Note that the quantum state arises *after the moment of measurement*.

From (87) it turns out that the average value of the projection of the spin on the direction $\mathbf{r}$ is given by the formula

$$\langle S(\mathbf{r})\rangle = \frac{1}{2}(\mathbf{rn}). \quad (88)$$

A schematic diagram of a computer program that makes it possible to come to the result (88) is as follows:

(1) The two unit-vectors $\mathbf{n}$ and $\mathbf{r}$ are fixed.

(2) The random number generator generates a unit vector $\mathbf{R} \in \Re_{\mathbf{n}}^{+}$ and $\varepsilon(\mathbf{r}) \in (-1/2, +1/2)$.

(3) One calculates $|\mathbf{Rr} + \varepsilon(\mathbf{r})|$.

(4) If inequality (77) is satisfied, then $\mathbf{Rr}$ is calculated. If $\mathbf{Rr} > 0$, then the number 1 is added to memory M^{+}, if $\mathbf{Rr} < 0$, then the number 1 is added to memory M^{-}. On this, the experiment with this squbit is considered completed.

(5) If inequality (78) is satisfied in item (3), then the program returns to the step (2), and so on.

(6) After the loop breaks (or is artificially broken by a limiter), the program proceeds to the study of the next squbit.

(7) After processing a sufficient number of squbits, the average value of the spin projection is calculated using the formula

$$\langle S(\mathbf{r})\rangle = \frac{1}{2}\frac{M_{+} - M_{-}}{M_{+} + M_{-}}.$$

Here, the symbols $M_{\pm}$ denote the numbers accumulated in the respective memories.

In this scheme, the vector $\mathbf{n}$ should be considered not as a characteristic of an individual squbit, but as a characteristic of a subensemble $\Psi_{\mathbf{n}}^{+}$ of all squbits for which $\mathbf{R} \in \Re_{\mathbf{n}}^{+}$, since for an individual squbit it is impossible to calculate the average value of $\langle S(\mathbf{r})\rangle$. Thus, as a representative of a quantum state, we consider here a subensemble $\Psi_{\mathbf{n}}^{+}$ rather than a Hilbert space vector.

The actual implementation of this program is described in [82].

At first glance, it seems that the subensemble $\Psi_{\mathbf{n}}^{+}$ has nothing to do with the Hilbert space vector, using which the quantum state is described in the standard mathematical apparatus of quantum mechanics. However, the canonical Gelfand–Naimark–Segal (GNS) construction provides this link. In the case, when the observables are the C^*-algebra elements and a linear positive normalized functional is given on this algebra, the GNS construction allows one to construct a Hilbert space (see, e.g., [6, 13]), in which self-adjoint operators uniquely correspond to the observables, while linear functionals are represented as mathematical expectations of operators on vectors of this space.

In our case, these linear functionals are given by Eq. (88). On the right-hand side, the vector $\mathbf{n}$ distinguishes one functional from another, while the vector $\mathbf{r}$ is an argument of this functional. Since, by construction, functionals (88) have the meaning of the average values of observables over the corresponding subensembles of squbits, there is no need for the Born axiom. In the standard mathematical apparatus of

quantum mechanics, the primary concept is the Hilbert space, while the average values of observables are secondary. In our scheme, the values of observables are primary, and the Hilbert space, through the GNS construction, is a secondary element. While the values of the observables have a visual physical meaning, neither the Hilbert space nor the Born rule have a visual physical meaning.

Let us pay attention to the fact that when calculating the average value $\langle S(\mathbf{r})\rangle$, we checked inequalities (77) and (78), in which the vector $\mathbf{r}$ appears. This means that for the same set of vectors $\mathbf{R}$ ($\mathbf{R} \in \mathfrak{R}^+_{\mathbf{n}}$), a scheme for calculating the average values for different $S(\mathbf{r})$ will be different. In terms of probability theory, this means that the set of elementary events is the same, but the probability measures are different. Recall that in the Kolmogorov theory, the probability measures depend not only on the set of elementary events, but also on the σ-algebra with which this set is equipped. In our case, the observables describing the spin projections onto different directions r are incompatible, and for these observables, a common σ-algebra cannot be constructed.

