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Conventional Quantum Statistics with a Probability Distribution Describing Quantum System States

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Abstract—The review of a new probability representation of quantum states is presented, where the states are described by conventional probability distribution functions. The invertible map of the probability distribution onto density operators in the Hilbert space is found using the introduced operators called a quantizer–dequantizer, which specify the invertible map of operators of quantum observables onto functions and a product of the operators onto an associative product (star product) of the functions. Examples of a quantum oscillator and a spin-1/2 particle are considered. The kinetic equations for probabilities, specifying the evolution of the states of a quantum system, which are equivalent to Schrödinger and von Neumann equations, are derived explicitly.

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INTRODUCTION

In quantum mechanics, quantum field theory [1], and quantum statistics, the physical system states are described by wavefunctions [2], density matrices [3, 4], as well as by vectors in a Hilbert space [5]. In classical statistical mechanics, the system states are described by probability distributions [6] in the phase space. The evolution of classical states is described by the Liouville equation for the probability density $f(q, p, t)$ in the phase space of the system or by the Boltzmann equation [7, 8]. The evolution of pure quantum states, identified with wavefunctions of systems with a Hamiltonian, is described by the Schrödinger equation, while the von Neumann equation describes the evolution of a density matrix of mixed states. The description of classical states by probability distribution functions and the description of quantum states by wavefunctions or by density matrices are very different. The description of quantum states intuitively requires the additional interpretation from the standpoint of equivalence with the classical picture of physical phenomena. In this connection, the other descriptions of quantum states were suggested, e.g., with the employment of the quasiprobability distributions in the phase space of systems: Wigner functions [9], Husimi–Kano functions [10, 11], and Glauber–Sudarshan functions [12, 13]. The description of states by a function in the phase state was proposed by Blokhintsev [14]. All these quantum state descriptions deal with functions in the formal phase space of the system, but these functions are not the probability distributions. It is not possible to describe a particle state by the joint coordinate and momentum distribution function due to the uncertainty relation [15–17]. Indeed, its quantum nature is such that it is impossible to specify (to measure) simultaneously both the particle coordinate and momentum. Therefore, a joint distribution function of these stochastic quantities does not exist. However, the uncertainty relation allows the state to be described by the probability distribution of a single quantity, e.g., only of a coordinate. This description was proposed [18, 19] based on the experimental approach to measuring the photon state, identified with the Wigner function [20], by using of a homodyne detector. The latter makes it possible to measure the photon optical tomogram—a distribution function of the quadrature component of a photon at the fixed phase of the local oscillator. It is known [21, 22] that, using the Radon transformation [23], from the optical tomogram, measured in the conducted experiments, the Wigner function, identified with a quantum state, is reconstructed. From the standpoint of the mechanical model of electromagnetic field oscillations (photon state), this corresponds to an oscillation. A state of a quantum oscillator in this case is specified by the probability distribution of only one random oscillator coordinate which is measured in the ensemble of reference frames in the phase space. Thus, there are no conflicts with the Heisenberg uncertainty relation, since a random momentum of the oscillator is not an argument of the probability distribution function, which only depends on a random coordinate and on values of parameters of the reference frame, in which this random coordinate is measured.

A goal of this work is to show that quantum system states can be specified by standard probability distributions for both systems with continuous variables (oscillator) and with discrete variables (electron spin). This implies that there exist the invertible maps of density operators onto probability distributions. These maps are linear and, therefore, linear equations of quantum mechanics (e.g., the von Neumann equation for the density operator [4]) are mapped to kinetic equations for probability distributions [18, 24], which specify the quantum system states in the probability representation of quantum mechanics. The problem of describing quantum states, connected with probability distributions, was discussed, e.g., in [25–27], see also the recent review [28] and references therein. Aspects relating to this problem were also touched upon in [29, 30].

STATES OF A CLASSICAL HARMONIC OSCILLATOR IN TOMOGRAPHIC PROBABILITY REPRESENTATION

To clarify the physical meaning of the probability distribution specifying a quantum system state, we consider an example of the classical oscillator state with the Hamiltonian $H_{cl} = p^2/2 + q^2/2$ in the presence of fluctuations within classical statistical mechanics [1]. The oscillator state at any instant is described on the phase space of the system by the joint probability distribution function $f(q,p,t) \geq 0$ which satisfies the normalization condition $\int f(q,p,t)dqdp = 1$. We introduce a new coordinate system in the phase space of the system, using the symplectic transformation of the coordinate $q \to X$ and momentum $p \to \mathcal{P}$, specified by the matrix of the form

$$\begin{pmatrix} X \\ \mathcal{P} \end{pmatrix} = \begin{pmatrix} s\cos\theta & s^{-1}\sin\theta \\ -s\sin\theta & s^{-1}cos\theta \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}. \quad (1)$$

