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Ellipsoid Space Charge Model for Electron Beam Dynamics Simulations

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Abstract—The pulsed beam current in linear accelerators can approach significant values up to tens of Amperes. Such high currents cause many specific adverse effects in the accelerators. One of such effects is the repelling forces of the space charge as they become comparable to the forces of the electromagnetic accelerating fields, and can influence the stability of phase and radial particle motion. In the numerical analysis of the beam dynamics in linear accelerators, it is necessary to choose one of the different space charge models depending on the desired accuracy, complexity, speed and computer resources availability. The most accurate results are achieved by the numerical solution of the Poisson equation. However, this method may require significant resources and time and may not be suitable when fast analysis is required in the linac design stages. Analytical methods, on the other hands, base on the analytical solution of Poisson equations for the predefined shape of the particles distribution inside the beam. One of the most popular analytical space charge models is the ellipsoidal beam approximations. Despite being a well-developed model, many published approaches lack some important features as fully three-dimensional ellipsoid asymmetry and multi-bunch model. In this paper, we will derive the equations of the space charge field for the full-3D non-relativistic ellipsoid bunch step by step, starting from the Poisson equation, and compare this model with the other known models.

Keywords: electron linac, beam dynamics, space charge, charged ellipsoid, Poisson equation

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1. INTRODUCTION

The problem of space charge forces inside the accelerated beam and their influence on the particle motion is in practice identical to the problem of the gravitating mass distribution. In other words, it is related to the theory of Newtonian potential. The classics of astronomy and mathematics addressed this problem more than two centuries ago, and achieved the significant progress of its solution in case when the gravitating mass can be represented by a three-dimensional ellipsoid. The obtained results were published in the works [1–4] with the different level of details. However, these results haven't become well-known in accelerator physics, which was studying the space charge problems for a long time. In this case, it is worth to mention the works [5, 6]. Unfortunately, in these papers the beam is represented as a charge ellipsoid of rotation, or in the other word is practically suitable only for 2D case. Likewise, the potential (and the field) of the beam, provided in the work [5] is only applicable for the particles located inside the ellipsoid (or the beam core). On the other hand, the expressions from the work [6] for the fields outside the ellipsoid were found only in the multipole approximation.

In the following sections, we will derive the derive the equations of the space charge fields for the full-3D relativistic ellipsoid bunch inside and outside the beam core. We will start from the Poisson equation and continue to solve it for different particle distributions. Appendixes present the solutions of some mathematical problems used in the text of this paper and are presented for the self-consistency of the provided solution.

2. POTENTIAL OF A UNIFORMLY CHARGED 3D ELLIPSOID

We will start from the potential of the charged 3D ellipsoid based on the Dirichlet's approach [7]. Thus, the potential of a uniformly charged 3D ellipsoid with the density ρ and semi-axes (a_x, a_y, a_z) in a random point of space (x, y, z), as shown in Fig. 1, is:

$$\varphi(\vec{r}) = \rho\iiint\limits_V \frac{dx'\,dy'\,dz'}{|\vec{r} - \vec{r}'|} \tag{1}$$
$$= \rho\iiint\limits_V \frac{dx'\,dy'\,dz'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}},$$

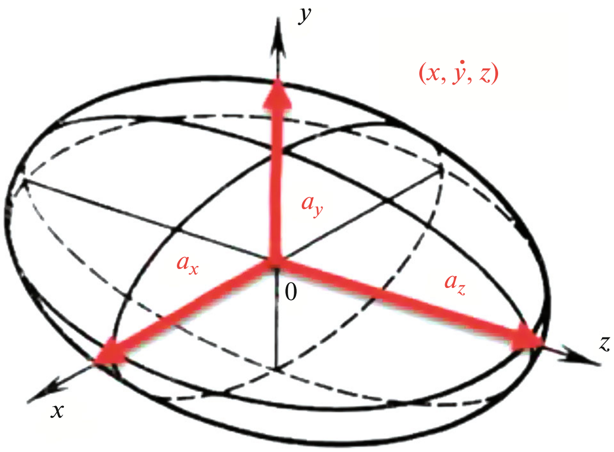

Fig. 1. Ellipsoid beam geometrical representation and dimensions.

where the integration is expanded to the volume of the ellipsoid:

$$s(\vec{r}\,') = \frac{x'^2}{a_x^2} + \frac{y'^2}{a_x^2} + \frac{z'^2}{a_x^2} \le 1. \tag{2}$$

To overcome the problems related to the variability of the integration boundaries, Dirichlet proposed to use the special form-factor (so called "discontinuous factor"):

$$F_\mathrm{D}(f) = \frac{2}{\pi}\int_0^\infty \frac{\sin t}{t}\cos(ft)dt = \begin{cases} 1 & \text{for } f < 1, \\ 1/2 & \text{for } f = 1, \\ 0 & \text{for } f > 1. \end{cases} \tag{3}$$

Figure 2 demonstrates the distribution of Dirichlet discontinuous factor for the argument values from 0 to 0.95 (top) and 0.95 to 1.0 (bottom). It is clearly seen that the behivour of the form-factor is in the good accordance with the expression (3).

For this problem, we chose the discontinuous factor in the form of

$$F_\mathrm{D}(\vec{r}\,') = \frac{2}{\pi}\int_0^\infty \frac{\sin t}{t}\cos\left[\left(\frac{x'^2}{a_x^2} + \frac{y'^2}{a_x^2} + \frac{z'^2}{a_x^2}\right)t\right]dt. \tag{4}$$

Then,

$$\varphi(\vec{r}) = \rho\iiint_V \frac{dx'\,dy'\,dz'}{|\vec{r} - \vec{r}\,'|} = \rho\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_\mathrm{D}(\vec{r}\,')\frac{dx'\,dy'\,dz'}{|\vec{r} - \vec{r}\,'|}$$
$$= \frac{2\rho}{\pi}\int_0^\infty \frac{\sin t}{t}dt \tag{5}$$
$$\times\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos\left[\left(\frac{x'^2}{a_x^2} + \frac{y'^2}{a_x^2} + \frac{z'^2}{a_x^2}\right)t\right]\frac{dx'\,dy'\,dz'}{|\vec{r} - \vec{r}\,'|}\right\},$$

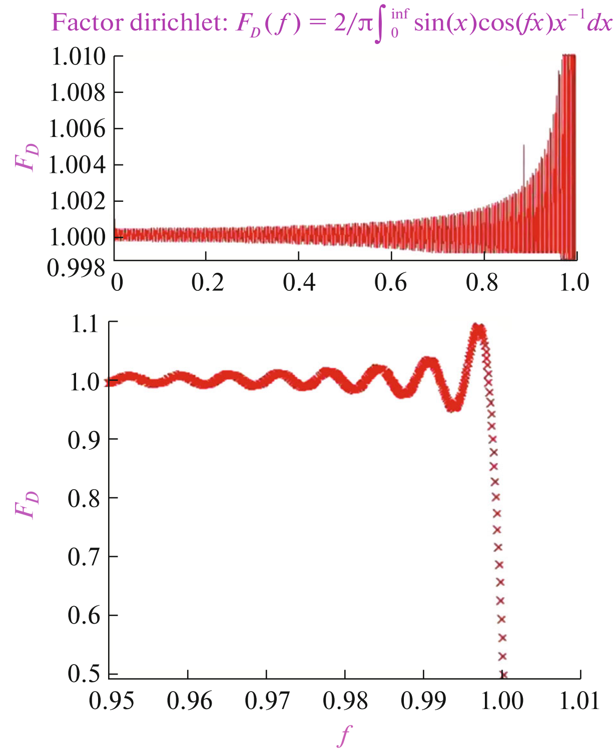

Fig. 2. Dirichlet discontinuous factor as a function of argument.

or

$$\varphi(\vec{r}) = \mathrm{Re}\left\{\frac{2\rho}{\pi}\int_0^\infty \frac{\sin t}{t}dt\right.$$
$$\left.\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[i\left(\frac{x'^2}{a_x^2} + \frac{y'^2}{a_x^2} + \frac{z'^2}{a_x^2}\right)t\right]\frac{dx'\,dy'\,dz'}{|\vec{r} - \vec{r}\,'|}\right\}. \tag{6}$$

Here, we will use the following expression:

$$\int_0^\infty \frac{e^{ir^2\zeta}}{\sqrt{\zeta}}d\varsigma = \left(\sqrt{\zeta} = x;\ d\zeta = 2xdx\right) = 2\int_0^\infty e^{ir^2x^2}dx$$
$$= \frac{2}{\sqrt{-ir}}\int_0^\infty e^{-(\sqrt{-i}rx)^2}d(\sqrt{-i}rdx) = \frac{2}{r}\frac{\sqrt{\pi e^{i\pi/2}}}{2} = \frac{\sqrt{\pi}}{r}e^{i\pi/4}, \tag{7}$$

so that

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \frac{e^{-i\pi/4}}{\sqrt{\pi}}\int_0^\infty \frac{\exp\left(i|\vec{r} - \vec{r}\,'|^2\zeta\right)}{\sqrt{\zeta}}d\zeta. \tag{8}$$

Then

$$\varphi(\vec{r}) = \mathrm{Re}\left\{\frac{2\rho e^{-i\frac{\pi}{4}}}{\pi\sqrt{\pi}}\int_0^\infty d\zeta\int_0^\infty \frac{\sin t}{t\sqrt{\zeta}}dt\right.$$
$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[i\left(\frac{x'^2}{a_x^2} + \frac{y'^2}{a_x^2} + \frac{z'^2}{a_x^2}\right)t + i|\vec{r} - \vec{r}\,'|^2\zeta\right] \tag{9}$$
$$\left.\times\, dx'\,dy'\,dz'\right\}.$$

Since

$$|\vec{r}-\vec{r}\,'|^2 = x^2+y^2+z^2-2xx' - 2yy' - 2zz' + x'^2 + y'^2 + z'^2. \quad (10)$$

And by introducing the integrals

$$Q_u(\zeta,t_u) = \int_{-\infty}^{\infty} \exp\left\{i\left[(\zeta+t_u)u'^2 - 2uu'\zeta\right]\right\}du', \quad (11)$$

where $u = x$, y, or z, and $t_u = t/a_u^2$. Then,

$$\varphi(\vec{r}) = \mathrm{Re} \times \left\{\frac{2\rho e^{-i\frac{\pi}{4}}}{\pi\sqrt{\pi}}\int_0^{\infty} d\zeta\left[\int_0^{\infty}\frac{\sin t}{t\sqrt{\zeta}}\mathrm{e}^{i\zeta(x^2+y^2+z^2)}Q_xQ_yQ_z dt\right]\right\}. \quad (12)$$

To calculate the integrals Q_u, we must consider that

$$\begin{aligned}(\zeta+t_u)u'^2 - 2uu'\zeta &= (\zeta+t_u)\left[u'^2 - 2\frac{u\zeta}{\zeta+t_u}u'\right]\\ &= (\zeta+t_u)\left[\left(u' - \frac{u\zeta}{\zeta+t_u}\right)^2 - \left(\frac{u\zeta}{\zeta+t_u}\right)^2\right]\\ &= (\zeta+t_u)\left(u' - \frac{u\zeta}{\zeta+t_u}\right)^2 - \frac{u^2\zeta^2}{\zeta+t_u}.\end{aligned} \quad (13)$$

Then

$$\begin{aligned}Q_u(\zeta,t_u) &= \exp\left(-i\frac{u^2\zeta^2}{\zeta+t_u}\right) \times \int_{-\infty}^{\infty}\exp\left[i(\zeta+t_u)\left(u' - \frac{u\zeta}{\zeta+t_u}\right)^2\right]du'\\ &= \mathrm{e}^{-i\frac{u^2\zeta^2}{\zeta+t_u}}\int_{-\infty}^{\infty}\mathrm{e}^{i(\zeta+t_u)u'^2}du' = \sqrt{\frac{i}{\zeta+t_u}}\,\mathrm{e}^{-i\frac{u^2\zeta^2}{\zeta+t_u}}\int_{-\infty}^{\infty}\mathrm{e}^{u'^2}du'\\ &= \sqrt{\frac{\pi}{\zeta+t_u}}\,\mathrm{e}^{i\left(\frac{\pi}{4}-\frac{u^2\zeta^2}{\zeta+t_u}\right)}.\end{aligned} \quad (14)$$

Thus,

$$Q_x(\zeta,t_x)Q_y(\zeta,t_y)Q_z(\zeta,t_z) = \pi^{\frac{3}{2}}e^{\frac{3\pi}{4}}\frac{\exp\left[-i\zeta^2\left(\frac{x^2}{\zeta+t_x}+\frac{y^2}{\zeta+t_y}+\frac{z^2}{\zeta+t_z}\right)\right]}{\sqrt{(\zeta+t_x)(\zeta+t_y)(\zeta+t_z)}}. \quad (15)$$

This results means that since the exponent of integration function in the expression (12) can be presented as

$$\begin{aligned}&\zeta(x^2+y^2+z^2) - \zeta^2\left(\frac{x^2}{\zeta+t_x}+\frac{y^2}{\zeta+t_y}+\frac{z^2}{\zeta+t_z}\right)\\ &= \zeta\left[x^2\left(1-\frac{\zeta}{\zeta+t/a_x^2}\right)+y^2\left(1-\frac{\zeta}{\zeta+t/a_y^2}\right)+z^2\left(1-\frac{\zeta}{\zeta+t/a_z^2}\right)\right]\\ &= \zeta\left(x^2\frac{t/a_x}{\zeta+t/a_x^2}+y^2\frac{t/a_y}{\zeta+t/a_y^2}+z^2\frac{t/a_z}{\zeta+t/a_z^2}\right) = \zeta t\left(\frac{x^2}{t+a_x^2\zeta}+\frac{y^2}{t+a_y^2\zeta}+\frac{z^2}{t+a_z^2\zeta}\right)\\ &= t\left(\frac{x^2}{t/\zeta+a_x^2}+\frac{y^2}{t/\zeta+a_y^2}+\frac{z^2}{t/\zeta+a_z^2}\right) \equiv tS,\end{aligned} \quad (16)$$

then the expression for the potential becomes:

$$\begin{aligned}\varphi(\vec{r}) &= \mathrm{Re}\left\{2\rho e^{i\frac{\pi}{2}}\int_0^{\infty}\int_0^{\infty}\frac{\sin t}{t\sqrt{\zeta}}\frac{e^{itS}}{\sqrt{(\zeta+t/a_x^2)(\zeta+t/a_y^2)(\zeta+t/a_z^2)}}dtd\zeta\right\}\\ &= 2\rho\,\mathrm{Re}\left\{e^{i\frac{\pi}{2}}\int_0^{\infty}\int_0^{\infty}\frac{\sin t}{t\zeta^2}\frac{e^{itS}}{\sqrt{\left(1+\frac{t/\zeta}{a_x^2}\right)\left(1+\frac{t/\zeta}{a_y^2}\right)\left(1+\frac{t/\zeta}{a_z^2}\right)}}dtd\zeta\right\}.\end{aligned} \quad (17)$$

