

Landau–Khalatnikov–Fradkin Transformation and Hatted ζ -Values

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Abstract—We show an exact formula obtained in [1], which relates hatted and standard ζ -values to all orders of perturbation theory. The formula is based on the Landau–Khalatnikov–Fradkin (LKF) transformation between the massless propagators of charged particles interacting with gauge fields, in two different gauges.

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1. INTRODUCTION

Consider the multi-loop structure of propagator-type functions (p -functions¹). About three decades ago, it was noticed that all contributions proportional to $\zeta_4 = \pi^4/90$ mysteriously cancel out in the Adler function at three-loops [3]. Two decades later, it was shown that the four-loop contribution is also π -free and that a similar fact holds for the coefficient function of the Bjorken sum rule [4]. There is by now mounting evidence, see, e.g., [5–8], that various massless Euclidean physical quantities demonstrate striking regularities in terms proportional to even ζ -function values, ζ_{2n} , e.g., to π^{2n} with n being a positive integer². Such puzzling facts have recently given rise to the “no- π theorem”. The latter is based on the observation [10, 11] that the ε -dependent transformation of the ζ -values:

$$\hat{\zeta}_3 \equiv \zeta_3 + \frac{3\varepsilon}{2}\zeta_4 - \frac{5\varepsilon^3}{2}\zeta_6, \quad \hat{\zeta}_5 \equiv \zeta_5 + \frac{5\varepsilon}{2}\zeta_6, \quad \hat{\zeta}_7 \equiv \zeta_7, \quad (1)$$

eliminates even zetas from the expansion of four-loop p -integrals. A generalization of (1) to 5-, 6- and 7-loops is available in [12–14]. The results (1) and their extensions in [13, 14] give a possibility to predict the terms $\sim \pi^{2n}$ in higher orders of perturbation theory

(see their evaluation in [12–14]). Note that the results [12–14] also contain multi-zeta values the consideration of which is beyond the scope of the present study.

Remarkably, in [1], an all order generalization of (1) could be achieved in a rather unexpected way: with the help of the LKF transformation [15]. The latter elegantly relates the QED fermion propagator in two different ξ -gauges (and similarly for the fermion-photon vertex). Its most important applications (see [1] and references therein) are related to the study of the gauge covariance of QED Schwinger–Dyson equations and their solutions. Other applications [16] are focused on estimating large orders of perturbation theory. Indeed, and this will play a crucial role in what follows, the non-perturbative nature of the LKF transformation allows to fix some of the coefficients of the all-order expansion of the fermion propagator. Starting with a perturbative propagator in some fixed gauge, say η , all the coefficients depending on the difference between the gauge fixing parameters of the two propagators, $\xi-\eta$, get fixed by a weak coupling expansion of the LKF-transformed initial one. Such estimations have been carried out for QED in various dimensions [16], for generalizations to brane worlds [17] and for more general SU(N) gauge theories [18].

Here we review the results [1] of usage of the LKF transformation in order to study general properties of the coefficients of the propagator. We show how the transformation naturally reveals the existence of the hatted transcendental basis. Moreover, it allows us to extend the results of Eq. (1) to any order in ε .

¹ Following [2], by p -functions we understand ($\overline{\text{MS}}$ -renormalized) Euclidean 2-point functions (that can also be obtained from 3-point functions by setting one external momentum to zero with the help of infra-red rearrangement) expressible in terms of massless propagator-type Feynman integrals also known as p -integrals.

² Notice also that, within a Schwinger–Dyson equation approach in fixed dimension, renormalized Euclidean massless correlators were shown to be expressed only in terms of odd zeta-values [9].

2. LKF TRANSFORMATION

In the following, we shall consider QED in an Euclidean space of dimension d ($d = 4 - 2\varepsilon$). The general forms of the fermion propagator in the momentum and x -space representations, $S_F(p, \xi)$ and $S_F(x, \xi)$, in some gauge ξ read:

$$S_F(p, \xi) = \frac{1}{i\hat{p}} P(p, \xi), \quad S_F(x, \xi) = \hat{x}X(x, \xi), \quad (2)$$

where the tensorial structure, e.g., the factors \hat{p} and \hat{x} containing Dirac γ -matrices, have been extracted. The two representations, $S_F(x, \xi)$ and $S_F(p, \xi)$, are related by the Fourier transform which is defined as:

$$\begin{aligned} S_F(p, \xi) &= \int \frac{d^d x}{(2\pi)^{d/2}} e^{ipx} S_F(x, \xi), \\ S_F(x, \xi) &= \int \frac{d^d p}{(2\pi)^{d/2}} e^{-ipx} S_F(p, \xi). \end{aligned} \quad (3)$$

The famous LKF transformation connects in a very simple way the fermion propagator in two different gauges, e.g., ξ and η . In dimensional regularization, it reads [1]:

$$S_F(x, \xi) = S_F(x, \eta) e^{iD(x)}. \quad (4)$$

We may now proceed in calculating $D(x)$. In order to do so, it is possible to use the following simple formulas for the Fourier transform of massless propagators (see, e.g., [19]):

$$\begin{aligned} \int d^d x \frac{e^{ipx}}{x^{2\alpha}} &= \frac{2^{2\alpha} \pi^{d/2} a(\alpha)}{p^{2\alpha}}, \quad \int d^d p \frac{e^{-ipx}}{p^{2\alpha}} = \frac{2^{2\alpha} \pi^{d/2} a(\alpha)}{x^{2\alpha}}, \\ a(\alpha) &= \frac{\Gamma(\tilde{\alpha})}{\Gamma(\alpha)}, \quad \tilde{\alpha} = \frac{d}{2} - \alpha. \end{aligned} \quad (5)$$