The meaning of this transformation is that we first change the scale of spatial coordinates $q \to q' = sq$, $p \to p' = s^{-1}p$, and then rotate the abscissa and ordinate axes: $X = q'\cos\theta + p'\sin\theta$, $\mathcal{P} = -q'\sin\theta + p'\cos\theta$. We introduce the variables $\mu = s\cos\theta$, $\nu = s^{-1}\sin\theta$. Let us consider the following problem. If at the time t the oscillator state is given by the joint distribution function $f(q, p, t)$ of two random variables q and p, then which is the distribution function of a single random function $X = \mu q + \nu p$, denoted $w_k(X|\mu,\nu,t)$? It is easy to verify that this function is given by the Radon transformation [23] of the distribution function $f(q,p,t)$, that is,

$$w_k(X|\mu,\nu,t) = \int f(q,p,t)\delta(X - \mu q - \nu p)dqdp. \quad (2)$$

The function $w_k(X|\mu,\nu,t)$ is called the symplectic tomographic probability distribution or the symplectic tomogram. This transformation is invertible, and the probability distribution of two random quantities $f(q,p,t)$ is reconstructed if the symplectic tomogram is known, namely,

$$f(q,p,t) = \frac{1}{4\pi^2}\int w_k(X|\mu,\nu,t) \times \exp[i(X - \mu q - \nu p)]\,dXd\mu d\nu. \quad (3)$$

It is evident that symplectic tomogram (2) is nonnegative and normalized: $\int w_k(X|\mu,\nu,t)dX = 1$. The variable X has the meaning of a particle coordinate, measured in the reference frame on the phase space of the particle with coordinate axes rotated by the angle θ; in this case before the rotation, the scale transformation of the coordinate $q \to sq$ and the velocity (momentum) $p \to s^{-1}p$ is performed. The transformation under study can be interpreted as a transformation in the space of coordinates q and velocities $\dot{q}$, at which the coordinate scale changes, while the velocity scale is preserved. The symplectic tomogram $w_k(X|\mu,\nu,t)$ of the classical particle is a conditional probability distribution of the coordinate X, when specified, the parameters μ and ν of the reference frame are determined in the phase space of the particle where this coordinate is measured. By the Bayes formula, the joint probability distribution function $w_k(X,\mu,\nu,t)$ of three random variables X, μ, and ν, which both define the coordinate X and the random parameters μ and ν characterizing the reference frame in the phase space of the particle, can be introduced. This function can be shown by the formula

$$w_k(X,\mu,\nu,t) = w_k(X|\mu,\nu,t)P(\mu,\nu), \quad (4)$$

where $P(\mu,\nu) \geq 0$. This function is the distribution function of two random quantities μ and ν, which is normalized by the condition

$$\int \mathcal{P}(\mu,\nu)d\mu d\nu = 1. \quad (5)$$

In particular, it can be specified by the normal probability distribution

$$\mathcal{P}_G(\mu,\nu) = \pi^{-1}\exp(-\mu^2 - \nu^2). \quad (6)$$

The distribution function $f(q,p,t)$ is reconstructed by using (4):

$$f(q,p,t) = \frac{1}{4\pi^2}\int w_K(X,\mu,\nu,t)\mathcal{P}^{-1}(\mu,\nu) \times \exp(i(X-\mu q-\nu p))dXd\mu d\nu. \quad (7)$$

For a classical particle, the following distribution functions are admissible:

$$f(q,p,t) = \delta(q-q_0(t))\delta(p-p_0(t)), \quad (8)$$

where $q_0(t)$ and $p_0(t)$ describe a particle trajectory. The tomogram of the particle state in the classical mechanics with distribution function (8) is written as

$$w_k(X|\mu,\nu,t) = \delta(X-\mu q_0(t)-\nu p_0(t)). \quad (9)$$

In (9), $q_0(t)$ and $p_o(t)$ describe a trajectory of the particle in its phase space. For instance, in the case of the mentioned classical oscillator, the symplectic tomogram

$$w_k(X|\mu,\nu,t) = \delta[X-\mu(A\cos t+B\sin t) - \nu(B\cos t - A\sin t)] \quad (10)$$

is admissible with arbitrary initial values of the coordinate A and momentum B. For a quantum oscillator, due to the Heisenberg uncertainty relation [15], the distribution function $f(q,p,t)$ does not exist. Therefore, for the quantum oscillator, the tomogram of form (10), which is connected with distribution (8) and violates the uncertainty relation, does not exist. However, distribution (10) depends only on the coordinate X and is independent of the momentum $\mathcal{P}$; therefore, the existence of tomograms, which describe the states not violating the uncertainty relation, is possible. Below, these cases are considered by using the quantization procedure based on the method of star product of functions. Using a particular case of the tomogram $w_k(X|\mu,\nu,t)$ with $\mu=\cos\theta$, $\nu=\sin\theta$, we derive the probability distribution $w_k(X|\theta,t)$, called the optical tomogram. It specifies also a symplectic tomogram due to the property of the Dirac delta function $\delta(\lambda y) = |\lambda|^{-1}\delta(y)$, which leads to the formula

$$w_k(X|\mu,\nu,t) = \frac{1}{\sqrt{\mu^2+\nu^2}}w_k\left(X|\theta=\arctan\frac{\nu}{\mu},t\right). \quad (11)$$