Now, let's substitute the variable ζ for $\xi = t/\zeta$. Then $d\zeta/\zeta^2 = -d\xi/t$, so that

$$\begin{aligned}
&\varphi(\vec{r}) \\
&= 2\rho\,\mathrm{Re}\left\{e^{i\frac{\pi}{2}}\int_0^\infty \frac{d\xi}{\sqrt{(1+\xi/a_x^2)(1+\xi/a_y^2)(1+\xi/a_z^2)}} \times \int_0^\infty \frac{\sin t}{t^2} e^{itS} dt\right\} \\
&= 2\rho\int_0^\infty \frac{d\xi}{\sqrt{(1+\xi/a_x^2)(1+\xi/a_y^2)(1+\xi/a_z^2)}} \\
&\times \int_0^\infty \frac{\sin t}{t^2}\mathrm{Re}\left\{e^{i(\pi/2+tS)}\right\} dt \\
&= -2\rho\int_0^\infty \frac{d\xi}{\sqrt{(1+\xi/a_x^2)(1+\xi/a_y^2)(1+\xi/a_z^2)}} \\
&\times \int_0^\infty \frac{\sin t}{t^2}\cos(tS)dt = -2\rho\int_0^\infty \frac{d\xi}{D(\xi)}\int_0^\infty \frac{\sin t}{t^2}\cos(tS)dt,
\end{aligned} \tag{18}$$

where

$$D(\xi) = \sqrt{(1+\xi/a_x^2)(1+\xi/a_y^2)(1+\xi/a_z^2)} = \frac{\sqrt{(a_x^2+\xi)(a_y^2+\xi)(a_z^2+\xi)}}{a_x a_y a_z}. \tag{19}$$

And

$$S = \frac{x^2}{t/\zeta + a_x^2} + \frac{y^2}{t/\zeta + a_y^2} + \frac{z^2}{t/\zeta + a_z^2} = \frac{x^2}{a_x^2+\xi} + \frac{y^2}{a_y^2+\xi} + \frac{z^2}{a_z^2+\xi} = S(\xi). \tag{20}$$

To calculate the integral with respect to the variable t, we will first find it's derivate with respect to $u = x, y$, or z:

$$\begin{aligned}
\frac{\partial\varphi(\vec{r})}{\partial u} &= -2\rho\int_0^\infty \frac{d\xi}{D(\xi)}\int_0^\infty \frac{\sin t}{t^2}\frac{\partial}{du}[\cos(tS)]dt \\
&= -4u\rho\int_0^\infty \frac{d\xi}{(a_u^2+\xi)D(\xi)}\underbrace{\int_0^\infty \frac{\sin t}{t}\cos(tS)dt}_{I(S)}.
\end{aligned} \tag{21}$$

Now, to calculate the integral $I(S)$, we must use the definition of the Dirichlet form-factor

$$\int_0^\infty \frac{\sin t}{t}\cos(ft)dt = \begin{cases}\pi/2 & \text{for } f < 1,\\ 0 & \text{for } f > 1.\end{cases} \tag{22}$$

When the variable ζ changes in the interval [0, ∞], the variable ξ stays within the same interval. For all points inside the ellipsoid $x^2/a_x^2 + y^2/a_y^2 + z^2/a_z^2 < 1$ and ξ inside the whole range of its values, we can claim that

$$S(\xi) = \frac{x^2}{a_x^2+\xi} + \frac{y^2}{a_y^2+\xi} + \frac{z^2}{a_z^2+\xi} < 1. \tag{23}$$

Therefore, for all points (x, y, z) inside the ellipsoid $S(\xi) < 1$, the integral $I(S)$ equals to $\pi/2$ and thus, for all internal points the derivative will be:

$$\begin{aligned}
\frac{\partial\varphi(\vec{r})}{\partial u} &= -2\pi u\rho\int_0^\infty \frac{d\xi}{(a_u^2+\xi)D(\xi)} \\
&= -2\pi u\rho\int_0^\infty \frac{d\xi}{(a_u^2+\xi)\sqrt{\prod_{u=x,y,z}(1+\xi/a_u^2)}}.
\end{aligned} \tag{24}$$

Now, let's assume that the point (x, y, z) is located outside of the ellipsoid. This point belongs to the other ellipsoid, confocal to the original. The surface equation of this new ellipsoid will be

$$\frac{x^2}{a_x^2+\lambda} + \frac{y^2}{a_y^2+\lambda} + \frac{z^2}{a_z^2+\lambda} = 1. \tag{25}$$

Thus, for $\xi < \lambda$ and $\lambda > 0$,

$$\begin{aligned}
S(\xi) &= \frac{x^2}{a_x^2+\xi} + \frac{y^2}{a_y^2+\xi} + \frac{z^2}{a_z^2+\xi} \\
&> \frac{x^2}{a_x^2+\lambda} + \frac{y^2}{a_y^2+\lambda} + \frac{z^2}{a_z^2+\lambda} > 1.
\end{aligned} \tag{26}$$

And the integral $I(S) = 0$. On the other hand, when $\xi > \lambda$,

$$\begin{aligned}
S(\xi) &= \frac{x^2}{a_x^2+\xi} + \frac{y^2}{a_y^2+\xi} + \frac{z^2}{a_z^2+\xi} \\
&< \frac{x^2}{a_x^2+\lambda} + \frac{y^2}{a_y^2+\lambda} + \frac{z^2}{a_z^2+\lambda} < 1.
\end{aligned} \tag{27}$$

And the $I(S) = \pi/2$. Thus, for all external points of the ellipsoid

$$I(S(\xi)) = \begin{cases}0 & \text{for } \xi < \lambda,\\ \pi/2 & \text{for } \xi > \lambda.\end{cases} \tag{28}$$

For these points, the potential derivative will be

$$\begin{aligned}
\frac{\partial\varphi(\vec{r})}{\partial u} &= -2\pi u\rho\int_\lambda^\infty \frac{d\xi}{(a_u^2+\xi)D(\xi)} \\
&= -2\pi u\rho\int_\lambda^\infty \frac{d\xi}{(a_u^2+\xi)\sqrt{\prod_{u=x,y,z}(1+\xi/a_u^2)}}.
\end{aligned} \tag{29}$$

With λ satisfying the characteristic equation (25). But at the same time

$$d\varphi = \frac{\partial\varphi}{dx}dx + \frac{\partial\varphi}{dy}dy + \frac{\partial\varphi}{dz}dz. \tag{30}$$

And therefore,

$$\left.\begin{aligned}\varphi_{\text{in}}(\vec{r}) &= \pi\rho\int_0^\infty\left(C_{\text{in}} - \sum_{u=x,y,z}\frac{u^2}{a_u^2+\xi}\right)\frac{d\xi}{D(\xi)};\\ \varphi_{\text{out}}(\vec{r}) &= \pi\rho\int_\lambda^\infty\left(C_{\text{out}} - \sum_{u=x,y,z}\frac{u^2}{a_u^2+\xi}\right)\frac{d\xi}{D(\xi)}.\end{aligned}\right\}\quad(31)$$

To provide the continuity of the potential due to the transition through the surface of the ellipsoid, the integration constants must be equal ($C_{\text{in}} = C_{\text{out}} = C$). This constant C is defined the following way: for an infinitely distant point from the ellipsoid, the potential equals to zero and $\lambda \to \infty$. In this case, the sum with respect to $u = x, y, z$ can be neglected, so we automatically get $C = 1$.

Thus, the final expressions for the potential will be:

$$\begin{aligned}\varphi(\vec{r}) &= \pi\rho\int_\lambda^\infty\left(1 - \sum_{u=x,y,z}\frac{u^2}{a_u^2+\xi}\right)\frac{d\xi}{D(\xi)}\\ &= \pi\rho a_x a_y a_z\int_\lambda^\infty\left(1 - \sum_{u=x,y,z}\frac{u^2}{a_u^2+\xi}\right)\frac{d\xi}{\sqrt{\prod_{u=x,y,z}(a_u^2+\xi)}},\end{aligned}\quad(32)$$

for the points inside the ellipsoid, $\lambda = 0$, and for the outside points, λ is a root of a characteristic equation (25).

The integrals in the expression (32) can be expresses using the reduced elliptical integrals (as shown, for example, in § 6.2 of [3]), but they seem to be impractical in the numerical analysis.

3. CHARACTERISTIC EQUATION

In this section, we will show that the characteristic equation, which can be presented in a form of

$$w(\lambda) = \frac{x^2}{a_x^2+\lambda} + \frac{y^2}{a_y^2+\lambda} + \frac{z^2}{a_z^2+\lambda} - 1 = 0, \quad (33)$$

always has three real roots, one of which is always positive. Let's assume that the semi-axe of the ellipsoid satisfies the following condition: $a_x > a_y > a_z$, and plot the graph of the function $y = w(\lambda)$, as shown in Fig. 3. This function has four asymptotes: one horizontal ($y = -1$) and three vertical ($\lambda = -a_x^2$, $\lambda = -a_y^2$, and $\lambda = -a_z^2$). The function $w(\lambda)$ has following specifics: when $\lambda \to -\infty$ it approaches the horizontal asymptote from the bottom; when $\lambda \to \infty$ – from the top; it approaches the vertical asymptotes from the bottom when $\lambda \to -a_u^2$ from the left, and from the top when $\lambda \to -a_u^2$ from the right; finally, $w(0) = 1$.

It is immediately clear from the plot in Fig. 3 that the characteristic equation has a single positive root λ_1 and two negative $\lambda_{2,3}$, which are located inside the intervals $(-a_y^2, -a_z^2)$ and $(-a_x^2, -a_y^2)$ respectively. These roots can be analytically described with the Cardan's formulas:

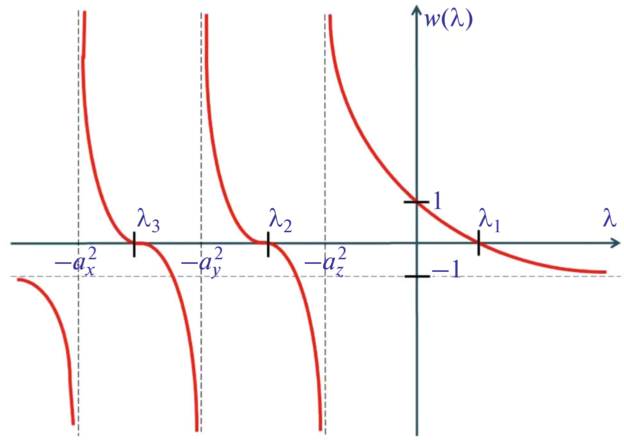

Fig. 3. The roots of the characteristic equation.

$$\lambda = a_x^2\left(\left\{\begin{matrix}2\sqrt[3]{\tau}\cos(\varphi)\\ 2\sqrt[3]{\tau}\cos(\varphi+2\pi/3)\\ 2\sqrt[3]{\tau}\cos(\varphi+4\pi/3)\end{matrix}\right\} - \frac{B}{3A}\right). \quad (34)$$

Which uses the following sequence of the expressions:

$$\begin{gathered}A = \frac{a_x^4}{a_y^2 a_z^2},\\ B = \frac{a_x^2}{a_y^2 a_z^2}\left[\left(a_x^2+a_y^2+a_z^2\right) - \left(x^2+y^2+z^2\right)\right],\\ C = 1 + \frac{a_x^2(a_y^2+a_z^2)}{a_y^2 a_z^2}\\ - \frac{x^2(a_y^2+a_z^2)+y^2(a_x^2+a_z^2)+z^2(a_x^2+a_y^2)}{a_y^2 a_z^2},\\ D = 1 - \left(\frac{x^2}{a_x^2}+\frac{y^2}{a_y^2}+\frac{z^2}{a_z^2}\right);\end{gathered}\quad(35)$$

and

$$\begin{gathered}p = \frac{C}{A} - \frac{B^2}{3A^2}, \quad q = \frac{2B^3}{27A^3} - \frac{BC}{3A^2} + \frac{D}{A};\\ \tau = \sqrt{-\frac{p^3}{27}}, \quad \varphi = \arccos\left(-\frac{q}{2\tau}\right).\end{gathered}\quad(36)$$

This qualitative analysis of the function $w(\lambda)$ demonstrates that for any values of the parameters of the characteristic equation (25), it will always be that $p \le 0$ and

$$2\sqrt[3]{\tau}\cos(\varphi) > B/(3A) \quad (37)$$

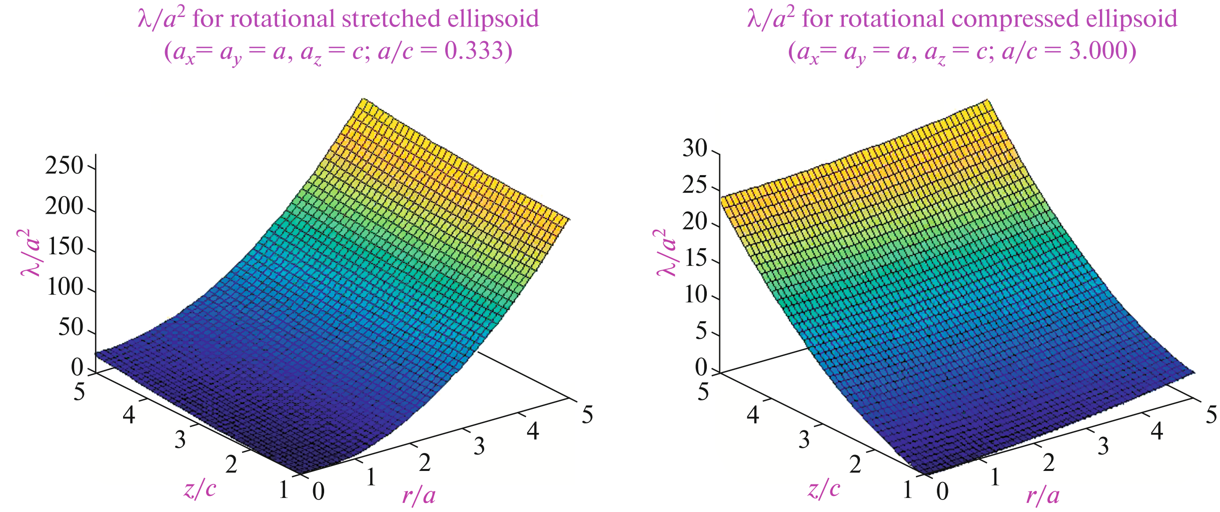

Fig. 4. Value maps for of characteristic equation root for rotational ellipsoid (left, for a case of stretched ellipsoid; right, for a case of compressed ellipsoid).

by closer investigation of the possible values of the characteristic equation root for a rotational ellipsoid, we found that λ does not exceed values of a few hundreds for stretched ellipsoid and several tens for compressed ellipsoid, for reasonable range of beam shape parameters ratios, as shown in Fig. 4.