25. SINGLET STATE OF TWO SQUBITS

A characteristic property of the singlet state of two particles is that when measuring the spin projections of the first and second particles $S_1(\mathbf{n})$ and $S_2(\mathbf{n})$ onto any direction $\mathbf{n}$, the following equality always holds:

$$S_1(\mathbf{n}) + S_2(\mathbf{n}) \equiv S(\mathbf{n}) = 0. \tag{89}$$

In the approach under consideration, equality (89) assumes a strong correlation between the elementary states of squbits 1 and 2. This correlation can be implemented by requiring the fulfillment of the equalities

$$\mathbf{R}_1^{(k)} + \mathbf{R}_2^{(k)} = 0, \quad \varepsilon_1^{(k)}(\mathbf{r}) = \varepsilon_2^{(k)}(-\mathbf{r}). \tag{90}$$

Here, $\mathbf{R}_1^{(k)}$ and $\mathbf{R}_2^{(k)}$ are the orientation vectors of the kth layer for the first and second squbits, $\varepsilon_i^{(k)}(\mathbf{r})$ ($i = 1, 2$) are the functions $\varepsilon^{(k)}(\mathbf{r})$ for these squbits. It immediately follows from equalities (90) that the numbers of active layers for the first and second squbits coincide, and relation (89) is satisfied for any direction, regardless of the distance between the squbits at the time of measurement. In other words, the situation of the Einstein–Podolsky–Rosen paradox occurs [28]. Correlation (89) arises not at the moment of measurement of spin projections, but at the moment of preparation of a singlet two-squbit state. Particles with this correlation are often referred to as an EPR-pair.

Let us see to what correlations this leads for the squbit spin projections $S_1(\mathbf{r}_1)$, $S_2(\mathbf{r}_2)$ on the directions $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. Just as for a one-squbit system, no probability measure can be assigned to the elementary state of a two-squbit system. Therefore, we consider an event in which $\mathbf{r}_1$, $\mathbf{r}_2$ and $d\varepsilon$ are fixed, while $\mathbf{R}_1^{(k)} \in \mathfrak{R}$, $\varepsilon_1^{(k)}(\mathbf{r}_1) \in d\varepsilon$ or $-\varepsilon_1^{(k)}(\mathbf{r}_1) \in d\varepsilon$. We also introduce two parameters j_1 and j_2: $j_1 = +1$, if $\mathbf{R}_1^{k'}\mathbf{r}_1 > 0$; $j_1 = -1$, if $\mathbf{R}_1^{k'}\mathbf{r}_1 < 0$; $j_2 = +1$, if $\mathbf{R}_2^{k'}\mathbf{r}_2 > 0$; $j_2 = -1$, if $\mathbf{R}_2^{k'}\mathbf{r}_2 < 0$. Both squbits have the same active layer number.

Let $\mathbf{r}_1$, $\mathbf{r}_2$, $d\varepsilon$, j_1 and j_2 be fixed. Let $k = 1$. Then the probability of implementation of inequality (77) for squbit 1 is described by the expression

$$P^{(1)}(\mathbf{r}_1, \mathbf{r}_2, \varepsilon, j_1, j_2)d\varepsilon = d\varepsilon\, N/2 \int d\mathbf{R}\, \hat{T}[-j_2\mathbf{R}\mathbf{r}_2] \times [\hat{T}[j_1(\mathbf{R}\mathbf{r}_1 + \varepsilon) - 1/2] + \hat{T}[j_1(\mathbf{R}\mathbf{r}_1 - \varepsilon) - 1/2]]. \tag{91}$$

From here we get

$$P^{(1)}(\mathbf{r}_1, \mathbf{r}_2, \varepsilon) = \sum_{j_1, j_2} P^{(1)}(\mathbf{r}_1, \mathbf{r}_2, \varepsilon, j_1, j_2) = N 2\pi. \tag{92}$$