The optical tomogram of the classical oscillator state, defined by Eq. (10), has the form of probability distribution

$$w_k(X|\theta,t) = \delta\{X-\cos\theta(A\cos t+B\sin t) - \sin\theta(B\cos t - A\sin t)\}. \quad (12)$$

QUANTIZATION USING THE STAR-PRODUCT FORMALISM EMPLOYING THE QUANTIZER AND DEQUANTIZER

Let us pass to a description of the quantum system states. All possible quantum state representations can be formulated using the quantization procedure, understandable as a rule of the invertible mapping of operators $\hat{A}$, acting on a Hilbert space $\mathcal{H}$, onto the functions $f_A(\chi)$, called symbols of operators $\hat{A}$, depending on the set of discrete or continuous variables $\chi=(\chi_1,\chi_2,\ldots,\chi_N)$, where some numbers $\chi_k, k=1,2,\ldots,N$ run over the continuous set of values, while certain numbers run over the discrete set of values. The invertible mapping is given by two relations [31]

$$\hat{A}\to f_A(\chi) = \mathrm{Tr}\left(\hat{A}\hat{U}(\chi)\right), \quad (13)$$

$$f_A(\chi)\to\hat{A} = \int f_A(\chi)\hat{D}(\chi)d\chi. \quad (14)$$

The operator $\hat{U}(\chi)$ is called a dequantizer. The operator $\hat{D}(\chi)$ is called a quantizer. If a part of variables χ_k is discrete, the relevant integral in (14) is replaced by the sum over these variables. The dequantizer and quantizer $\hat{U}(\chi)$, $\hat{D}(\chi)$ should satisfy the condition that for all operators $\hat{A}$, acting on $\mathcal{H}$, this equality holds

$$\int \mathrm{Tr}\hat{U}(\chi)\hat{D}(\chi')\mathrm{Tr}\left(\hat{U}(\chi')\hat{A}\right)d\chi' = \mathrm{Tr}\left(\hat{A}\hat{U}(\chi)\right). \quad (15)$$

In the particular case, the following relation can be satisfied: $\mathrm{Tr}\left(\hat{U}(\chi)\hat{D}(\chi')\right) = \delta(\chi-\chi')$. With the fulfillment of (13), (14), and (15), the symbol of the operator product $f_{AB}(\chi)$, where $f_{AB}(\chi) = \mathrm{Tr}\left(\hat{A}\hat{B}\hat{U}(\chi)\right)$, is specified by the star product of symbols of the operators $\hat{A}$ and $\hat{B}$, i.e.,

$$f_{AB}(\chi) = (f_A\star f_B)(\chi) = \int f_A(\chi_1)f_B(\chi_2)K(\chi_1,\chi_2,\chi)d\chi_1 d\chi_2. \quad (16)$$

The operator product is associative, i.e., $(\hat{A}\hat{B})\hat{C} = \hat{A}(\hat{B}\hat{C})$, and the star product of functions $f_A(\chi)$, $f_B(\chi)$ is also associative: $((f_A\star f_B)\star f_C)(\chi) = (f_A\star(f_B\star f_C))(\chi)$. Since the product of the operators $\hat{A}$ and $\hat{B}$ in a general case is noncommutative, the star product of their symbols in general is also noncommutative. Substituting (13) and (14) to (16), we derive the kernel of the star product of functions, expressed through the quantizer and dequantizer:

$$\mathcal{K}(\chi_1,\chi_2,\chi_3) = \mathrm{Tr}\left(\hat{D}(\chi_1)\hat{D}(\chi_2)\hat{U}(\chi_3)\right). \quad (17)$$

If we take two sets of operators $\hat{U}_1(\chi)$, $\hat{D}_1(\chi)$ and $\hat{U}_2(\xi)$, $\hat{D}_2(\xi)$, which are quantizers and dequantizers, satisfying (13), (14), and (15), then symbols of the operators—functions $f_A^{(1)}(\chi)$ and $f_A^{(2)}(\xi)$, where $\chi = (\chi_1,\chi_2,\dots,\chi_N)$, $\xi = (\xi_1,\xi_2,\dots,\xi_M)$,—are related by the integral relationship

$$f_A^{(1)}(\chi) = \int K^{(1,2)}(\chi,\xi) f_A^{(2)}(\xi)d\xi. \quad (18)$$

The kernel of the integral transformation of symbols of the operators $\hat{A}$ is given by the expression

$$\mathcal{K}^{(1,2)}(\chi,\xi) = \mathrm{Tr}\left(\hat{U}_1(\chi)\hat{D}_2(\xi)\right). \quad (19)$$

All available representations of the density operators of quantum states and the observables specified by the Hermitian operators, acting on a Hilbert space $\mathcal{H}$, are described by their pairs of quantizers and dequantizers [31, 32].