4. POTENTIAL OF A ROTATIONAL ELLIPSOID

In order to find the potential of the uniformly charged rotational ellipsoid, it is the most convenient to use the expression (32) for the potential outside the 3D ellipsoid. By using $\lambda = 0$ in the found expression, we will acquire the formulas for the points inside the ellipsoid.

Now, we will consider the rotation ellipsoid that can be described with the equation

$$\frac{x^2 + y^2}{a^2 + \lambda} + \frac{z^2}{c^2 + \lambda} = 1. \tag{38}$$

Where $a_x = a_y = a$, $a_z \equiv c$. In this case, the expression (32) for the potential outside the ellipsoid can be written as

$$\begin{aligned} \varphi_{\text{out}}(\vec{r}) &= \pi\rho a^2 c \int_{\lambda}^{\infty} \left(1 - \frac{x^2 + y^2}{a^2 + \xi} - \frac{z^2}{c^2 + \xi}\right) \\ &\times \frac{d\xi}{(a^2 + \xi)\sqrt{c^2 + \xi}} \\ &\equiv \pi\rho a^2 c[I_0 - (x^2 + y^2)I_{xy} - z^2 I_z]. \end{aligned} \tag{39}$$

The values of integrals I_0, I_{xy}, and I_z, as shown in Appendixes A and B, are different for the cases of "compressed" (along the rotation axis) and "stretched" ellipsoids. Now, for the "compressed" ellipsoid, $a > c$ and

$$\left.\begin{aligned} I_0 &= \frac{2}{\sqrt{a^2 - c^2}} \arccos\sqrt{\frac{c^2 + \lambda}{a^2 + \lambda}}; \\ I_{xy} &= \frac{1}{(a^2 - c^2)^{3/2}} \\ &\times \left[\arccos\sqrt{\frac{c^2 + \lambda}{a^2 + \lambda}} - \frac{\sqrt{(a^2 - c^2)(c^2 + \lambda)}}{a^2 + \lambda}\right]; \\ I_z &= \frac{2}{(a^2 - c^2)^{3/2}} \left(\sqrt{\frac{a^2 - c^2}{c^2 + \lambda}} - \arccos\sqrt{\frac{c^2 + \lambda}{a^2 + \lambda}}\right), \end{aligned}\right\} \tag{40}$$

so that

$$\begin{aligned} \varphi_{\text{out}}^{\text{compressed}}(\vec{r}) &= \pi\rho a^2 c[I_0 - (x^2 + y^2)I_{xy} - z^2 I_z] \\ &= \pi\rho a^2 c \left\{\frac{2}{\sqrt{a^2 - c^2}} \arccos\sqrt{\frac{c^2 + \lambda}{a^2 + \lambda}}\right. \\ &- \frac{x^2 + y^2}{(a^2 - c^2)^{3/2}} \\ &\times \left[\arccos\sqrt{\frac{c^2 + \lambda}{a^2 + \lambda}} - \frac{\sqrt{(a^2 - c^2)(c^2 + \lambda)}}{a^2 + \lambda}\right] \\ &+ \left.\frac{2z^2}{(a^2 - c^2)^{3/2}} \left(\arccos\sqrt{\frac{c^2 + \lambda}{a^2 + \lambda}} - \sqrt{\frac{a^2 - c^2}{c^2 + \lambda}}\right)\right\}. \end{aligned} \tag{41}$$

In case of the "stretched" ellipsoid, $c > a$ and then

$$\left.\begin{aligned} I_0 &= \frac{1}{\sqrt{c^2-a^2}}\log\frac{1+\sqrt{(c^2-a^2)/(c^2+\lambda)}}{1-\sqrt{(c^2-a^2)/(c^2+\lambda)}};\\ I_{xy} &= \frac{1}{(c^2-a^2)^{3/2}}\\ &\times\left(\frac{\sqrt{(c^2-a^2)(c^2+\lambda)}}{a^2+\lambda} - \frac{1}{2}\log\frac{1+\sqrt{(c^2-a^2)/(c^2+\lambda)}}{1-\sqrt{(c^2-a^2)/(c^2+\lambda)}}\right);\\ I_z &= \frac{2}{(c^2-a^2)^{3/2}}\\ &\times\left[\frac{1}{2}\log\frac{1+\sqrt{(c^2-a^2)/(c^2+\lambda)}}{1-\sqrt{(c^2-a^2)/(c^2+\lambda)}} - \sqrt{\frac{c^2-a^2}{c^2+\lambda}}\right], \end{aligned}\right\} \tag{42}$$

providing

$$\begin{aligned} \varphi_{\rm out}^{\rm stretched}(\vec{r}) &= \pi\rho a^2 c[I_0 - (x^2+y^2)I_{xy} - z^2 I_z]\\ &= \pi\rho a^2 c\left\{\frac{1}{\sqrt{c^2-a^2}}\log\frac{1+\sqrt{(c^2-a^2)/(c^2+\lambda)}}{1-\sqrt{(c^2-a^2)/(c^2+\lambda)}}\right.\\ &- \frac{x^2+y^2}{(c^2-a^2)^{3/2}}\left[\frac{\sqrt{(c^2-a^2)(c^2+\lambda)}}{a^2+\lambda}\right.\\ &\left.- \frac{1}{2}\log\frac{1+\sqrt{(c^2-a^2)/(c^2+\lambda)}}{1-\sqrt{(c^2-a^2)/(c^2+\lambda)}}\right] - \frac{2z^2}{(c^2-a^2)^{3/2}}\\ &\left.\times\left[\frac{1}{2}\log\frac{1+\sqrt{(c^2-a^2)/(c^2+\lambda)}}{1-\sqrt{(c^2-a^2)/(c^2+\lambda)}} - \sqrt{\frac{c^2-a^2}{c^2+\lambda}}\right]\right\}. \end{aligned} \tag{43}$$

The potentials inside the rotational ellipsoid can be obtained when $\lambda = 0$ is inserted into the expressions (41) and (43), result ing in the expressions for the "compressed" ellipsoid:

$$\begin{aligned} \varphi_{\rm in}^{\rm compressed}(\vec{r}) &= \pi\rho a^2 c\left\{\frac{2}{\sqrt{a^2-c^2}}\arccos\frac{c}{a}\right.\\ &- \frac{x^2+y^2}{(a^2-c^2)^{3/2}}\left[\arccos\frac{c}{a} - \frac{\sqrt{(a^2-c^2)c^2}}{a^2}\right]\\ &\left.+ \frac{2z^2}{(a^2-c^2)^{3/2}}\left(\arccos\frac{c}{a} - \sqrt{\frac{a^2-c^2}{c^2}}\right)\right\} \end{aligned} \tag{44}$$

or

$$\begin{aligned} \varphi_{\rm in}^{\rm compressed}(\vec{r}) &= \pi\rho ac\left\{\frac{2}{\sqrt{1-(c/a)^2}}\arccos\frac{c}{a}\right.\\ &- \frac{(x^2+y^2)/a^2}{[1-(c/a)^2]^{3/2}}\left[\arccos\frac{c}{a} - \frac{c}{a}\sqrt{1-\left(\frac{c}{a}\right)^2}\right]\\ &\left.+ \frac{2z^2/a^2}{[1-(c/a)^2]^{3/2}}\left(\arccos\frac{c}{a} - \left(\frac{c}{a}\right)^{-1}\sqrt{1-\left(\frac{c}{a}\right)^2}\right)\right\}. \end{aligned} \tag{45}$$

And for the stretched ellipsoid:

$$\begin{aligned} &\varphi_{\rm in}^{\rm stretched}(\vec{r})\\ &= \pi\rho a^2 c\left\{\frac{1}{\sqrt{c^2-a^2}}\log\frac{1+\sqrt{(c^2-a^2)/c^2}}{1-\sqrt{(c^2-a^2)/c^2}}\right.\\ &- \frac{x^2+y^2}{(c^2-a^2)^{3/2}}\\ &\times\left[\frac{\sqrt{(c^2-a^2)c^2}}{a^2} - \frac{1}{2}\log\frac{1+\sqrt{(c^2-a^2)/c^2}}{1-\sqrt{(c^2-a^2)/c^2}}\right]\\ &- \frac{2z^2}{(c^2-a^2)^{3/2}}\\ &\left.\times\left[\frac{1}{2}\log\frac{1+\sqrt{(c^2-a^2)/c^2}}{1-\sqrt{(c^2-a^2)/c^2}} - \sqrt{\frac{c^2-a^2}{c^2}}\right]\right\}. \end{aligned} \tag{46}$$

Or

$$\begin{aligned} &\varphi_{\rm in}^{\rm stretched}(\vec{r})\\ &= \pi\rho ac\left\{\frac{1}{\sqrt{(c/a)^2-1}}\log\frac{c/a+\sqrt{(c/a)^2-1}}{c/a-\sqrt{(c/a)^2-1}}\right.\\ &- \frac{(x^2+y^2)/a^2}{[(c/a)^2-1]^{3/2}}\\ &\times\left[\sqrt{\left(\frac{c}{a}\right)^2-1} - \frac{1}{2}\log\frac{c/a+\sqrt{(c/a)^2-1}}{c/a-\sqrt{(c/a)^2-1}}\right]\\ &- \frac{2z^2/a^2}{[(c/a)^2-1]^{3/2}}\\ &\left.\times\left[\frac{1}{2}\log\frac{c/a+\sqrt{(c/a)^2-1}}{c/a-\sqrt{(c/a)^2-1}} - \left(\frac{c}{a}\right)^{-1}\sqrt{\left(\frac{c}{a}\right)^2-1}\right]\right\}. \end{aligned} \tag{47}$$

The formulas (41), and (43)–(47) can be found in [4]. The expressions (45) and (47) for the potential of the rotational ellipsoid were described in [5] (however, with unfortunate typos), and published in [6]. The same paper also provides the expressions for the potential outside the ellipsoid, however only in multipole approximation (analog to (41), (43)).

5. POTENTIAL OF A SPHERE

The potential of a uniform sphere both inside and outside, can be easily found. Nevertheless, it is useful to derive it from the expressions for the rotational ellipsoid by using a limiting transition $c = a$, since the expressions (41), and (43)–(47) don't work in this case. To do this, it is sufficient to modify only the formula (41), and use $\lambda = 0$ for the case inside the sphere. Thus, the potential outside the sphere based on the expression for the "compressed" rotational ellipsoid will be the following:

$$\varphi_{\text{out}}^{\text{sphere}}(\vec{r}) \underset{a\to c}{=} \pi\rho a^2 c\left\{\underbrace{\frac{2}{\sqrt{a^2-c^2}}\arccos\sqrt{\frac{c^2+\lambda}{a^2+\lambda}}}_{(K1)} - (x^2+y^2)\underbrace{\frac{1}{(a^2-c^2)^{3/2}}\left[\arccos\sqrt{\frac{c^2+\lambda}{a^2+\lambda}} - \frac{\sqrt{(a^2-c^2)(c^2+\lambda)}}{a^2+\lambda}\right]}_{(K2)}\right. \\ \left. + 2z^2\underbrace{\frac{1}{(a^2-c^2)^{3/2}}\left(\arccos\sqrt{\frac{c^2+\lambda}{a^2+\lambda}} - \sqrt{\frac{a^2-c^2}{c^2+\lambda}}\right)}_{(K3)}\right\}. \quad (48)$$

Assume $c = a - \delta$ and $\delta/2a \ll 1$. Then, by using the expressions for coefficients $K_1 - K_3$, described in the Appendix C, we get

$$\varphi_{\text{out}}^{\text{sphere}}(\vec{r}) = \pi\rho a^3\left[\frac{2}{\sqrt{a^2+\lambda}} - \frac{2}{3}\frac{x^2+y^2+z^2}{(a^2+\lambda)^{3/2}}\right] \\ = \frac{2\pi\rho a^3}{\sqrt{a^2+\lambda}}\left(1 - \frac{1}{3}\frac{x^2+y^2+z^2}{a^2+\lambda}\right). \quad (49)$$

By using $\lambda = 0$, we can find the potential inside the sphere

$$\varphi_{\text{in}}^{\text{sphere}}(\vec{r}) = 2\pi\rho a^2\left(1 - \frac{1}{3}\frac{x^2+y^2+z^2}{a^2}\right) \\ = 2\pi\rho a^2\left(1 - \frac{1}{3}\frac{r^2}{a^2}\right). \quad (50)$$

To obtain the final expression for the potential outside the sphere, we will take into account the fact that the expression (25) can be simplified:

$$a^2 + \lambda = x^2 + y^2 + z^2 = r^2. \quad (51)$$

Which, therefore, simplifies the expression (49) to its final form:

$$\varphi_{\text{out}}^{\text{sphere}}(\vec{r}) = \frac{4\pi\rho a^3}{3}\frac{1}{r} = \frac{Q}{r}. \quad (52)$$

Obviously, this formula can be trivially found from the Gauss theorem.