The probability of implementation of inequality (78) is described by the expression

$$\tilde{P}^{(1)}(\mathbf{r}_1, \mathbf{r}_2, \varepsilon)d\varepsilon = d\varepsilon N/2 \int dR[\hat{T}[1/2 + \mathbf{R}\mathbf{r}_1 + \varepsilon] \times \hat{T}[1/2 - \mathbf{R}\mathbf{r}_1 - \varepsilon] + \hat{T}[1/2 + \mathrm{R}\mathrm{r}_1 - \varepsilon] \times \hat{T}[1/2 - \mathbf{R}\mathbf{r}_1 + \varepsilon]] = N 2\pi d\varepsilon. \tag{93}$$

From (92) and (93) we obtain

$$N = (4\pi)^{-1}, \quad \tilde{P}^{(1)}(\mathbf{r}_1, \mathbf{r}_2, \varepsilon) = 1/2.$$

Repeating calculations for a one-squbit system, we get that for fixed $\mathbf{r}_1$, $\mathbf{r}_2$, $d\varepsilon$, j_1, j_2, the probability of the active layer to have a number $k = k'$ is described by the formula

$$P^{(k')}(\mathbf{r}_1, \mathbf{r}_2, \varepsilon, j_1, j_2) = \frac{1}{4\pi}\frac{1}{2^{k'}} \int d\mathbf{R}\, \hat{T} \times [-j_1 j_2 \mathbf{R}\mathbf{r}_2][\hat{T}[\mathbf{R}\mathbf{r}_1 + \varepsilon - 1/2] + \hat{T}[\mathbf{R}\mathbf{r}_1 - \varepsilon - 1/2]], \tag{94}$$

while the probability of finding a higher number than k for the active layer is described by the expression

$$\tilde{P}^{(k)}(\mathbf{r}_1, \mathbf{r}_2, \varepsilon) = 2^{-k}.$$

Further, following a procedure similar to that which led to Eqs. (85), (86), and (87), we obtain

$$P(\mathbf{r}_1, \mathbf{r}_2, j_1, j_2) = \frac{1}{4}(1 - j_1 j_2(\mathbf{r}_1\mathbf{r}_2)). \tag{95}$$

This expression describes the probability of finding the spin projection of the first squbit in the direction $\mathbf{r}_1$ equal to $j_1/2$, and the probability to detect the spin projection of the second squbit in the direction $\mathbf{r}_2$ which is equal to $j_2/2$.

Probability (95) coincides with that obtained in the standard approach to quantum mechanics. As expected, formula (95) is symmetrical with respect to

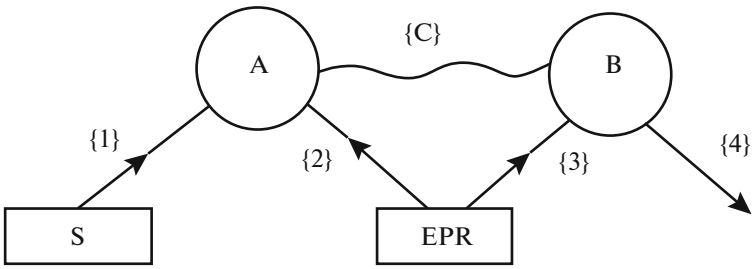

Fig. 11. Scheme of quantum teleportation.

squbits 1 and 2, although the original formula (91) is not symmetrical. This is explained by the fact that when fixing a probability measure, we must somehow combine elementary events (elementary states) into events that can be assigned a probability measure. In formula (91), this unification is made according to the features $\mathbf{R}_1^{(k)} \in \mathfrak{R}$, $\pm\varepsilon_1^{(k)} \in d\varepsilon$. Owing to equalities (90), it is no longer necessary to fix $\mathbf{R}_2^{(k)}$ and $\varepsilon_2^{(k)}$. Note that each of the functions must satisfy inequalities (76), (77), and (78). In addition, functions $\varepsilon_i^{(k)}$ are needed only to select the active layer, and their specific values are not of interest. When combining by attribute $\pm\varepsilon_1^{(k)} \in d\varepsilon$, we combined elementary events according to the distribution of active layers for squbit 1. Automatically, the same distribution will be for squbit 2. On the other hand, in order to restore the distribution of values of the functions $\varepsilon_2^{(k)}(\mathbf{r}_2)$ from this distribution, the explicit form of all functions $\varepsilon_i^{(k)}(\mathbf{r})$ should be set, which is a very difficult task, which is of no practical interest in this case.