SYMPLECTIC TOMOGRAPHIC PROBABILITY DISTRIBUTION OF QUANTUM STATES BY EXAMPLE OF AN OSCILLATOR

As shown in [21, 22], the Radon transformation [23] of the Wigner function of the photon state (optical tomogram $w(X\mid\theta)$), measured by the homodyne detection [20], is the probability distribution function of the random quantity X, called the photon quadrature. The optical tomogram depends also on the angular variable θ, called the phase of the local oscillator. The Radon transformation of the Wigner function specifies the expression for the optical tomogram [21, 22]. In this case, if the pure state $|\psi\rangle$ with the wavefunction $\psi(y)$ is considered, then the optical tomogram is expressed through the wavefunction in the following way:

$$w_\psi(X|\theta) = \frac{1}{2\pi|\sin\theta|} \times \left|\int \psi(y)\exp\left(\frac{i\cot\theta y^2}{2} - \frac{iXy}{\sin\theta}\right)dy\right|^2. \quad (20)$$

Tomogram (20) is nonnegative and normalized with all values of the phase θ of the local oscillator. Since the density operator of the mixed state is the convex sum of density operators of pure states, the conditions of nonnegativity and normalization for the tomogram of the mixed state are also satisfied. Thus, an optical tomogram is the conditional probability of the X quadrature with the given parameter θ. The Radon transformation is invertible; therefore, the knowing of the optical tomogram allowed the Wigner function of the photon, identified with its quantum state, to be reconstructed. In [22], a tomogram is considered as a technical method for finding a quantum state which is identified with the Wigner function. In [18], a notion of the symplectic probability distribution was introduced and the quantum states are proposed to be identified with this distribution as well as with the optical tomographic distribution, regarding them as primary objects.

We define a dequantizer by the expression $\hat{U}(\chi) \equiv \hat{U}(X,\mu,\nu)$, where the real variables $0 < X,\mu,\nu < \infty$ specify the operator

$$\hat{U}(X,\mu,\nu) = \delta(X\hat{1} - \mu\hat{q} - \nu\hat{p}). \quad (21)$$

The density operator symbol $\hat{\rho}$, called the symplectic tomogram, e.g., of oscillator state, is given by

$$w(X|\mu,\nu) = \mathrm{Tr}\hat{\rho}\delta(X\hat{1} - \mu\hat{q} - \nu\hat{p}). \quad (22)$$

We define the quantizer $\hat{D}(\chi) \equiv \hat{D}(X,\mu,\nu)$ by the expression

$$\hat{D}(X,\mu,\nu) = \frac{1}{2\pi}\exp\left[i(X\hat{1} - \mu\hat{q} - \nu\hat{p})\right]. \quad (23)$$

From delta function properties, the normalization relation follows. Since $\mathrm{Tr}\hat{\rho} = 1$,

$$\int w(X|\mu,\nu)dX = 1 \quad (24)$$

for any values of the parameters μ and ν. An optical tomogram is a particular case of the symplectic tomogram with the parameter values $\mu = \cos\theta$, $\nu = \sin\theta$. For pure states $|\psi\rangle$, as in the case of optical tomogram, the symplectic tomogram can be expressed through the wavefunction $\psi(y)$ [34]

$$w_\psi(X|\mu,\nu) = \frac{1}{2\pi|\nu|}\left|\int \psi(y)\exp\left(\frac{i\mu y^2}{2\nu} - \frac{iXy}{\nu}\right)dy\right|^2. \quad (25)$$

From this expression, the nonnegativity follows, while from properties of the Dirac delta function the normalizability of symplectic tomogram results. Similar properties are also valid for the convex sum of density operators; consequently, the symplectic tomogram $w(X\mid\mu,\nu)$ of any mixed state is also nonnegative and normalized. Although the symplectic tomogram depends on three variables (X,μ,ν), while the optical tomogram is dependent of two variables (X,θ), they are expressed, as in the classical case, through each other (see Eq. (11)).