6. FORM-FACTORS FOR ELECTRIC FIELD CALCULATIONS

The found expressions (32) for the potential of the charged ellipsoid, as well as the formulas (41), and (43)–(47) for the rotational ellipsoid, demonstrate that in all cases the electric field grows linearly along the corresponding coordinate, while the more complex dependence on the parameters and coordinates is accounted by the behavior of the form-factors M_r:

$$\vec{E}(\vec{r}) = -\frac{\partial\varphi(\vec{r})}{\partial\vec{r}} = 2\pi\rho\left(\int_\lambda^\infty \frac{d\xi}{(a_u^2+\xi)D(\xi)}\right)\cdot\vec{r} \\ \equiv 2\pi\rho M_{\vec{r}}(\vec{a};\lambda)\vec{r}. \quad (53)$$

This expression shows that in the general case, form-factors can be represented as single integrals. But in case of rotational ellipsoid, they can be expressed with the corresponding elementary functions. Here, we present the obvious expressions for all form-factors.

(A) Inside ellipsoid:

$$M_u(\vec{a};0) = \int_0^\infty \frac{d\xi}{(a_u^2+\xi)D(\xi)} \\ = a_x a_y a_z \int_0^\infty \frac{d\xi}{(a_u^2+\xi)\sqrt{(a_x^2+\xi)(a_y^2+\xi)(a_z^2+\xi)}}. \quad (54)$$

The dependences of form-factors inside ellipsoid on different ratio of a_x/a_z for different ratio a_y/a_x for typical realistic beam dimensions are shown in Fig. 5.

(B) Outside ellipsoid:

$$M_u(\vec{a};\lambda) = \int_\lambda^\infty \frac{d\xi}{(a_u^2+\xi)D(\xi)} \\ = a_x a_y a_z \int_\lambda^\infty \frac{d\xi}{(a_u^2+\xi)\sqrt{(a_x^2+\xi)(a_y^2+\xi)(a_z^2+\xi)}}, \quad (55)$$

where λ is the maximal positive root of a characteristic equation.

(C) Inside the "compressed" rotational ellipsoid ($a_x = a_y = a$, $a_z = c < a$):

$$\left.\begin{aligned} &M_{x,y}^{\text{in;compressed}}(a,c;0) \\ &= \frac{c/a}{[1-(c/a)^2]^{3/2}}\left[\arccos\frac{c}{a} - \frac{c}{a}\sqrt{1-\left(\frac{c}{a}\right)^2}\right]; \\ &M_z^{\text{in;compressed}}(a,c;0) \\ &= \frac{2}{[1-(c/a)^2]^{3/2}}\left[\sqrt{1-\left(\frac{c}{a}\right)^2} - \frac{c}{a}\arccos\frac{c}{a}\right]. \end{aligned}\right\} \quad (56)$$

(D) Outside the "compressed" rotational ellipsoid ($a_x = a_y = a$, $a_z = c < a$):

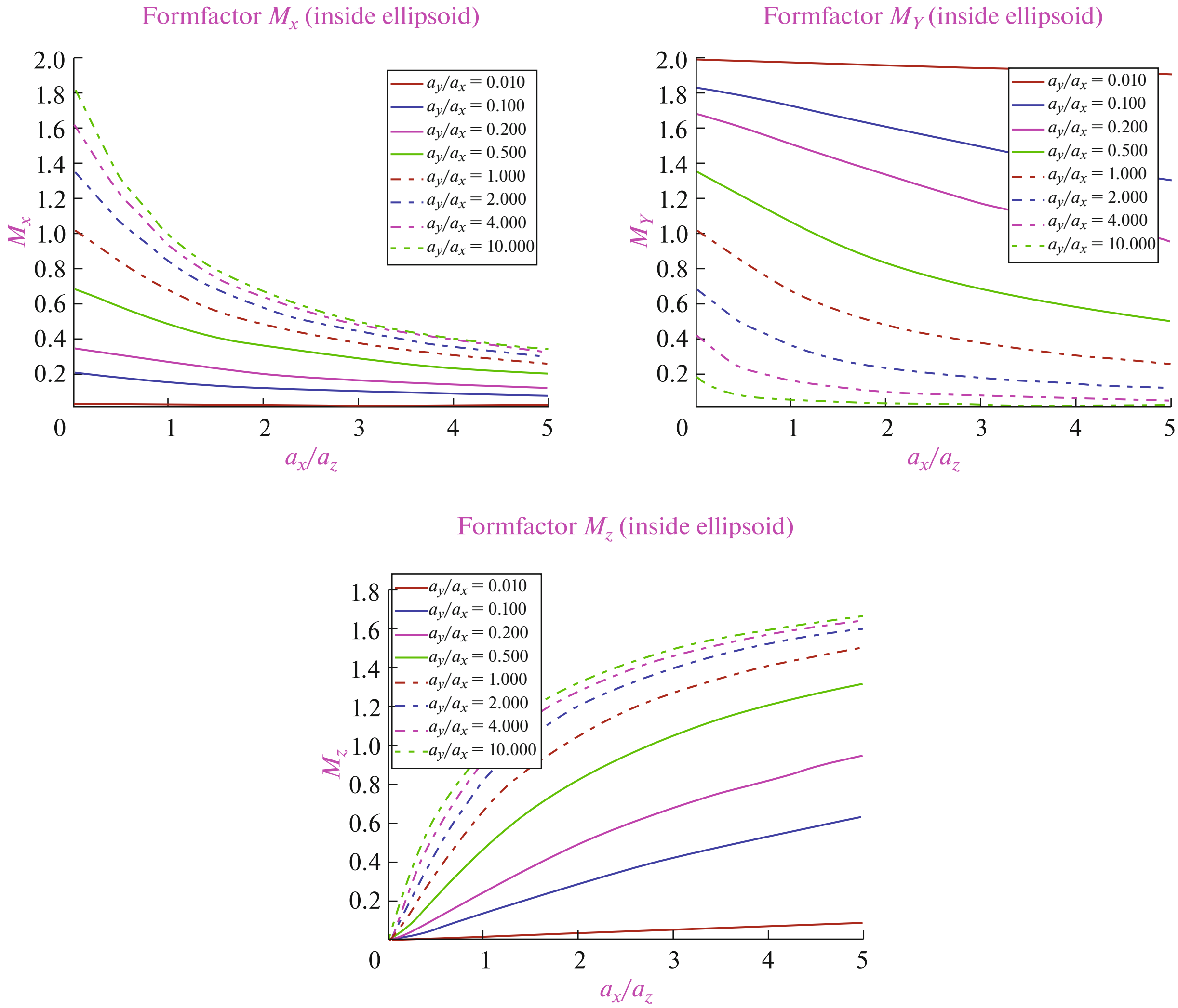

Fig. 5. Form-factors (x, y, and z) inside the ellipsoid as functions of ratios of a_x/a_z and a_y/a_x.

$$\left.\begin{aligned} M_{x,y}^{\text{out;compressed}}(a,c;\lambda) &= \frac{c/a}{[1-(c/a)^2]^{3/2}}\left[\arccos\sqrt{\frac{c^2+\lambda}{a^2+\lambda}} - \frac{\sqrt{(a^2-c^2)(c^2+\lambda)}}{a^2+\lambda}\right]; \\ M_z^{\text{out;compressed}}(a,c;\lambda) &= \frac{2\,c/a}{[1-(c/a)^2]^{3/2}}\left(\sqrt{\frac{a^2-c^2}{c^2+\lambda}} - \arccos\sqrt{\frac{c^2+\lambda}{a^2+\lambda}}\right). \end{aligned}\right\} \quad (57)$$

(E) Inside the "stretched" rotational ellipsoid ($a_x = a_y = a$, $a_z = c > a$):

$$\left.\begin{aligned} M_{x,y}^{\text{in;stretched}}(a,c;0) &= \frac{c/a}{[(c/a)^2-1]^{3/2}}\left[\frac{c}{a}\sqrt{\left(\frac{c}{a}\right)^2-1} - \frac{1}{2}\log\frac{c/a+\sqrt{(c/a)^2-1}}{c/a-\sqrt{(c/a)^2-1}}\right]; \\ M_z^{\text{in;stretched}}(a,c;0) &= \frac{2}{[(c/a)^2-1]^{3/2}}\left[\frac{c/a}{2}\log\frac{c/a+\sqrt{(c/a)^2-1}}{c/a-\sqrt{(c/a)^2-1}} - \sqrt{\left(\frac{c}{a}\right)^2-1}\right]. \end{aligned}\right\} \quad (58)$$

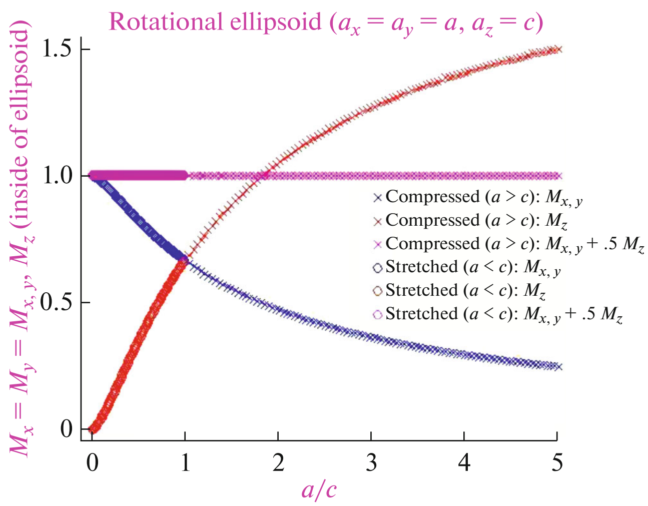

Fig. 6. Form-factors inside the rotational ellipsoid as function of longitudinal compression factor.

(F) Outside the "stretched" rotational ellipsoid ($a_x = a_y = a$, $a_z = c > a$):

$$\left.\begin{aligned} M_{x,y}^{\text{out;stretched}}(a,c;\lambda) &= \frac{c/a}{[(c/a)^2 - 1]^{3/2}} \\ &\times \left[\frac{\sqrt{(c^2 - a^2)(c^2 + \lambda)}}{a^2 + \lambda} \right. \\ &\left. - \frac{1}{2}\log\frac{1 + \sqrt{(c^2 - a^2)/(c^2 + \lambda)}}{1 - \sqrt{(c^2 - a^2)/(c^2 + \lambda)}}\right]; \\ M_z^{\text{out;stretched}}(a,c;\lambda) &= \frac{2c/a}{[(c/a)^2 - 1]^{3/2}} \\ \times \left[\frac{1}{2}\log\frac{1 + \sqrt{(c^2 - a^2)/(c^2 + \lambda)}}{1 - \sqrt{(c^2 - a^2)/(c^2 + \lambda)}} \right. &\left. - \sqrt{\frac{c^2 - a^2}{c^2 + \lambda}}\right]. \end{aligned}\right\} \quad (59)$$

It is easy to see that the pairs of form-factors for the rotational ellipsoid are connected with each other by the simple relationships. Outside the ellipsoid, regardless of compression or stretching:

$$\left.\begin{aligned} &M_{x,y}(a,c;\lambda) + \frac{1}{2}M_z(a,c;\lambda) \\ &= \frac{a^2 c}{(a^2 + \lambda)\sqrt{c^2 + \lambda}} \quad \rightarrow \\ &M_z(a,c;\lambda) \\ &= 2\left[\frac{a^2 c}{(a^2 + \lambda)\sqrt{c^2 + \lambda}} - M_{x,y}(a,c;\lambda)\right]. \end{aligned}\right\} \quad (60)$$

Inside the ellipsoid ($\lambda = 0$) this relationship is simplified even further:

$$\begin{aligned} &M_{x,y}(a,c;0) + \frac{1}{2}M_z(a,c;0) = 1 \\ &\rightarrow M_z(a,c;0) = 2[1 - M_{x,y}(a,c;0)]. \end{aligned} \quad (61)$$

The dimensional dependencies of form-factor values inside the ellipsoid, calculated by using the expressions (56), (58), and (61), are shown in Fig. 6.

Correspondingly, the dimensional dependencies of form-factor values outside the ellipsoid, calculated by using the expressions (57) and (59) for different λ values, are shown in Figs. 6, 7.

7. MULTI-BUNCH REGIME

In case of a multi-bunch regime, each bunch can be regarded as a uniformly charged ellipsoid, and then this "reference" bunch will suffer from the fields of all other bunches that will be external in respect to it. The forces, thus, can be accounted by using the expression (32). It is however, necessary to solve the characteristic equation (25) for each external bunch. Typically, the distance between the bunches (L) is large comparing to its dimensions a_x, a_y, a_z. Also, in this case, the transverse coordinates inside the reference bunch are small relative to the distance L: $x, y \ll L$, while the longitudinal coordinate z is comparable to L.

It is possible to demonstrate that in this case, the maximal root of the characteristic equation will be at the order of z. If so, then it is possible to omit the terms with x^2 and y^2 in the characteristic equation (25), so that it takes the form:

$$z^2/(a_z^2 + \lambda) = 1. \quad (62)$$

With $a_z \ll \lambda \approx L$, it yields with $\lambda \approx z$. For the more strict solution, it is necessary to solve the characteristic equation considering a_x, a_y, a_z, x, $y \ll z \approx L$. In this case, the coefficients (35) become:

$$\begin{gathered} A = \frac{a_x^4}{a_y^2 a_z^2}, \quad B \approx -\frac{a_x^2 z^2}{a_y^2 a_z^2}, \quad C \approx -\frac{z^2(a_x^2 + a_y^2)}{a_y^2 a_z^2}, \\ D \approx -\frac{z^2}{a_z^2}. \end{gathered} \quad (63)$$

Next, we need to calculating the coefficients (36) using expressions (63) and obtain:

$$\begin{gathered} p \approx -\frac{1}{3}\frac{z^4}{a_x^4}, \quad q \approx -\frac{2}{27}\frac{z^6}{a_x^6} \rightarrow \tau = \sqrt{-\frac{p^3}{27}} \approx \frac{z^6}{27 a_x^6}, \\ \varphi = \arccos\left(-\frac{q}{2\tau}\right) \approx 0. \end{gathered} \quad (64)$$

Then, the maximal root of the characteristic equation, according to (34) becomes:

$$\begin{aligned} \lambda &= a_x^2\left(2\sqrt[3]{\tau}\cos\varphi - \frac{B}{3A}\right) \\ &\approx a_x^2\left(\frac{2}{3}\frac{z^2}{a_x^2} - \left(-\frac{1}{3}\frac{z^2}{a_x^2}\right)\right) = z^2. \end{aligned} \quad (65)$$

Figure 8 demonstrates the dependence of a mximal root of the characteristic equation on the distance between leading and neighboring bunch for round and flat beam.