From Eq. (95), for the correlation function, we obtain the expression

$$E(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{4}\sum_{j_1, j_2} j_1 j_2 P(\mathbf{r}_1, \mathbf{r}_2, j_1, j_2) = -\frac{1}{4}(\mathbf{r}_1\mathbf{r}_2), \qquad (96)$$

which violates Bell's inequality.

Correlation (96) has a nonlocal character, since it connects the measured values of the projections of squbit spins, which at the time instant of the measurement can be at any distance. However, this correlation does not miraculously occur at the measurement moment. It arose at the time moment of preparation of the singlet state.

26. COMPUTER MODEL OF TELEPORTATION

Let us briefly recall the main provisions of the standard description of quantum teleportation. This phenomenon is associated with entangled quantum states. As such, the so-called Bell states are most often used:

$$\begin{aligned}
|\Psi^{(-)}\rangle_{12} &= \frac{1}{\sqrt{2}}\big[|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2\big],\\
|\Psi^{(+)}\rangle_{12} &= \frac{1}{\sqrt{2}}\big[|+\rangle_1|-\rangle_2 + |-\rangle_1|+\rangle_2\big],\\
|\Phi^{(-)}\rangle_{12} &= \frac{1}{\sqrt{2}}\big[|+\rangle_1|+\rangle_2 - |-\rangle_1|-\rangle_2\big],\\
|\Phi^{(+)}\rangle_{12} &= \frac{1}{\sqrt{2}}\big[|+\rangle_1|+\rangle_2 + |-\rangle_1|-\rangle_2\big].
\end{aligned} \qquad (97)$$

These states form an orthonormal basis in a two-particle Hilbert space. They are built from single-particle states $|\pm\rangle_i$ $(i = 1,2)$. The state $|\Psi^{(-)}\rangle_{12}$ is singlet, the states $|\Psi^{(+)}\rangle_{12}$, $|\Phi^{(-)}\rangle_{12}$, and $|\Phi^{(+)}\rangle_{12}$ are triplet.

The physical implementation of these states can be different. In theoretical reasoning, spin states of a particle with spin 1/2 are usually considered as $|\pm\rangle_i$. In experiments, photons with different polarizations are typically used. We will use the spin terminology.

Figure 11 shows a schematic diagram of quantum teleportation: *S* is the source of the initial state; *EPR* is the source of EPR pairs; *A* is the Bell state analyzer (Alice); *B* is the unitary converter (Bob); {*C*} is the classical communication channel; {1} is the carrier of the initial teleportable state; {2}–{3} is the EPR pair; {4} is the carrier of the final teleported state.

The standard description of the teleportation scheme is as follows. The source *S* emits a particle {1} in the quantum state $|\Psi\rangle_1 = \alpha|+\rangle + \beta|-\rangle$, where α and β are complex numbers ($|\alpha|^2 + |\beta|^2 = 1$). Particle {1} is sent to Alice. The EPR source emits the EPR pair of {2} and {3} in the singlet state $|\Psi^{(-)}\rangle_{23}$. One particle ({2}) of the pair is sent to Alice, the other particle ({3}) is sent to Bob.

According to the rules of quantum mechanics, the state of a system of three particles ({1}, {2}, {3}) is described by the vector $|\Psi\rangle_{123} = |\Psi\rangle_1 \otimes |\Psi^{(-)}\rangle_{23}$. This

vector can be decomposed into Bell states (97) of particles {1} and {2}:

$$|\Psi\rangle_{123} = \frac{1}{2}\{|\Psi^{(-)}\rangle_{12}(-\alpha|+\rangle_3 - \beta|-\rangle_3) + |\Psi^{(+)}\rangle_{12}(-\alpha|+\rangle_3 + \beta|-\rangle_3) + |\Phi^{(-)}\rangle_{12} \times (\alpha|-\rangle_3 + \beta|+\rangle_3) + |\Phi^{(+)}\rangle_{12}(\alpha|-\rangle_3 - \beta|+\rangle_3)\}. \quad (98)$$

Alice decides in which of the four states (97) the particles {1} and {2} are. Experimentally, this is relatively easy to do for the state $|\Psi^{(-)}\rangle_{12}$. Therefore, other states are usually simply discarded. This reduces the teleportation effectiveness by four times, but this is accepted. Alice communicates the result of her observation to Bob via the channel $\{C\}$.