Let us consider an example of a harmonic oscillator with the Hamiltonian $\hat{H} = p^2/2 + \hat{q}^2/2 = \hat{a}^\dagger\hat{a} + 1/2$. For the coherent state $|\alpha\rangle$, which is the proper normalized state of the annihilation operator $\hat{a}$, i.e., $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ with the complex eigenvalue $\alpha = \mathrm{Re}\alpha + i\mathrm{Im}\alpha$, we derive the symplectic tomogram in the form of the normal distribution of the random variable X, which depends on μ and ν

$$w_\alpha(X|\mu,\nu) = [\pi(\mu^2+\nu^2)]^{-\frac{1}{2}}\exp\left[-\frac{(X-\bar{X})^2}{\mu^2+\nu^2}\right], \quad (26)$$

with a mean value $\bar{X} = \mu\sqrt{2}\mathrm{Re}\alpha + \nu\sqrt{2}\mathrm{Im}\alpha$ and variance $\sigma = (\mu^2+\nu^2)/2$. For the Fock state $|n\rangle$, the tomogram has a form of the distribution

$$w_n(X|\mu,\nu) = [\pi(\mu^2+\nu^2)]^{-1/2}[2^n n!]^{-1} \times \exp\left[-\frac{X^2}{\mu^2+\nu^2}\right] H_n^2\left(\frac{X}{\mu^2+\nu^2}\right), \quad (27)$$

where H_n is the Hermitian polynomial. The optical tomogram of the coherent state $|\alpha\rangle$ is derived from (27) by substitution of $\mu = \cos\theta$, $\nu = \sin\theta$ and has the form of a normal distribution. The optical tomogram of the Fock state $|n\rangle$ is independent of the phase of the local oscillator and specified by the probability distribution of the random quantity X, that is, $w_n(X|\theta) = \pi^{-1/2}(2^n n!)^{-1}\exp(-X^2)H_n^2(X)$.

EVOLUTION OF QUANTUM STATES IN PROBABILITY REPRESENTATION

The density operator $\hat{\rho}(t)$ of quantum state in describing systems with the Hamiltonian $\hat{H}$ obeys the von Neumann equation

$$\frac{\partial\hat{\rho}(t)}{\partial t} + i[\hat{H},\hat{\rho}(t)] = 0. \quad (28)$$

In this case, the density operator $\hat{\rho}(t)$ can be expressed through the density operator $\hat{\rho}(0)$ and the unitary evolution operator $\hat{u}(t)$, specifying the vector $|\psi(t)\rangle = \hat{u}(t)|\psi(0)\rangle$ in the following form:

$$\hat{\rho}(t) = \hat{u}(t)\rho(0)\hat{u}^\dagger(t). \quad (29)$$

In this case, the operators of coordinate $\hat{q}_H(t)$ and momentum $\hat{p}_H(t)$ in the Heisenberg representation can be also expressed through the operators of coordinate $\hat{q}$ and momentum $\hat{p}$ in the Schrödinger representation: $\hat{q}_H(t) = \hat{u}^\dagger(t)\hat{q}\hat{u}(t)$, $\hat{p}_H(t) = \hat{u}^\dagger(t)\hat{p}\hat{u}(t)$. The evolution operation $\hat{u}(t)$ satisfies the Schrödinger equation

$$i\frac{\partial\hat{u}(t)}{\partial t} = \hat{H}\hat{u}(t), \quad \hat{u}(0) = \hat{1}. \quad (30)$$

Using definition (22) for the symplectic tomogram of the oscillator, we derive a value of the symplectic tomographic probability distribution at time t in the form

$$w(X|\mu,\nu,t) = \mathrm{Tr}\left[\hat{u}(t)\hat{\rho}(0)\hat{u}^\dagger(t)\delta(X\hat{1}-\mu\hat{q}-\nu\hat{p})\right]. \quad (31)$$

Using the properties of the operation for taking the trace of the operator product in (31), we derive the expression for probability distribution

$$w(X|\mu,\nu,t) = \mathrm{Tr}[\hat{\rho}(0)[\hat{u}^\dagger(t)\delta(X\hat{1}-\mu\hat{q}-\nu\hat{p})\hat{u}(t)]]. \quad (32)$$

Thus, this probability distribution can be expressed using the coordinate and momentum operators in the Heisenberg representation:

$$w(X|\mu,\nu,t) = \mathrm{Tr}\left[\hat{\rho}(0)\delta(X\hat{1}-\mu\hat{q}_H(t)-\nu\hat{p}_H(t))\right]. \quad (33)$$

For $t = 0$ we have the tomogram $w(X|\mu,\nu,t=0)$, where $\hat{q}_H(0) = \hat{q}$, $\hat{p}_H(0) = \hat{p}$. For the harmonic oscillator ($m = \omega = 1$),

$$\hat{q}_H(t) = \hat{q}\cos t + \hat{p}\sin t, \quad \hat{p}_H(t) = -\hat{q}\sin t + \hat{p}\cos t. \quad (34)$$

Therefore, the tomographic probability distribution, which is equal for $t = 0$ to $w_{in}(X|\mu,\nu)$, evolves as follows:

$$w(X|\mu,\nu,t) = w_{in}(X|\mu_H(t),\nu_H(t)). \quad (35)$$

Due to Eq. (34), we derive the evolution of oscillator state tomograms, knowing initial values of distribution function for $t = 0$, that is,

$$w(X|\mu,\nu,t) = w_{in}(X|\mu_H(t) = \mu\cos t - \nu\sin t,\ \nu_H(t) = \mu\sin t + \nu\cos t). \quad (36)$$