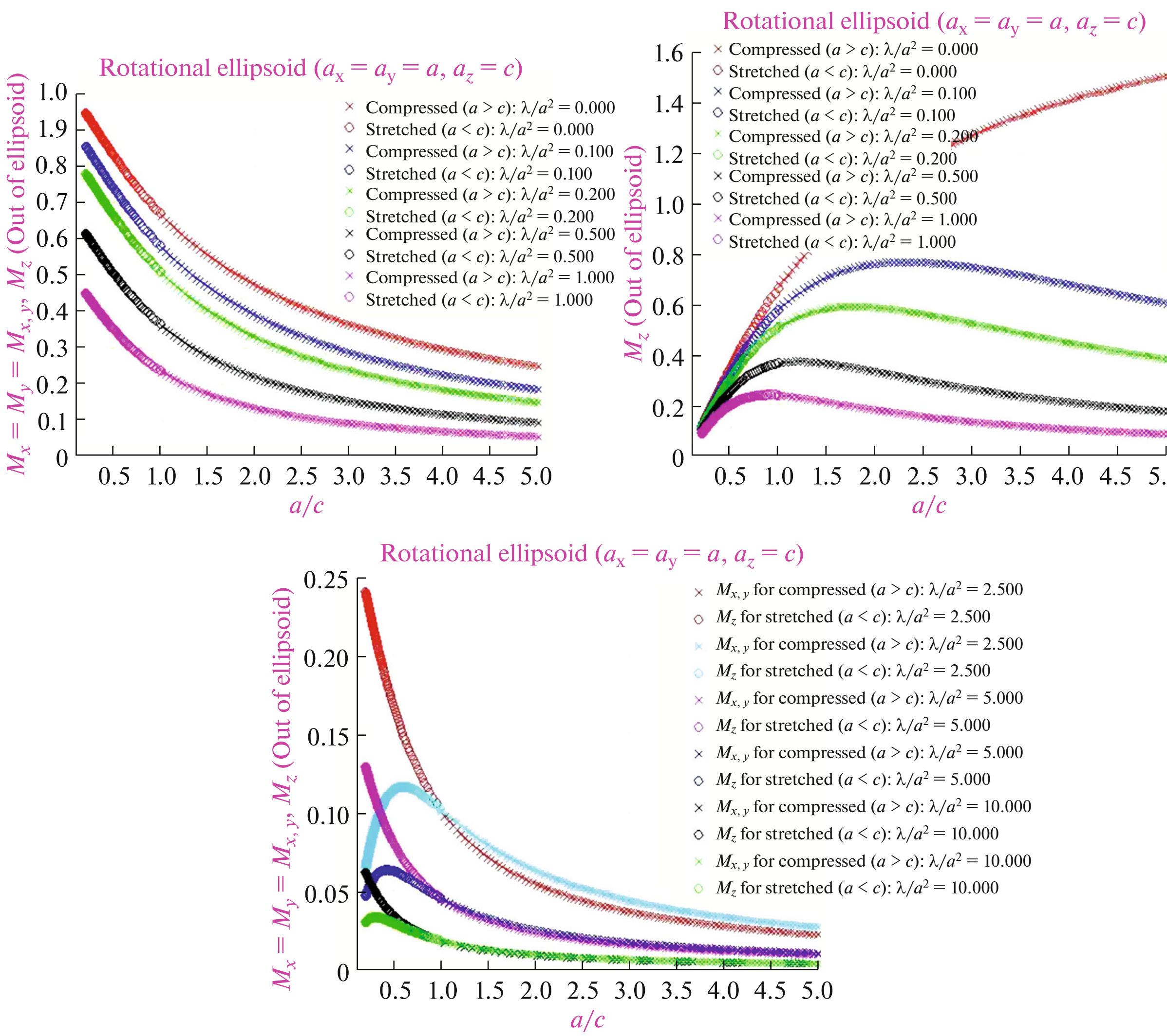

Fig. 7. Form-factors outside the rotational ellipsoid as function of longitudinal compression factor.

Now, it is easy to estimate the asymptotic values of the form-factor (55) for the neighbor bunches in respect to the reference bunch (a_x, a_y, a_z, $\ll z$ and $\xi \geq z$):

$$\begin{aligned} &M_u(\vec{a}; z^2) \\ &= a_x a_y a_z \int_{z^2}^{\infty} \frac{d\xi}{(a_u^2+\xi)\sqrt{(a_x^2+\xi)(a_y^2+\xi)(a_z^2+\xi)}} \\ &\approx a_x a_y a_z \int_{z^2}^{\infty} \frac{d\xi}{\xi^{5/2}} = -\frac{2}{3} a_x a_y a_z \, \xi^{-3/2}\Big|_{z^2}^{\infty} = \frac{2}{3}\frac{a_x a_y a_z}{z^3}. \end{aligned} \tag{66}$$

This result means that all form-factors are indentical in this case. Thus, the fields from the bunches with the charge Q, located at the distance $\pm L$ are equal to:

$$\begin{aligned} \vec{E}(\vec{r}) &= 2\pi\rho M_{\vec{r}}(\vec{a}; (z \pm L)^2) \cdot \vec{r} \\ &= \frac{4}{3}\pi\rho a_x a_y a_z \frac{\vec{r}}{(z \pm L)^3} = \frac{Q\vec{r}}{(z \pm L)^3}. \end{aligned} \tag{67}$$

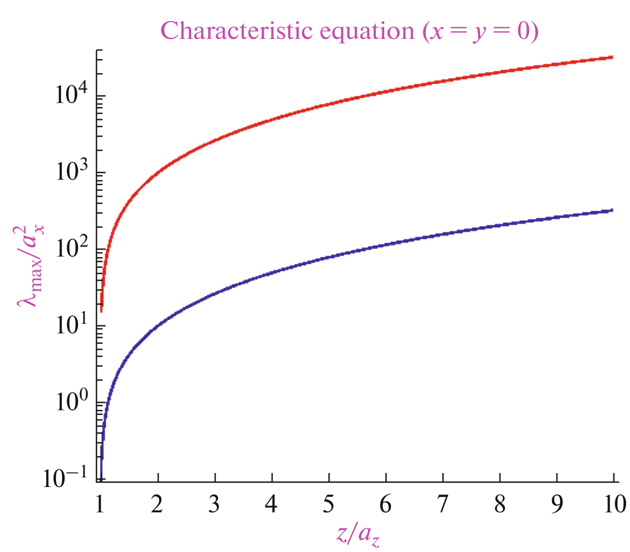

Fig. 8. The roots of the characteristic equation in multibunch regime as a function of normalized longitudinal coordinate (red—round beam; blue—flat beam).

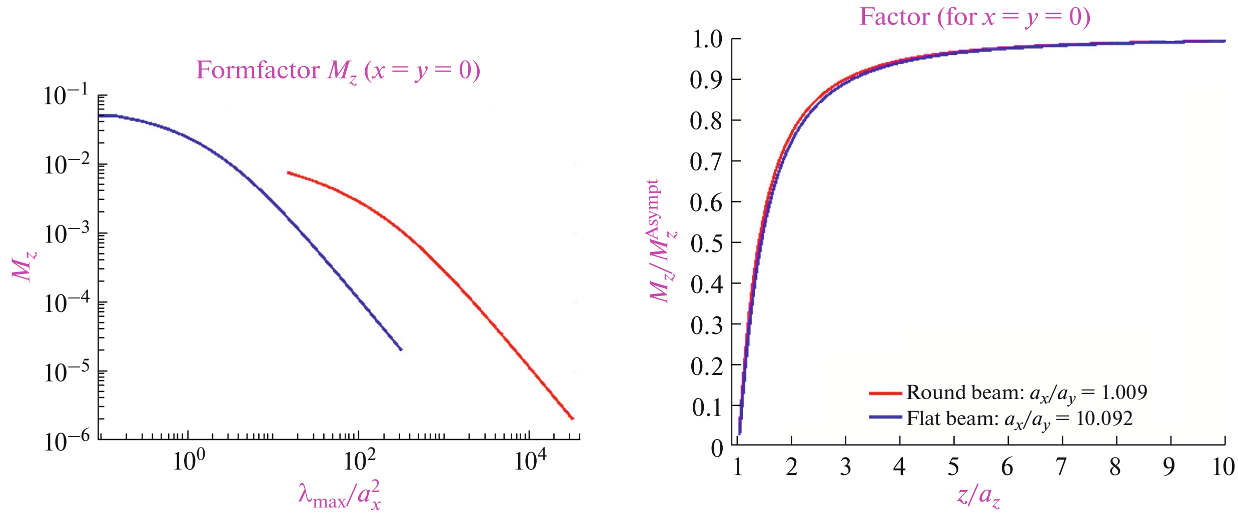

Fig. 9. Absolute (left) and relative (right) values of the form factor M_z (red—round beam; blue—flat beam).

The expression (67) shows that the asymptotic values of the form factors are identical. Figure 9 demonstrates the values of the "vertical" form factor M_z, calculated with this formula and compares it with the absolute values calculated with the expression (55).

It can be seen, that the difference in the values of the factor and its asymptotic value is of the order of 20% at a distance between the bunches $z/z_{rms} \approx 2$ and does not exceed 10% for $z/z_{rms} \geq 3$.

8. GAUSSIAN ELLIPSOID DISTRIBUTION

Finally, we will consider the case when the charge distribution of the bunch is Gaussian in the whole space and is characterized with the square deviation parameters (σ_x, σ_y, σ_z), so that it has the following form:

$$\rho(\sigma_x, \sigma_y, \sigma_z; \vec{r}\,') = \frac{Q}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z}\exp\left(-\frac{x'^2}{2\sigma_x^2} - \frac{y'^2}{2\sigma_y^2} - \frac{z'^2}{2\sigma_z^2}\right). \quad (68)$$

Here Q is the full charge of the bunch. The potential of the bunch in any point will be the following:

$$\varphi(\vec{\sigma}; \vec{r}) = \frac{Q}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z} \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp\left(-\frac{x'^2}{2\sigma_x^2} - \frac{y'^2}{2\sigma_y^2} - \frac{z'^2}{2\sigma_z^2}\right)}{\sqrt{|\vec{r} - \vec{r}\,'|}} dx'\,dy'\,dz'. \quad (69)$$

However,

$$\int_{-\infty}^{\infty} e^{-|\alpha^2 t^2|} dt = \sqrt{\frac{\pi}{\alpha}} \to \frac{1}{\sqrt{|\vec{r} - \vec{r}\,'|}} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-|\vec{r} - \vec{r}\,'|^2 t^2} dt. \quad (70)$$

Which yields to the expression

$$\varphi(\vec{\sigma}; \vec{r}) = \frac{Q/\sqrt{\pi}}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z} \times \int_{-\infty}^{\infty} dt \prod_{\substack{u = x,y,z \\ u' = x',y',z'}} \underbrace{\int_{-\infty}^{\infty} \exp\left[-\frac{u'^2}{2\sigma_u^2} - (u - u')^2 t^2\right] du'}_{I(u,\sigma_u;t)} \quad (71)$$
$$= \frac{Q/\sqrt{\pi}}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z}\int_{-\infty}^{\infty} I(x, \sigma_x; t) I(y, \sigma_y; t) I(z, \sigma_z; t) dt.$$

The integrals $I(u, \sigma_u; t)$ can be found with a help of a simple transformation:

$$\frac{u'^2}{2\sigma_u^2} + (u - u')^2 t^2 = \left(\frac{1}{2\sigma_u^2} + t^2\right)u'^2 - 2t^2 uu' + t^2 u^2$$
$$= t^2u^2 = \left(\frac{1}{2\sigma_u^2} + t^2\right)\left[u'^2 - 2\frac{t^2 u}{1/2\sigma_u^2 + t^2}u' + \left(\frac{t^2 u}{1/2\sigma_u^2 + t^2}\right)^2 - \left(\frac{t^2 u}{1/2\sigma_u^2 + t^2}\right)^2\right]$$
$$= t^2 u^2 - \frac{t^4 u^2}{1/2\sigma_u^2 + t^2} + \left(\frac{1}{2\sigma_u^2} + t^2\right)\left(u' - \frac{t^2 u}{1/2\sigma_u^2 + t^2}\right)^2 \quad (72)$$
$$= \frac{t^2 u^2/2\sigma_u^2}{1/2\sigma_u^2 + t^2}\left(\frac{1}{2\sigma_u^2} + t^2\right) + \left(u' - \frac{t^2 u}{1/2\sigma_u^2 + t^2}\right)^2.$$

Then

$$I(u,\sigma_u;t) = \int\limits_{-\infty}^{\infty} \exp\left[-\frac{u'^2}{2\sigma_u^2} - (u-u')^2 t^2\right] du' = \int\limits_{-\infty}^{\infty} \exp\left[-\frac{t^2u^2/2\sigma_u^2}{1/2\sigma_u^2 + t^2} - \underbrace{\left(\frac{1}{2\sigma_u^2} + t^2\right)}_{\alpha}\right.$$
$$\left.\times\left(u' - \frac{t^2 u}{1/2\sigma_u^2 + t^2}\right)^2\right] du' = \exp\left(-\frac{t^2u^2/2\sigma_u^2}{1/2\sigma_u^2 + t^2}\right)\int\limits_{-\infty}^{\infty} e^{-\alpha u'^2} du' \tag{73}$$
$$= \exp\left(-\frac{t^2u^2/2\sigma_u^2}{1/2\sigma_u^2 + t^2}\right)\sqrt{\frac{\pi}{\alpha}} = \sqrt{2\pi\sigma_u^2}\,\frac{\exp\left(-\frac{t^2u^2}{1+2\sigma_u^2t^2}\right)}{\sqrt{1+2\sigma_u^2t^2}}.$$