If Alice recorded this result, then, according to the projection principle, the three-particle system collapses into the state $|\Psi'\rangle_{123} = |\Psi^{(-)}\rangle_{12}(-\alpha|+\rangle_3 - \beta|-\rangle_3)$. In this case, particle {3} is already in the same state as particle {1} (see (97)) and Bob has nothing to do. If Alice has recorded one of the triplet states, then Bob will have to perform some unitary transformations of the state of particle {3} (to change some signs of the coefficients α and β) to obtain the state (of particle {4}) that matches the state of particle {1}. In real experiments, a simple beam splitter BS is usually used as an analyzer of the two-particle states of particles {1} and {2}. It singles out a singlet state but cannot distinguish between triplet states.

Now let us see how this phenomenon can be implemented using squbits. First, let us describe the program that will act as the device BS. We will also call this program BS. The geometry of the device BS allocates in space a definite Cartesian coordinate system with direction vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. We require that the device BS be able to distinguish between a singlet and three triplet states. These states can be characterized by the products of the spin projections for the first and second particles. These observables are compatible. The combination $[S_1(\mathbf{x})S_2(\mathbf{x}) < 0,\ S_1(\mathbf{z})S_2(\mathbf{z}) < 0]$ is characteristic of the singlet state. The combinations $[S_1(\mathbf{x})S_2(\mathbf{x}) > 0,\ S_1(\mathbf{z})S_2(\mathbf{z}) > 0]$, $[S_1(\mathbf{x})S_2(\mathbf{x}) > 0,\ S_1(\mathbf{z})S_2(\mathbf{z}) < 0]$, $[S_1(\mathbf{x})S_2(\mathbf{x}) < 0,\ S_1(\mathbf{z})S_2(\mathbf{z}) > 0]$ are typical for three triplet states.

Let squbits {1} and {2} arrive at the device BS. We assume that the BS device responds to the active layers of squbits {1}, {2} and measures the signs of $S_1(\mathbf{x})S_2(\mathbf{x})$, $S_1(\mathbf{z})S_2(\mathbf{z})$. Thus, our device BS distinguishes all four orthogonal two-particle states. Therefore, with its help, you can achieve one hundred percent efficiency of teleportation.

Note that we only use projections on two axes, not on all three. The point is this. Consider, e.g., a singlet state. In the standard formalism of quantum mechanics, it corresponds to the following combinations of spin projection values: $S_1(\mathbf{x})S_2(\mathbf{x}) < 0$, $S_1(\mathbf{z})S_2(\mathbf{z}) < 0$, $S_1(\mathbf{y})S_2(\mathbf{y}) < 0$, though the first two conditions are sufficient to uniquely fix a quantum state. For two squbits, in addition to the specified set, the following is also possible: $S_1(\mathbf{x})S_2(\mathbf{x}) < 0$, $S_1(\mathbf{z})S_2(\mathbf{z}) < 0$, $S_1(\mathbf{y})S_2(\mathbf{y}) > 0$. This set does not correspond to any quantum state of the two-particle system. Similar reasoning is also valid for the three variants corresponding to the triplet state. Thus, we cannot assert that the pair under consideration is in any one of the quantum states. We can only state that this pair belongs to one of the four *prequantum* states. The latter means that an instrument that measures the products of spin projections onto only two directions gives a result that is characteristic of the corresponding quantum state.