For example, the tomographic probability distribution $w_n(X|\mu,\nu,t)$ of the Fock stationary state $|n\rangle$ is given by (27). Relation (27) corresponds to the condition

$$\hat{u}^\dagger(t)|n\rangle\langle n|\hat{u}(t) = |n\rangle\langle n|,\ n = 0,1,2,\ldots, \quad (37)$$

which is satisfied, since the oscillator evolution operator $\hat{u}(t) = \exp\left[-it(\hat{a}^\dagger\hat{a}+1/2\hat{1})\right]$ commutes with the operator $|n\rangle\langle n|$. The optical tomogram, given at the initial moment as the function $w_{in}(X|\theta)$ turns at time t into the function of the form

$$w(X|t) = w_{in}(X|(\theta+t)). \quad (38)$$

The Heisenberg uncertainty relations impose the integral conditions on the tomogram of quantum states. For the optical tomogram $w(X|\theta)$, we have the inequality

$$\left\{\left[\int X^2 w(X|\theta)dX\right] - \left[\int Xw(X|\theta)dX\right]^2\right\} \times \left\{\left[\int X^2 w\left(X|\theta+\frac{\pi}{2}\right)dX\right] - \left[\int Xw(X|\theta)dX\right]^2\right\} \geq \frac{1}{4}. \quad (39)$$

This condition is not satisfied by tomogram (9) of the classical oscillator state, which violates the Heisenberg uncertainty relations. Since in the experiment [20] for the determination of the Wigner function of photon state, the optical tomogram $w(X|\theta)$ is measured, inequality (39) can be verified immediately.

EVOLUTION EQUATION FOR THE DENSITY OPERATOR SYMBOL IN QUANTIZATION SCHEMES WITH DIFFERENT DEQUANTIZERS AND QUANTIZERS

In this section, we consider the equation of quantum state evolution, using the quantizer and dequantizer [24], and the kinetic equation for probability distribution, which specifies a quantum state. The von Neumann equation for the unitary evolution of the density operator is given by the Hamiltonian $\hat{H}(t)$ acting on the Hilbert space $\mathcal{H}$, and it has the form of (28). We multiply this operator equation by the dequantizer $\hat{U}(\chi)$ and take the trace of the derived operator relation. For the density operator symbol $f_\rho(\chi,t)$, we derive a linear integral equation of the form

$$\frac{\partial}{\partial t}f_\rho(\chi,t)+\int \mathcal{K}_H(\chi,\chi',t)f_\rho(\chi',t)d\chi' = 0. \quad (40)$$

The kernel of this equation is given by the Hamiltonian, as well as by the quantizer $\hat{D}(\chi')$ and dequantizer $\hat{U}(\chi)$, and is written as

$$\mathcal{K}_H(\chi,\chi',t) = i\mathrm{Tr}\left([\hat{U}(\chi),\hat{H}]\hat{D}(\chi')\right),$$
$$\text{or } \mathcal{K}_H(\chi,\chi',t) = i\mathrm{Tr}\left([\hat{D}(\chi'),\hat{U}(\chi)]\hat{H}\right). \quad (41)$$

If the probability representation of quantum states is used, then Eq. (40) is the kinetic equation for the probability distribution $f_\rho(\chi,t)$, specifying a quantum state. As an example, for the symplectic tomographic probability distribution, the kernel of the integral kinetic equation, which describes the evolution of quantum state given by the tomogram $w(X|\mu,\nu,t) = f_\rho(\chi,t)$, where $\chi = (X,\mu,\nu)$, is specified by the expression

$$\mathcal{K}_H(X,\mu,\nu,t,X',\mu',\nu',t) = \frac{i}{2\pi}\mathrm{Tr}\{[\delta(X\hat{1}-\mu\hat{q} - \nu\hat{p}),\hat{H}(t)]\exp(i(X'\hat{1}-\mu'\hat{q}-\nu'\hat{p}))\}$$
$$= \frac{i}{2\pi}\mathrm{Tr}\{\hat{H}(t)[\exp\left(i(X'\hat{1}-\mu'\hat{q}-\nu'\hat{p})\right), \delta(X\hat{1}-\mu\hat{q}-\nu\hat{p})]\}. \quad (42)$$

Thus, the kinetic equation for the quantum state evolution—probability distribution $w(X|\mu,\nu,t)$—is given by

$$\frac{\partial}{\partial t}w(X|\mu,\nu,t)+\int \mathcal{K}_H(X,\mu,\nu,X',\mu',\nu',t) \times w(X'|\mu',\nu',t)dX'd\mu'd\nu' = 0 \quad (43)$$