And therefore

$$\varphi(\vec{\sigma};\vec{r}) = \frac{Q/\sqrt{\pi}}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z}\int\limits_{-\infty}^{\infty} I(x,\sigma_x;t)I(y,\sigma_y;t)I(z,\sigma_z;t)dt$$
$$= \frac{Q/\sqrt{\pi}}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z}\int\limits_{-\infty}^{\infty}\prod_{u=x,y,z}\sqrt{2\pi\sigma_u^2}\,\frac{\exp\left(-\frac{t^2u^2}{1+2\sigma_u^2t^2}\right)}{\sqrt{1+2\sigma_u^2t^2}}dt \tag{74}$$
$$= \frac{2Q}{\sqrt{\pi}}\int\limits_0^{\infty}\frac{\exp\left(-\frac{x^2t^2}{1+2\sigma_x^2t^2} - \frac{y^2t^2}{1+2\sigma_y^2t^2} - \frac{z^2t^2}{1+2\sigma_z^2t^2}\right)}{\sqrt{(1+2\sigma_x^2t^2)(1+2\sigma_y^2t^2)(1+2\sigma_z^2t^2)}}dt.$$

From this equation it is easy to find the expression for the space charge field in any point of the space:

$$\vec{E}_u(\sigma_x,\sigma_y,\sigma_z;\vec{r}) = -\frac{\partial\varphi(\sigma_x,\sigma_y,\sigma_z;\vec{r})}{\partial\vec{r}_u} = \frac{4Q}{\sqrt{\pi}}\vec{r}_u\int\limits_0^{\infty}\frac{\exp\left(-\frac{x^2t^2}{1+2\sigma_x^2t^2} - \frac{y^2t^2}{1+2\sigma_y^2t^2} - \frac{z^2t^2}{1+2\sigma_z^2t^2}\right)}{(1+2\sigma_u^2t^2)\sqrt{(1+2\sigma_x^2t^2)(1+2\sigma_y^2t^2)(1+2\sigma_z^2t^2)}}t^2dt. \tag{75}$$

It is very convenient to make the following substitution $\xi = 1/t^2$, and get the final expression for the field:

$$\vec{E}_u(\vec{\sigma};\vec{r}) = \frac{2Q}{\sqrt{\pi}}$$
$$\times\,\vec{r}_u\int\limits_0^{\infty}\frac{\exp\left(-\frac{x^2}{2\sigma_x^2+\xi} - \frac{y^2}{2\sigma_y^2+\xi} - \frac{z^2}{2\sigma_z^2+\xi}\right)}{(2\sigma_u^2+\xi)\sqrt{(2\sigma_x^2+\xi)(2\sigma_y^2+\xi)(2\sigma_z^2+\xi)}}d\xi. \tag{76}$$

This expression for the Gaussian distribution of the bunch charge is actively used for beam-beam effects studies [8, 9]. It is convenient to introduce the form factors M_u, similar to the ones introduced in the previous Sections, so that:

$$\vec{E}_u(\vec{\sigma};\vec{r}) = \frac{Q}{\sqrt{2\pi}\sigma_x\sigma_y\sigma_z}M_u\vec{r}_u, \tag{77}$$

where

$$M_u(\vec{r};\vec{\sigma}) = 2\sqrt{2}\sigma_x\sigma_y\sigma_z$$
$$\times\int\limits_0^{\infty}\frac{\exp\left(-\frac{x^2}{2\sigma_x^2+w} - \frac{y^2}{2\sigma_y^2+w} - \frac{z^2}{2\sigma_z^2+w}\right)}{(2\sigma_u^2+w)\sqrt{(2\sigma_x^2+w)(2\sigma_y^2+w)(2\sigma_z^2+w)}}dw. \tag{78}$$

In the case of axially-symmetrical distribution ($\sigma_x = \sigma_y \equiv \sigma_r$), the expressions for the form-factors are simplified:

$$M_x = M_y \equiv M_{xy} = 2\sqrt{2}\sigma_r^2\sigma_z\int\limits_0^{\infty}\frac{\exp\left(-\frac{r^2}{2\sigma_r^2+w} - \frac{z^2}{2\sigma_z^2+w}\right)}{(2\sigma_r^2+w)^2\sqrt{(2\sigma_z^2+w)}}dw,$$
$$M_z = 2\sqrt{2}\sigma_r^2\sigma_z\int\limits_0^{\infty}\frac{\exp\left(-\frac{r^2}{2\sigma_r^2+w} - \frac{z^2}{2\sigma_z^2+w}\right)}{(2\sigma_r^2+w)(2\sigma_z^2+w)^{3/2}}dw. \tag{79}$$

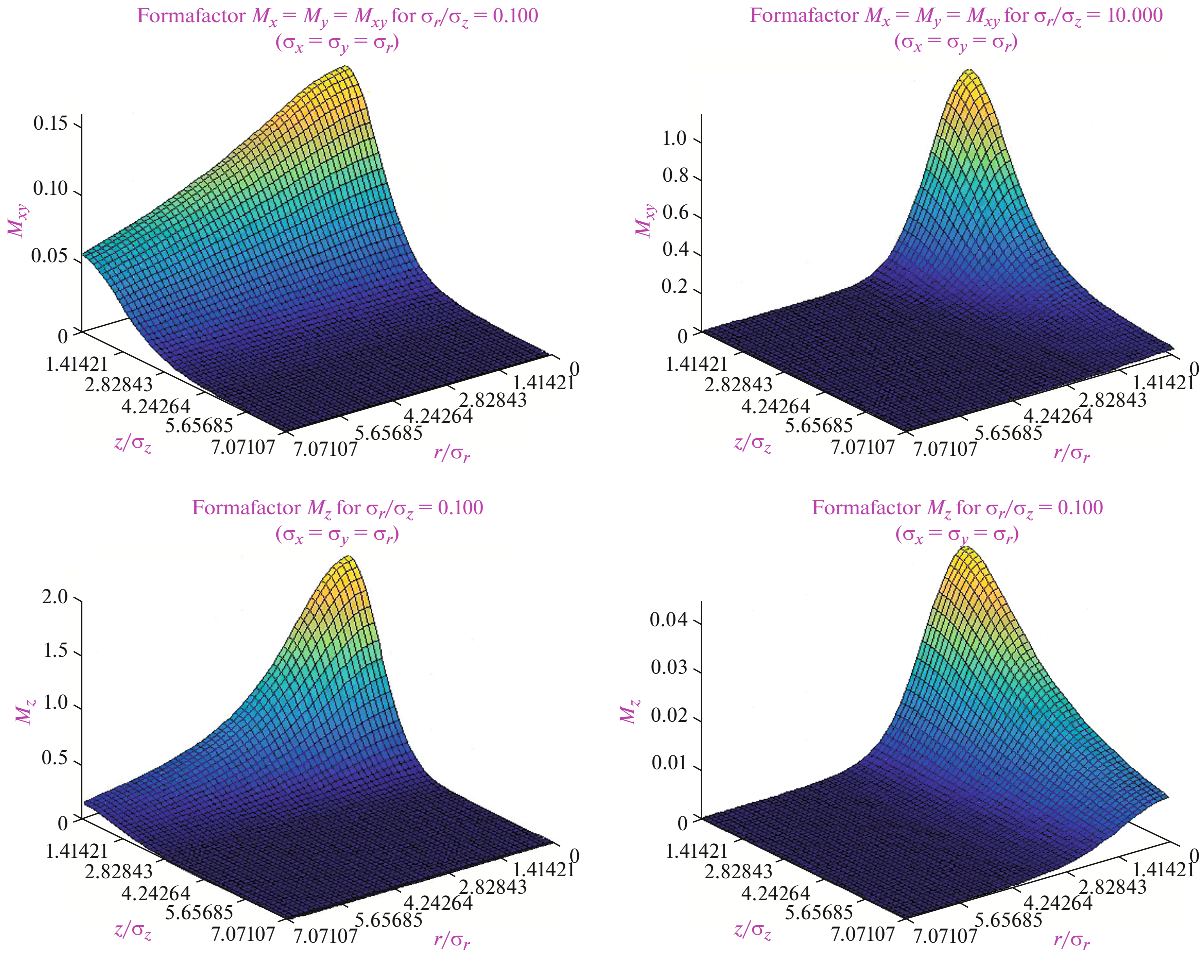

Fig. 10. Form-factor values maps in z–r space for different values of beam compression factors.

By introducing the dimensionless integration variable $\xi = w/2\sigma_z^2$, both form-factors are found to be dependent on the parameters $\alpha = \sigma_z^2/\sigma_r^2$, which characterizes the degree of beam compression, and can be written as following:

$$M_{xy}\left(\frac{r}{\sqrt{2}\sigma_r}, \frac{z}{\sqrt{2}\sigma_z}; \alpha\right) = \alpha \int_0^\infty \frac{\exp\left(-\frac{r^2/2\sigma_r^2}{1+\alpha\xi} - \frac{z^2/2\sigma_z^2}{1+\xi}\right)}{(1+\alpha\xi)^2\sqrt{1+\xi}} d\xi,$$

$$M_z\left(\frac{r}{\sqrt{2}\sigma_r}, \frac{z}{\sqrt{2}\sigma_z}; \alpha\right) = \int_0^\infty \frac{\exp\left(-\frac{r^2/2\sigma_r^2}{1+\alpha\xi} - \frac{z^2/2\sigma_z^2}{1+\xi}\right)}{(1+\alpha\xi)(1+\xi)^{3/2}} d\xi. \quad (80)$$

The Fig. 10 demonstrate the typical behavior of both form-factors for large and small values of the parameter α. It is clearly seen that the largest values of the form-factors are naturally located in the center of the beam.

Finally, the Fig. 11 demonstrate the dependence of form-factors in the beam center as a function of beam compression parameter. It is clearly seen, that in case of the Gaussian density distribution the expression (61) is fulfilled for the whole space.

9. COMPARISON OF THE MODELS

In this paragraph we will compare the accuracy of the computer simulations, performed using the well-known Lapostolle method [6], and the method described in this paper. The simulations were done in the recently upgraded [10], and benchmarked in Hellweg code [11] that support both models. We have simulated the beam propagation of 10 A electron beams with the energy of 100 keV in the 50 cm drift space and compared the output emittance growth for two models. The input Twiss parameters for all cases are identical ($\alpha_x = 0.0$, $\beta_x = 3.0$ cm/rad, $\varepsilon_x = 15$ μm, $\Delta z = 2.5$ cm). The following cases were simulated (see Fig. 12):

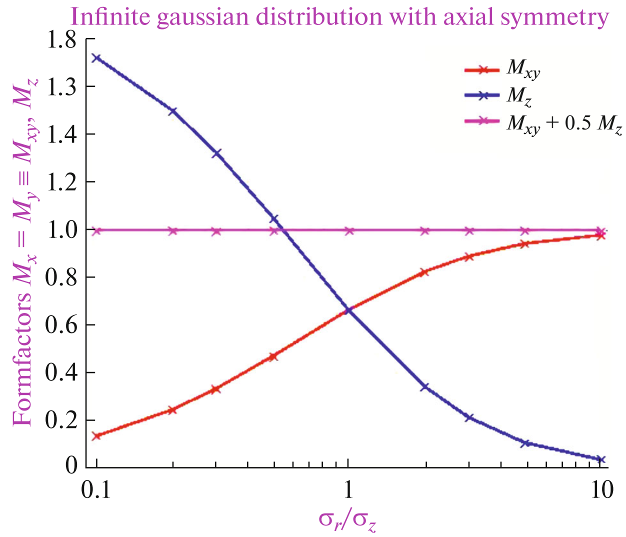

Fig. 11. Form-factor values in the beam center as a function of beam compression factor.

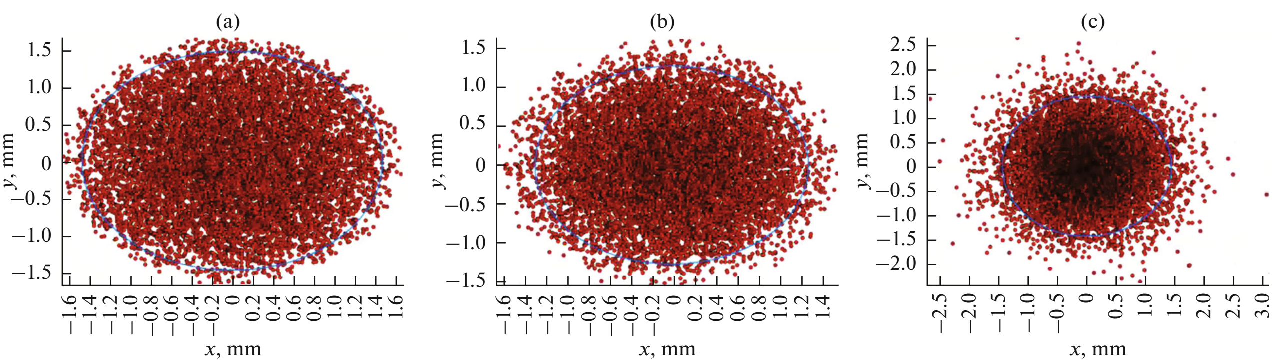

Fig. 12. Initial distribution of the simulated beams ((a) Round KV, (b) Round waterbag, (c) Round gaussian, (d) Elliptic gaussian).

(a) Round beam with 4D KV distribution;

(b) Round beam with 6D waterbag distribution;

(c) Round beam with gaussian distribution.