On the other hand, a BS can be not only a measuring device, but can also prepare a singlet quantum state. In this case, the device BS, having received an elementary state of squbits with the set $S_1(\mathbf{x})S_2(\mathbf{x}) < 0$, $S_1(\mathbf{z})S_2(\mathbf{z}) < 0$, at the input, should give an elementary state of squbits with the set $S_1(\mathbf{x})S_2(\mathbf{x}) < 0$, $S_1(\mathbf{z})S_2(\mathbf{z}) < 0$, $S_1(\mathbf{y})S_2(\mathbf{y}) < 0$. This again indicates that the readings of the measuring device *are determined* by the elementary state of the squbits immediately *before the moment* of measurement and *determine* the quantum state *after the moment* of measurement. It can be expected that the latter is true not only for squbits, but also for qubits.

As noted in Section 23.1, if for the active layer $\mathbf{R}_1^{(k')} \in \mathfrak{R}^+(\mathbf{x})$, then $S_1(\mathbf{x}) = +1/2$. From here it follows that if $x_1 \equiv \mathbf{R}_1^{(k')}\mathbf{x} > 0$, then $S_1(\mathbf{x}) = +1/2$. It is similar for the second squbit and for projections onto $\mathbf{z}$. Therefore, the combination $[x_1x_2 < 0,\ z_1z_2 < 0]$ is characteristic of a singlet state, while the combinations $[x_1x_2 > 0,\ z_1z_2 > 0]$, $[x_1x_2 > 0,\ z_1z_2 < 0]$, $[x_1x_2 < 0,\ z_1z_2 > 0]$ are characteristic of triplet states. Thus, we can assume that the device BS directly sorts the squbits not by the spin projections, but by the projections x, z of the vectors $\mathbf{R}$ that specify the orientation of active layers of squbits. For a computer implementation, this is much more convenient.

Now let us turn to Fig. 24.1. The source S sends Alice a beam of squbits {1} polarized along the vector $\mathbf{n}$. Let the vectors $\mathbf{R}_1$ characterize the orientation of the active layers of squbits {1}, and let $\varepsilon_1 \equiv \varepsilon_1(\mathbf{n})$ be the values of the functions ε_1 for these layers. Then for the squbits sent by the source S, the inequality is valid:

$$\mathbf{R}_1\mathbf{n} + \varepsilon_1 - 1/2 > 0. \quad (99)$$

The source *EPR* emits EPR pairs of squbits {2}–{3}, the vectors $\mathbf{R}_2 = -\mathbf{R}_3$ characterize orientations of the active layers for these squbits, while $\varepsilon_2 = -\varepsilon_3$ are the values of the corresponding functions ε for the direction $\mathbf{n}$.

Reproducing the conditions of a real experiment, we assume that the device BS selects the pairs of squbits {1}–{2} for which it records a singlet prequantum state. In our case, these are pairs for which for active layers $x_1x_2 < 0$ and $z_1z_2 < 0$. This is equivalent to the fact that the BS selects pairs {1}–{3} that have

$$x_1x_3 > 0, \quad z_1z_3 > 0. \tag{100}$$

Note that the BS device are not in contact with squbits {3}. Therefore, it does not change their initial elementary states.

For squbits for whom

$$\mathbf{R}_3\mathbf{n} + \varepsilon_3 > 1/2, \tag{101}$$

teleportation was successful, while for squbits, for whom

$$\mathbf{R}_3\mathbf{n} + \varepsilon_3 < -1/2, \tag{102}$$

teleportation turned out to be unsuccessful.

Considering Eqs. (99)–(102), we obtain that the number of squbits, for which teleportation was successful, is given by the formula

$$\begin{aligned} N_+ &= N\int d\mathbf{R}_1\int d\mathbf{R}_3\int d\varepsilon_1\int d\varepsilon_3\hat{T}(\mathbf{R}_1\mathbf{n} + \varepsilon_1 - 1/2) \\ &\times \hat{T}(\mathbf{R}_1\mathbf{x}\cdot\mathbf{R}_3\mathbf{x})\hat{T}(\mathbf{R}_1\mathbf{z}\cdot\mathbf{R}_3\mathbf{z})\hat{T}(\mathbf{R}_3\mathbf{n} + \varepsilon_3 - 1/2), \end{aligned} \tag{103}$$