If $\hat{H} = \frac{\hat{q}^2}{2}+V(\hat{q})$, the kernel of the integral operator, which specifies the kinetic equation of evolution for the probability distribution $w(X|\mu,\nu,t)$, is given by the relation

$$\mathcal{K}(X,\mu,\nu,X',\mu',\nu',t) = \frac{i}{2\pi}\int dxdx' \times\left[-\frac{1}{2}\frac{d^2}{dx^2}\delta(x-x')+V(x',t)\delta(x-x')\right] \times \mathbb{O}_{x'x}(X,\mu,\nu,X',\mu',\nu',t), \quad (44)$$
$$\mathbb{O}_{x'x}(X,\mu,\nu,X',\mu',\nu',t) = \left\langle x'\middle|[\exp\left(iX'-i\mu'\hat{q}-i\nu'\hat{p}\right),\delta(X\hat{1}-\mu\hat{q}-\nu\hat{p})]\middle|x\right\rangle$$

Let us express the probability $P_{(1)}^{(2)} = \mathrm{Tr}(\hat{\rho}_1\hat{\rho}_2)$, derived according to the Born rule, through the quantizer and dequantizer. We derive

$$P_{(1)}^{(2)} = \int f_{\rho_1}(\chi_1)f_{\rho_2}(\chi_2) \times \mathrm{Tr}\left(\hat{D}(\chi_1)\hat{D}(\chi_2)\right)d\chi_1 d\chi_2. \quad (45)$$

In case of the symplectic probability quantum state representation, we derive the probability as

$$P_{(1)}^{(2)} = \frac{1}{2\pi}\int w_1(X|\mu,\nu)w_2 \times (Y|-\mu,-\nu)e^{i(X+Y)}dXdYd\mu d\nu. \quad (46)$$

For all pure states, $\hat{\rho}_\psi^2 = \hat{\rho}_\psi$ and the tomogram satisfies the condition

$$\frac{1}{2\pi}\int w_\psi(X|\mu,\nu)w_\psi \times (Y|-\mu,-\nu)e^{i(X+Y)}dXdYd\mu d\nu = 1. \quad (47)$$

With the given Hamiltonian $\hat{H}$, the stationary states $\hat{\rho}_E$, corresponding to the given energy E, satisfy the condition $\frac{\partial\hat{\rho}_E}{\partial t} = 0$ and $\hat{H}\hat{\rho}_E = \hat{\rho}_E\hat{H}$. The symbol $f_{\hat{\rho}_E}(\chi)$ of the operator $\hat{\rho}_E$ satisfies the integral matrix equation

$$\int f_{\rho_E}(\chi)\left[\hat{H},\hat{D}(\chi)\right]d\chi = 0. \quad (48)$$

Equation (48), written through the symbols of Hamiltonian, density operator, and kernel of star product of operator symbols is written as

$$\int f_H(\chi)f_{\rho_E}(\chi_1)f_{\rho_E}(\chi_2) \times \mathrm{Tr}\left(\left[\hat{D}(\chi_1),\hat{D}(\chi_2)\right]\hat{U}(\chi)\right) = 0. \quad (49)$$

In case of the symplectic probability representation, we derive the equation for the tomogram $w_E(X|\mu,\nu)$ of the state with the specified energy

$$\int w_E(X,\mu,\nu) \times\left[\hat{H},\exp\left(i(X\hat{1}-\mu\hat{q}-\nu\hat{p})\right)\right]d\chi d\mu d\nu = 0. \quad (50)$$

For the harmonic oscillator, $\hat{H} = \hat{a}^\dagger\hat{a} + 1/2)$, tomograms (27) satisfy Eq. (50). Equation (50) is reduced to the form

$$\begin{aligned}&\int w_E(X_1|\mu_1,\nu_1)w_H(X_2|\mu_2,\nu_2)\\&\times[\mathcal{K}(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2)\\&-\mathcal{K}(X_2,\mu_2,\nu_2,X_1,\mu_1,\nu_1)]\\&\times dX_1dX_2d\mu_1d\mu_2d\nu_1d\nu_2 = 0,\end{aligned} \quad (51)$$

where the kernel of the star product of tomographic symbols of operators is found in [31].

PROBABILITY REPRESENTATION OF QUBIT STATES (SPIN 1/2)

Let us consider the probability representation of system states with discrete variables using the example of spin 1/2. The Hilbert space is two-dimensional, and for matrices of operators, acting on this space, we consider four matrices (dequantizers) [35]

$$u(1) = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix},\ u(2) = \frac{1}{2}\begin{pmatrix}1 & -i\\i & 1\end{pmatrix},\quad u(3) = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\ u(4) = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \quad (52)$$

and four matrices (quantizers)

$$D(1) = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix},\ D(2) = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix},\quad D(3) = \begin{pmatrix}1 & \frac{-1+i}{2}\\\frac{-1-i}{2} & 0\end{pmatrix},\ D(4) = \begin{pmatrix}0 & \frac{-1+i}{2}\\\frac{-1-i}{2} & 1\end{pmatrix}. \quad (53)$$