The simulation results for these cases are presented in Fig. 13. As expected, the models produce similar results for uniform beams. However, when the halo is present, the Lapostolle model tends to overestimate

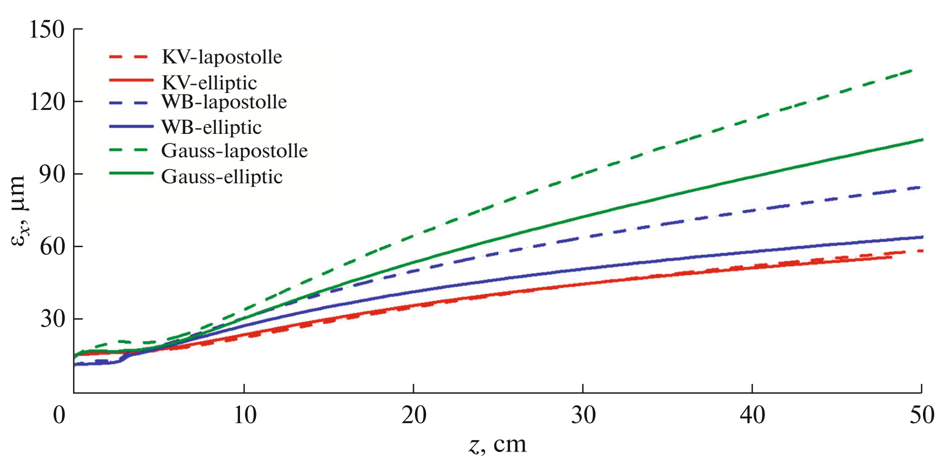

Fig. 13. Transverse emittance growth during the propagation in the drift space of the beam with different initial distributions (red—round KV, blue—round waterbag, green—round Gaussian) calculated in Hellweg with Lapostolle (dashed) and the described Elliptic (solid) space charge models.

the space charge effects due to the inaccurate field treatment outside of the beam core.

APPENDIX A

INTEGRALS FOR THE "COMPRESSED" ELLIPSOID

The integral I_0 can be calculated by the following way:

$$I_0 = \int_\lambda^\infty \frac{d\xi}{(a^2+\xi)\sqrt{c^2+\xi}} = c\int_\lambda^\infty \frac{d(\xi/c^2)}{(a^2+c^2\xi/c^2)\sqrt{1+\xi/c^2}} = c\int_{\lambda/c^2}^\infty \frac{dt}{(a^2+c^2t)\sqrt{1+t}}. \quad \text{(A1)}$$

After the variable substitution

$$y = \sqrt{1+t}. \quad \text{(A2)}$$

We get

$$I_0 = c\int_{\sqrt{1+\lambda/c^2}}^\infty \frac{2ydy}{[a^2+c^2(y^2-1)]y} = \frac{2c}{a^2-c^2}\int_{\sqrt{1+\lambda/c^2}}^\infty \frac{dy}{1+\frac{c^2}{a^2-c^2}y^2}. \quad \text{(A3)}$$

Next, the result depends on the fact whether the ellipsoid is "compressed" ($a > c$) or "stretched" along the rotation axis. In case of the "compressed ellipsoid, the multiplier before y^2 in the integral function is positive and thus:

$$I_0 = \frac{2c}{a^2-c^2}\sqrt{\frac{a^2-c^2}{c^2}}\int_{\sqrt{1+\lambda/c^2}}^\infty \frac{d\left(\sqrt{\frac{c^2}{a^2-c^2}}y\right)}{1+\left(\sqrt{\frac{c^2}{a^2-c^2}}y\right)^2} = \frac{2}{\sqrt{a^2-c^2}}\arctan\frac{c}{\sqrt{a^2-c^2}}y\Big|_{\sqrt{1+\lambda/c^2}}^\infty$$
$$= \frac{2}{\sqrt{a^2-c^2}}\left(\frac{\pi}{2}-\arctan\sqrt{\frac{c^2+\lambda}{a^2-c^2}}\right) = \frac{2}{\sqrt{a^2-c^2}}\arccos\sqrt{\frac{c^2+\lambda}{a^2-c^2}\Big/\left(1+\frac{c^2+\lambda}{a^2-c^2}\right)}. \quad \text{(A4)}$$

Or

$$I_0 = \frac{2}{\sqrt{a^2-c^2}}\arccos\sqrt{\frac{c^2+\lambda}{a^2+\lambda}}. \quad \text{(A5)}$$

Then, we'll find I_{xy} for the compressed ellipsoid by applying the variables substation used for I_0

$$I_{xy} = \int_\lambda^\infty \frac{d\xi}{(a^2+\xi)^2\sqrt{c^2+\xi}} = c\int_\lambda^\infty \frac{d(\xi/c^2)}{(a^2+c^2\xi/c^2)^2\sqrt{1+\xi/c^2}} = c\int_{\lambda/c^2}^\infty \frac{dt}{(a^2+c^2t)^2\sqrt{1+t}}$$
$$= \frac{2c}{(a^2-c^2)^2}\int_{\sqrt{1+\lambda/c^2}}^\infty \frac{dy}{\left(1+\frac{c^2}{a^2-c^2}y^2\right)^2}. \quad \text{(A6)}$$

Let's introduce the following parameters for the compressed ellipsoid:

$$\alpha = c^2/(a^2-c^2) > 0. \quad \text{(A7)}$$

And the integral I_1:

$$I_1(\alpha) = \int\frac{dy}{1+\alpha y^2} = \frac{1}{\sqrt{\alpha}}\int\frac{d(\sqrt{\alpha}y)}{1+(\sqrt{\alpha}y)^2} = \frac{1}{\sqrt{\alpha}}\arctan(\sqrt{\alpha}y). \quad \text{(A8)}$$

Then

$$-\frac{\partial I_1(\alpha)}{\partial\alpha} = \int\frac{y^2dy}{(1+\alpha y^2)^2} = \frac{1}{\alpha}\int\frac{[-1+(1+y^2)]dy}{(1+\alpha y^2)^2} = -\frac{1}{\alpha}\int\frac{dy}{(1+\alpha y^2)^2}+\frac{1}{\alpha}\int\frac{dy}{1+\alpha y^2}, \quad \text{(A9)}$$

or in the other words,

$$\int\frac{dy}{(1+\alpha y^2)^2} = I_1+\alpha\frac{\partial I_1(\alpha)}{\partial\alpha} = \frac{1}{\sqrt{\alpha}}\arctan(\sqrt{\alpha}y) + \alpha\frac{\partial}{\partial\alpha}\left[\frac{1}{\sqrt{\alpha}}\arctan(\sqrt{\alpha}y)\right] = \frac{1}{\sqrt{\alpha}}\arctan(\sqrt{\alpha}y)$$
$$-\frac{1}{2\sqrt{\alpha}}\arctan(\sqrt{\alpha}y)+\frac{\alpha}{\sqrt{\alpha}}\frac{1}{1+\alpha y^2}\frac{y}{2\sqrt{\alpha}} = \frac{1}{2\sqrt{\alpha}}\left[\arctan(\sqrt{\alpha}y)+\frac{\sqrt{\alpha}y}{1+\alpha y^2}\right] \quad \text{(A10)}$$

and thus,

$$I_{xy} = \frac{2c}{(a^2-c^2)^2}\int_{\sqrt{1+\lambda/c^2}}^\infty \frac{dy}{\left(1+\alpha y^2\right)^2} = \frac{1}{(a^2-c^2)^{3/2}}\left[\underbrace{\frac{\pi}{2}-\arctan\sqrt{\frac{c^2+\lambda}{a^2-c^2}}}_{\arccos\sqrt{(c^2+\lambda)/(a^2+\lambda)}} - \frac{\sqrt{\frac{c^2+\lambda}{a^2-c^2}}}{1+\frac{c^2+\lambda}{a^2-c^2}}\right] \quad \text{(A11)}$$

or

$$I_{xy} = \frac{1}{(a^2 - c^2)^{3/2}}\left[\arccos\sqrt{\frac{c^2+\lambda}{a^2+\lambda}} - \frac{\sqrt{(c^2+\lambda)(a^2-c^2)}}{a^2+\lambda}\right]. \tag{A12}$$

For the integral I_z, we will repeat the similar steps:

$$I_z = \int_\lambda^\infty \frac{d\xi}{(a^2+\xi)(c^2+\xi)^{3/2}} = \frac{1}{c}\int_\lambda^\infty \frac{d(\xi/c^2)}{(a^2+c^2\,\xi/c^2)(1+\xi/c^2)^{3/2}} = \frac{1}{c}\int_{\lambda/c^2}^\infty \frac{dt}{(a^2+c^2t)(1+t)^{3/2}}$$

$$= \frac{2}{c}\int_{\sqrt{1+\lambda/c^2}}^\infty \frac{dy}{\left[a^2+c^2(y^2-1)\right]y^2} = \frac{2}{c}\int_{\sqrt{1+\lambda/c^2}}^\infty \left(\frac{1}{y^2} - \frac{c^2}{a^2-c^2+c^2y^2}\right)\frac{dy}{a^2-c^2}$$

$$= \frac{2}{c(a^2-c^2)}\left[\frac{1}{y}\bigg|_\infty^{\sqrt{1+\lambda/c^2}} - \frac{c^2}{a^2-c^2}\int_{\sqrt{1+\lambda/c^2}}^\infty \frac{dy}{1+\frac{c^2}{a^2-c^2}y^2}\right] = \frac{2}{c(a^2-c^2)}\left[\frac{c}{\sqrt{c^2+\lambda}} - \frac{c^2}{a^2-c^2}\frac{\sqrt{a^2-c^2}}{c}\right. \tag{A13}$$

$$\left.\times\int_{\sqrt{(c^2+\lambda)/(a^2-c^2)}}^\infty \frac{dz}{1+z^2}\right] = \frac{2c}{c(a^2-c^2)}\left[\frac{1}{\sqrt{c^2+\lambda}} - \frac{1}{\sqrt{a^2-c^2}}\left(\underbrace{\frac{\pi}{2} - \arctan\sqrt{\frac{c^2+\lambda}{a^2-c^2}}}_{\arccos\sqrt{(c^2+\lambda)/(a^2+\lambda)}}\right)\right]$$

finally,

$$I_z = \frac{2}{(a^2-c^2)^{3/2}}\left(\sqrt{\frac{a^2-c^2}{c^2+\lambda}} - \arccos\sqrt{\frac{c^2+\lambda}{a^2+\lambda}}\right). \tag{A14}$$

APPENDIX B

INTEGRALS FOR THE "STRETCHED" ELLIPSOID

The integral I_0 as show in in Appendix A (A3), is equal to:

$$I_0 = \frac{2c}{a^2-c^2}\int_{\sqrt{1+\lambda/c^2}}^\infty \frac{dy}{1+\frac{c^2}{a^2-c^2}y^2}. \tag{B1}$$

For the stretched ellipsoid, $c > a$, so by introducing the parameter

$$\alpha = c^2/(c^2-a^2) > 1 \tag{B2}$$

we have:

$$I_0 = \frac{2c}{a^2-c^2}\int_{\sqrt{1+\lambda/c^2}}^\infty \frac{dy}{1-(\sqrt{\alpha}y)^2} = \frac{2c^2}{c(c^2-a^2)}\int_{\sqrt{1+\lambda/c^2}}^\infty \frac{dy}{(\sqrt{\alpha}y)^2-1} = \frac{2\alpha}{c\sqrt{\alpha}}\int_{\sqrt{1+\lambda/c^2}}^\infty \frac{d(\sqrt{\alpha}y)}{2}\left(\frac{1}{\sqrt{\alpha}y-1} - \frac{1}{\sqrt{\alpha}y+1}\right)$$

$$= \frac{1}{\sqrt{c^2-a^2}}\log\frac{\sqrt{\alpha}y-1}{\sqrt{\alpha}y+1}\bigg|_{\sqrt{1+\lambda/c^2}}^\infty = \frac{1}{\sqrt{c^2-a^2}}\log\frac{c\sqrt{(1+\lambda/c^2)/(c^2-a^2)}+1}{c\sqrt{(1+\lambda/c^2)/(c^2-a^2)}-1} \tag{B3}$$

$$= \frac{1}{\sqrt{c^2-a^2}}\log\frac{\sqrt{(c^2+\lambda)/(c^2-a^2)}+1}{\sqrt{(c^2+\lambda)/(c^2-a^2)}-1} = \frac{1}{\sqrt{c^2-a^2}}\log\frac{1+\sqrt{(c^2-a^2)/(c^2+\lambda)}}{1-\sqrt{(c^2-a^2)/(c^2+\lambda)}}.$$

Thus,

$$I_0 = \frac{1}{\sqrt{c^2-a^2}}\log\frac{1+\sqrt{(c^2-a^2)/(c^2+\lambda)}}{1-\sqrt{(c^2-a^2)/(c^2+\lambda)}}. \tag{B4}$$

Then, by using the expression (A5), we find

$$I_{xy} = \int_{\lambda}^{\infty} \frac{d\xi}{(a^2+\xi)^2\sqrt{c^2+\xi}} = \frac{2c}{(a^2-c^2)^2} \int_{\sqrt{1+\lambda/c^2}}^{\infty} \frac{dy}{\left(1+\frac{c^2}{a^2-c^2}y^2\right)^2} \tag{B5}$$

and for the stretched ellipsoid:

$$I_{xy} = \frac{2c}{(c^2-a^2)^2} \int_{\sqrt{1+\lambda/c^2}}^{\infty} \frac{dy}{\left(\frac{c^2}{a^2-c^2}y^2-1\right)^2} = \frac{2\alpha^2}{c^3\sqrt{\alpha}} \int_{\sqrt{1+\lambda/c^2}}^{\infty} \frac{d(\sqrt{\alpha}y)}{[(\underbrace{\sqrt{\alpha}y}_{x})^2-1]^2} = \frac{2}{(c^2-a^2)^{3/2}} \int_{\sqrt{(c^2+\lambda)/(c^2-a^2)}}^{\infty} \frac{dx}{(x^2-1)^2}. \tag{B6}$$

To calculate this integral, let's introduce the other integral:

$$I_1(\beta) = \int \frac{dx}{x^2-\beta^2} = \frac{1}{2\beta}\log\frac{x+\beta}{x-\beta}. \tag{B7}$$