and the number of squbits for which teleportation failed is given by the formula

$$\begin{aligned} N_- &= N\int d\mathbf{R}_1\int d\mathbf{R}_3\int d\varepsilon_1\int d\varepsilon_3\hat{T}(\mathbf{R}_1\mathbf{n} + \varepsilon_1 - 1/2) \\ &\times \hat{T}(\mathbf{R}_1\mathbf{x}\cdot\mathbf{R}_3\mathbf{x})\hat{T}(\mathbf{R}_1\mathbf{z}\cdot\mathbf{R}_3\mathbf{z})\hat{T}(-\mathbf{R}_3\mathbf{n} - \varepsilon_3 - 1/2). \end{aligned} \tag{104}$$

Here N is the normalization factor.

The teleportation quality is estimated by the number $F = N_+(N_+ + N_-)^{-1}$, which is commonly called fidelity.

The standard mathematical formalism of quantum mechanics predicts the fidelity $F = 1$ for all directions $\mathbf{n}$. A real experiment [83] gave fidelity values: $F = 0.92 \pm 0.02$ for $\mathbf{n} = \mathbf{z}$, $F = 0.90 \pm 0.03$ at $\mathbf{n} = \mathbf{x}$. The minimum fidelity $F = 0.84 \pm 0.02$ was achieved for $n = 2^{-1/2}(\mathbf{z} + \mathbf{x})$. The relative fidelity (a ratio to the maximum value) at the minimum had a value $F_r = 0.913$.

In our case, the quantities N_+ and N_- are determined by Eqs. (103) and (104).

Let us make a replacement in them:

$$\hat{T}(\mathbf{R}_1\mathbf{x}\cdot\mathbf{R}_3\mathbf{x}) \to \hat{T}(\mathbf{R}_1\mathbf{x})\hat{T}(\mathbf{R}_3\mathbf{x}) + \hat{T}(-\mathbf{R}_1\mathbf{x})\hat{T}(-\mathbf{R}_3\mathbf{x})$$

and a similar substitution for $\mathbf{x} \to \mathbf{z}$. After that, the integrals appearing in them, related to squbits {1} and {3}, are factorized and easily calculated. As a result, up to a common factor, it is obtained

$$N_-(\mathbf{n} = \mathbf{x}) = N_-(\mathbf{n} = \mathbf{z}) = 0,$$

$$N_+\left(\mathbf{n} = \frac{\mathbf{x} + \mathbf{z}}{\sqrt{2}}\right) = 4 + 2(\sqrt{2} - 1)^2,$$

$$N_-\left(\mathbf{n} = \frac{\mathbf{x} + \mathbf{z}}{\sqrt{2}}\right) = 2(\sqrt{2} - 1)^2.$$

Hence for fidelity we get

$$F(\mathbf{x}) = F(\mathbf{z}) = 1,$$

$$F\left(\mathbf{n} = \frac{\mathbf{x} + \mathbf{z}}{\sqrt{2}}\right)\frac{2 + (\sqrt{2} - 1)^2}{2 + 2(\sqrt{2} - 1)^2} \approx 0.927.$$

Considering the experimental errors, this is in good agreement with the experimental value for the relative fidelity. Note that we did not use any fitting parameters.

We did not make any assumptions about the dynamics of the system under consideration. All the quantum effects that we have considered are purely statistical. In this case, they are not consequences of any special quantum probability theory. They play perfectly into the standard Kolmogorov probability theory. The only thing to keep in mind is that the probability space must be equipped with a σ-algebra. There is a fundamental difference between classical and quantum systems. While for a classical system it is possible to introduce a single σ-algebra that can serve all the observables characteristic of the system, in a quantum system for each group of compatible observables, it is necessary to introduce its own σ-algebra and, accordingly, its own system of probability measures. In our case, in the construction of these measures, an important role was played by functions $\varepsilon(\mathbf{r})$ whose values depend on $\mathbf{r}$. Recall that the observables we are considering—the spin projections onto directions $\mathbf{r}$—are incompatible for different $\mathbf{r}$.

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