It can be verified that the density matrix $\rho = \begin{pmatrix}\rho_{11} & \rho_{12}\\\rho_{21} & \rho_{22}\end{pmatrix}$, such that $\rho = \rho^\dagger$, $\mathrm{Tr}\rho = 1$, $0 \le \rho_{11},\rho_{22} \le 1$, can be presented in the form [36–39]

$$\rho = \begin{pmatrix}p_3 & p_1 - 1/2 - i(p_2 - 1/2)\\p_1 - 1/2 + i(p_2 - 1/2) & 1 - p_3\end{pmatrix}. \quad (54)$$

Here $p_j = \mathrm{Tr}\rho(u(j))$, $j = 1,2,3$, $1 - p_3 = \mathrm{Tr}\rho(u(4))$. Moreover, $\rho = \sum_{j=1}^4(\mathrm{Tr}\rho u(j))D(j)$. The physical meaning of parameters p_1, p_2, and p_3 is that in the state with the density matrix ρ, according to the Born rule, they are probabilities of projections of the spin $m = +1/2$ onto the directions x, y, and z, respectively. Indeed, the matrices $u(1)$, $u(2)$, and $u(3)$ are the density matrices of pure states with the state vectors

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix},\ |\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix},\ |\psi_3\rangle = \begin{pmatrix}1\\0\end{pmatrix}. \quad (55)$$

These states are the eigenvectors of operators of spin projections onto the directions x, y, and z, which are specified by Pauli matrices $\sigma_x = D(1)$, $\sigma_y = D(2)$, $\sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ with eigenvalues $m = +1/2$. In accordance with the Born rule, numbers p_1, p_2, p_3 have the physical meaning of the relevant probabilities. Formula (54) is verified immediately. Therefore, the operators $U(j)$, $D(j)$, $j = 1,2,3,4$, are a dequantizer and a quantizer for any matrix observable. Thus, a state of the spin-1/2 particle can be completely specified by three probability distributions $(p_1, 1 - p_1)$, $(p_2, 1 - p_2)$, and $(p_3, 1 - p_3)$ of projections of the spin $m = \pm 1/2$ onto the directions x, y, and z, respectively. In the pure state with the density matrix ρ_ψ, such that $\rho_\psi^2 = \rho_\psi$, the probabilities p_1, p_2, and p_3 satisfy the condition

$$\sum_{j=1}^{3}\left(p_j - \frac{1}{2}\right)^2 = \frac{1}{4}. \quad (56)$$

The evolution of the density matrix ρ, which is described by the von Neumann equation

$$\begin{pmatrix}\dot\rho_{11} & \dot\rho_{12}\\\dot\rho_{21} & \dot\rho_{22}\end{pmatrix} + i\left[\begin{pmatrix}H_{11} & H_{12}\\H_{21} & H_{22}\end{pmatrix},\begin{pmatrix}\rho_{11} & \rho_{12}\\\rho_{21} & \rho_{22}\end{pmatrix}\right] = 0, \quad (57)$$

with using (54), leads to the linear kinetic equation for the probabilities $p_1(t)$, $p_2(t)$, and $p_3(t)$, regarded as components of the three-dimensional vector $\mathbf{p}(t)$, namely, $\frac{d\mathbf{p}(t)}{dt} = M\mathbf{p}(t) + \Gamma$. Here the matrix M and vector Γ are written as follows:

$$M = \begin{pmatrix}0 & (H_{22} - H_{11}) & -2\mathrm{Im}H_{12}\\H_{11} - H_{22} & 0 & -2\mathrm{Re}H_{12}\\2\mathrm{Im}H_{12} & 2\mathrm{Re}H_{12} & 0\end{pmatrix},\quad \Gamma = \begin{pmatrix}\frac{H_{11} - H_{22}}{2} + \mathrm{Im}H_{12}\\\frac{H_{22} - H_{11}}{2} + \mathrm{Re}H_{12}\\-\mathrm{Im}H_{12} - \mathrm{Re}H_{12}\end{pmatrix}. \quad (58)$$

Thus, the von Neumann quantum equation of evolution is equivalent to the kinetic equation of probabilities of dichotomic classically similar observables.

CONCLUSIONS

We emphasize the main results of this work. A review of the new probability representation of quantum mechanics is given. In this representation, the quantum system state is described by the probability distributions which obey kinetic equations. This representation is completely equivalent to other representations with the use of state vectors, belonging to a Hilbert space, and density operators, acting on the Hilbert space. This equivalence is associated with the existence of mappings of density operators onto probability distributions using the operators (quantizers and dequantizers, comparing the operators to their symbols—functions. There are quantizers and dequantizers which map the density operators onto quasi-distributions of the type of a Wigner function. However, as shown in this work, there are maps matching conventional probability distribution functions to the density operators. The similar construction can be expanded to the quantum statistics and quantum field theory developed in [1, 7, 8].[1]

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