Then

$$\begin{aligned} \int \frac{dx}{(x^2-\beta^2)^2} &= \frac{\partial}{\partial\beta^2}\int\frac{dx}{x^2-\beta^2} = \frac{1}{2\beta}\frac{\partial}{\partial\beta}\left(\frac{1}{2\beta}\log\frac{x+\beta}{x-\beta}\right) \\ &= -\frac{1}{4\beta^2}\log\frac{x+\beta}{x-\beta} + \frac{1}{4\beta^2}\frac{x-\beta}{x+\beta}\frac{(x-\beta)-(x+\beta)(-1)}{(x-\beta)^2} = \frac{1}{4\beta^2}\left(\log\frac{x-\beta}{x+\beta} - \frac{2x}{x^2-\beta^2}\right), \end{aligned} \tag{B8}$$

thus,

$$\begin{aligned} I_{xy} &= \frac{2}{(c^2-a^2)^{3/2}} \int_{\sqrt{(c^2+\lambda)/(c^2-a^2)}}^{\infty} \frac{dx}{(x^2-1)^2} = \frac{1}{2(c^2-a^2)^{3/2}}\left(\log\frac{x-1}{x+1} - \frac{2}{x^2-1}\right)\Bigg|_{\sqrt{(c^2+\lambda)/(c^2-a^2)}}^{\infty} \\ &= \frac{1}{(c^2-a^2)^{3/2}}\left(\frac{\sqrt{(c^2+\lambda)/(c^2-a^2)}}{(c^2+\lambda)/(c^2-a^2)-1} + \frac{1}{2}\log\frac{\sqrt{(c^2+\lambda)/(c^2-a^2)}-1}{\sqrt{(c^2+\lambda)/(c^2-a^2)}+1}\right) \end{aligned} \tag{B9}$$

or

$$I_{xy} = \frac{1}{(c^2-a^2)^{3/2}}\left(\frac{\sqrt{(c^2+\lambda)(c^2-a^2)}}{a^2+\lambda} - \frac{1}{2}\log\frac{1+\sqrt{(c^2-a^2)/(c^2+\lambda)}}{1-\sqrt{(c^2-a^2)/(c^2+\lambda)}}\right). \tag{B10}$$

Finally, using the intermediate result from the Appendix A, when calculated the integral I_z, we have:

$$\begin{aligned} I_z &= \int_{\lambda}^{\infty}\frac{d\xi}{(a^2+\xi)(c^2+\xi)^{3/2}} = \frac{2}{c(a^2-c^2)}\left[\frac{1}{y}\Bigg|_{\infty}^{\sqrt{1+\lambda/c^2}} - \frac{c^2}{a^2-c^2}\int_{\sqrt{1+\lambda/c^2}}^{\infty}\frac{dy}{1+\frac{c^2}{a^2-c^2}y^2}\right] = \frac{2}{(a^2-c^2)} \\ &\times\left[\frac{1}{\sqrt{c^2+\lambda}} - \frac{c}{c^2-a^2}\int_{\sqrt{1+\lambda/c^2}}^{\infty}\frac{dy}{1-\frac{c^2}{c^2-a^2}y^2}\right] = \frac{2}{(a^2-c^2)}\left[\frac{1}{\sqrt{c^2+\lambda}} + \frac{1}{\sqrt{c^2-a^2}}\right. \\ &\times\left.\int_{\sqrt{1+\lambda/c^2}}^{\infty}\frac{d\left(c/\sqrt{c^2-a^2}\,y\right)}{\left(c/\sqrt{c^2-a^2}\,y\right)^2-1}\right] = \left(\alpha = \frac{c^2}{c^2-a^2}\right) = -\frac{2}{(c^2-a^2)^{3/2}}\left[\sqrt{\frac{c^2-a^2}{c^2+\lambda}} + \int_{\sqrt{1+\lambda/c^2}}^{\infty}\frac{d(\sqrt{\alpha}y)}{(\sqrt{\alpha}y)^2-1}\right] \\ &= -\frac{2}{(c^2-a^2)^{3/2}}\left[\sqrt{\frac{c^2-a^2}{c^2+\lambda}} + \int_{\sqrt{1+\lambda/c^2}}^{\infty}\frac{d(\sqrt{\alpha}y)}{2}\left(\frac{1}{\sqrt{\alpha}y-1} - \frac{1}{\sqrt{\alpha}y+1}\right)\right] = -\frac{2}{(c^2-a^2)^{3/2}} \\ &\times\left[\sqrt{\frac{c^2-a^2}{c^2+\lambda}} + \frac{1}{2}\log\frac{\sqrt{\alpha}y-1}{\sqrt{\alpha}y+1}\Bigg|_{\sqrt{1+\lambda/c^2}}^{\infty}\right] = \frac{2}{(c^2-a^2)^{3/2}}\left[-\sqrt{\frac{c^2-a^2}{c^2+\lambda}} + \frac{1}{2}\log\frac{\sqrt{c\,(1+\lambda/c^2)/(c^2-a^2)}+1}{\sqrt{cc\,(1+\lambda/c^2)/(c^2-a^2)}-1}\right] \end{aligned} \tag{B11}$$

or

$$I_z = \frac{2}{(c^2-a^2)^{3/2}}\left[\frac{1}{2}\log\frac{1+\sqrt{(c^2-a^2)/(c^2+\lambda)}}{1-\sqrt{(c^2-a^2)/(c^2+\lambda)}} - \sqrt{\frac{c^2-a^2}{c^2+\lambda}}\right]. \quad \text{(B12)}$$

APPENDIX C

TRANSITION FROM A COMPRESSED ELLIPSOID TO SPHERE

Let's calculate the free term K_1 in the expression (48) for the potential outside the compressed ellipsoid in transition from ellipsoid to sphere and the coefficients K_2 and K_3 in the same equation. Assuming

$$c = a - \delta \quad \text{and} \quad \bar{\delta} = \delta/2a \ll 1 \quad \text{(C1)}$$

we can calculate K_1 as

$$K_1 = \frac{2\arccos\sqrt{\frac{(a-\delta)^2+\lambda}{a^2+\lambda}}}{\sqrt{a^2-(a-\delta)^2}} = \frac{2\arccos\sqrt{\frac{a^2+\lambda-2a\delta(1-\bar{\delta})}{a^2+\lambda}}}{\sqrt{2a\delta(1-\bar{\delta})}}$$
$$= \frac{2(1-\bar{\delta})^{-1/2}}{\sqrt{2a\delta}}\arccos\left(1-2a\delta\frac{1-\bar{\delta}}{a^2+\lambda}\right)^{1/2} \approx (\text{Zero's order over } \bar{\delta}) \approx \frac{2}{\sqrt{2a\delta}}\arccos\left(1-\frac{a\delta}{a^2+\lambda}\right). \quad \text{(C2)}$$

However, for small $\alpha \ll 1$, $\arccos(1-\alpha) = t$, and $1-\alpha = \cos t \approx 1 - t^2/2$, then

$$K_1 \approx \frac{2}{\sqrt{2a\delta}}\arccos\left(1-\frac{a\delta}{a^2+\lambda}\right) \approx \frac{2}{\sqrt{2a\delta}}\sqrt{\frac{2a\delta}{a^2+\lambda}} = \frac{2}{\sqrt{a^2+\lambda}}. \quad \text{(C3)}$$

Then, by using the substitutions and the relations, obtained during the calculation of K_1, we calculate K_2:

$$K_2 = \frac{\arccos\sqrt{\frac{(a-\delta)^2+\lambda}{a^2+\lambda}} - \frac{\sqrt{[a^2-(a-\delta)^2][(a-\delta)^2+\lambda]}}{a^2+\lambda}}{[a^2-(a-\delta)^2]^{3/2}} = \frac{1}{(2a\delta)^{3/2}(1-\bar{\delta})^{3/2}}$$
$$\times\left[\arccos\left(1-2a\delta\frac{1-\bar{\delta}}{a^2+\lambda}\right)^{1/2} - \frac{\sqrt{2a\delta(1-\bar{\delta})[a^2+\lambda-2a\delta(1-\bar{\delta})]}}{a^2+\lambda}\right] \quad \text{(C4)}$$
$$\approx \frac{(1-\bar{\delta})^{-3/2}}{(2a\delta)^{3/2}}\left\{\arccos\left[\underbrace{1-a\delta\frac{1-\bar{\delta}}{a^2+\lambda}-\frac{a^2\delta^2(1-\bar{\delta})^2}{2(a^2+\lambda)^2}}_{=-\alpha}\right]\sqrt{\frac{2a\delta}{a^2+\lambda}}(1-\bar{\delta})^{1/2}\left(1-2a\delta\frac{1-\bar{\delta}}{a^2+\lambda}\right)^{1/2}\right\}.$$

For small $\alpha \ll 1$, $\arccos(1-\alpha) = t$, but now we will keep the quadratic terms, so that $1-\alpha = \cos t \approx 1 - t^2/2 + t^4/24$, and

$$K_2 \approx \frac{(1-\bar{\delta})^{-3/2}}{(2a\delta)^{3/2}}\left\{\sqrt{2a\delta\frac{1-\bar{\delta}}{a^2+\lambda}+\frac{a^2\delta^2(1-\bar{\delta})^2}{(a^2+\lambda)^2}}\left[1+\frac{1}{12}\left(a\delta\frac{1-\bar{\delta}}{a^2+\lambda}+\frac{a^2\delta^2(1-\bar{\delta})^2}{2(a^2+\lambda)^2}\right)\right]\right.$$
$$\left.-\sqrt{\frac{2a\delta}{a^2+\lambda}}(1-\bar{\delta})^{1/2}\left(1-2a\delta\frac{1-\bar{\delta}}{a^2+\lambda}\right)^{1/2}\right\}$$
$$\approx \frac{(1-\bar{\delta})^{-3/2}}{(2a\delta)^{3/2}}\sqrt{\frac{2a\delta}{a^2+\lambda}}(1-\bar{\delta})^{1/2}\left\{\left[1+\frac{a\delta(1-\bar{\delta})}{2(a^2+\lambda)}\right]^{1/2}\left(1+\frac{a\delta}{12}\frac{1-\bar{\delta}}{a^2+\lambda}\right)-\left(1-2a\delta\frac{1-\bar{\delta}}{a^2+\lambda}\right)^{1/2}\right\} \quad \text{(C5)}$$
$$\approx \frac{1-\bar{\delta}}{2a\delta\sqrt{a^2+\lambda}}\left(1+\frac{a\delta}{4}\frac{1-\bar{\delta}}{a^2+\lambda}+\frac{a\delta}{12}\frac{1-\bar{\delta}}{a^2+\lambda}-1+a\delta\frac{1-\bar{\delta}}{a^2+\lambda}\right) \approx \frac{1-\bar{\delta}}{2a\delta\sqrt{a^2+\lambda}}\frac{4}{3}a\delta\frac{1-\bar{\delta}}{a^2+\lambda},$$

or

$$K_2 \approx \frac{2/3}{(a^2+\lambda)^{3/2}}, \tag{C6}$$

finally, by using all these results and methods:

$$K_3 = \frac{1}{[a^2 - c(a-\delta)^2]^{3/2}}\left(\arccos\sqrt{\frac{(a-\delta)^2+\lambda}{a^2+\lambda}} - \sqrt{\frac{a^2-(a-\delta)^2}{(a-\delta)^2+\lambda}}\right) = \frac{(1-\bar{\delta})^{-3/2}}{(2a\delta)^{3/2}}$$

$$\times\left[\arccos\left(1-2a\delta\frac{1-\bar{\delta}}{a^2+\lambda}\right)^{1/2} - \frac{\sqrt{2a\delta(1-\bar{\delta})}}{a^2+\lambda-2a\delta(1-\bar{\delta})}\right] \approx \frac{(1-\bar{\delta})^{-3/2}}{(2a\delta)^{3/2}}$$

$$\times\left\{\sqrt{2a\delta\frac{1-\bar{\delta}}{a^2+\lambda}+\frac{a^2\delta^2(1-\bar{\delta})^2}{(a^2+\lambda)^2}}\left[1+\frac{1}{12}\left(a\delta\frac{1-\bar{\delta}}{a^2+\lambda}+\frac{a^2\delta^2(1-\bar{\delta})^2}{2(a^2+\lambda)^2}\right)\right] - \sqrt{\frac{2a\delta}{a^2+\lambda}} - (1-\bar{\delta})^{1/2}\left(1-2a\delta\frac{1-\bar{\delta}}{a^2+\lambda}\right)^{-1/2}\right\} \tag{C7}$$

$$\approx \frac{(1-\bar{\delta})^{-3/2}}{(2a\delta)^{3/2}}\sqrt{\frac{2a\delta(1-\bar{\delta})}{a^2+\lambda}}\left[\left(1+\frac{a\delta(1-\bar{\delta})}{2(a^2+\lambda)}\right)^{1/2}\left(1+\frac{a\delta}{12}\frac{1-\bar{\delta}}{a^2+\lambda}\right) - \left(1+a\delta\frac{1-\bar{\delta}}{a^2+\lambda}\right)\right] \approx \frac{1-\bar{\delta}}{2a\delta\sqrt{a^2+\lambda}}$$

$$\times\left[1+\frac{a\delta}{4}\frac{1-\bar{\delta}}{a^2+\lambda}+\frac{a\delta}{12}\frac{1-\bar{\delta}}{a^2+\lambda}-1-a\delta\frac{1-\bar{\delta}}{a^2+\lambda}\right] = \frac{1-\bar{\delta}}{2a\delta\sqrt{a^2+\lambda}}\left(-\frac{2}{3}a\delta\frac{1-\bar{\delta}}{a^2+\lambda}\right) \approx \frac{-1/3}{(a^2+\lambda)^{3/2}}.$$

Or

$$K_3 = \frac{-1/3}{(a^2+\lambda)^{3/2}}. \tag{C8}$$

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SUMMARY

In this paper, we have provided the detailed analytical solution for the space charge forces inside the non-symmetrical three-dimensional ellipsoid-shaped bunch of the charged particles inside and outside of the bunch core. The presented results generalize and include the commonly used expressions for the space charge [5, 6] and can be readily used in the beam dynamics simulation codes.

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