This yields with the parameter ε made explicit:

$$D(x) = \frac{i\Delta A}{\varepsilon} \Gamma(1 - \varepsilon) (\pi\mu^2 x^2)^\varepsilon, \quad A = \frac{\alpha_{em}}{4\pi} = \frac{e^2}{(4\pi)^2}. \quad (6)$$

From Eq. (6), we see that $D(x)$ contributes with a common factor ΔA accompanied by the singularity ε^{-1} .

3. LKF TRANSFORMATION IN MOMENTUM SPACE

Let's assume that, for some gauge fixing parameter η , the fermion propagator $S_F(p, \eta)$ with external momentum p has the form (2) with $P(p, \eta)$ reading:

$$P(p, \eta) = \sum_{m=0}^{\infty} a_m(\eta) A^m \left(\frac{\tilde{\mu}^2}{p^2} \right)^{m\varepsilon}, \quad \tilde{\mu}^2 = 4\pi\mu^2, \quad (7)$$

where $a_m(\eta)$ are coefficients of the loop expansion of the propagator and $\tilde{\mu}$ is the renormalization scale, which lies somehow between the MS-scale μ and the $\overline{\text{MS}}$ -scale $\bar{\mu}$. Then, the LKF transformation shows

that, for another gauge parameter ξ , the fermion propagator can be expressed as:

$$P(p, \xi) = \sum_{m=0}^{\infty} a_m(\xi) A^m \left(\frac{\tilde{\mu}^2}{p^2} \right)^{m\varepsilon}, \quad (8)$$

where

$$\begin{aligned} a_m(\xi) &= a_m(\eta) \frac{\Gamma(2 - (m+1)\varepsilon)}{\Gamma(1 + m\varepsilon)} \\ &\times \sum_{l=0}^{\infty} \frac{\Gamma(1 + (m+l)\varepsilon) \Gamma^l(1 - \varepsilon) (\Delta A)^l}{l! \Gamma(2 - (m+l)\varepsilon) (-\varepsilon)^l} \left(\frac{\tilde{\mu}^2}{p^2} \right)^{l\varepsilon}. \end{aligned} \quad (9)$$

In order to derive (9), we used the fermion propagator $S_F(p, \eta)$ with $P(p, \eta)$ given by (7), did the Fourier transform to $S_F(x, \eta)$ and applied the LKF transformation (4). As a final step, we took the inverse Fourier transform and obtained $S_F(p, \xi)$ with $P(p, \xi)$ given by (8).

3.1. Scale Fixing

Following [1], we consider only the case of the so-called MS-like schemes. In such schemes, we need to fix specific terms coming from the application of dimensional regularization. Such a procedure will be called *scale fixing* and will play a crucial role in our analysis.

Let's first recall that the $\overline{\text{MS}}$ -scale $\bar{\mu}$ is related to the previously defined scale $\tilde{\mu}$ with the help of $\bar{\mu} = \tilde{\mu}^2 e^{-\gamma}$, where γ is the Euler constant. An advantage of the $\overline{\text{MS}}$ -scale is that it subtracts the Euler constant γ from the ε -expansion. Moreover, it is well known that, in calculations of two-point massless diagrams, the final results do not display any ζ_2^3 . So it is convenient to choose some scale which also subtracts ζ_2 in intermediate steps of the calculation. For this purpose, in [1] we considered two different scales.

The first one is the popular G -scale [20]. Actually, following [10], in [1] we used a slight modification of this scale that we refer to as the g -scale and in which an additional factor $1/(1 - 2\varepsilon)$ is subtracted from the one-loop result.

Moreover, in [1] we also introduced a new scale which is based on old calculations of massless diagrams performed by Vladimirov who added [21] an additional factor $\Gamma(1 - \varepsilon)$ to each loop contribution.

The latter corresponds to adding the factor $\Gamma^{-1}(1 - \varepsilon)$ to the corresponding scale. We shall refer to this scale

³ Strictly speaking, ζ_2 can appear in some formulas such as sum rules in deep-inelastic scattering. They originate from an analytic continuation [23] of certain special forms of p -integrals. We will not consider this case in the present study.

as the minimal Vladimirov-scale, or MV-scale, and define⁴:

$$\mu_{MV}^{2\varepsilon} = \frac{\tilde{\mu}^{2\varepsilon}}{\Gamma(1-\varepsilon)}. \tag{10}$$

The use of the MV-scale leads to simpler results in comparison with the g one. Hence, the MV-scale is more appropriate to our analysis and all our results are given in the MV-scale. Differences coming from the use of the g -scale can be found in [1].

In the MV-scale, we can rewrite the result (9) in the following general form⁵:

$$a_m(\xi) = a_m(\eta) \sum_{l=0}^{\infty} \frac{1-(m+1)\varepsilon}{1-(m+l+1)\varepsilon} \times \Phi_{MV}(m, l, \varepsilon) \frac{(\Delta A)^l}{(-\varepsilon)^l l!} \left(\frac{\mu_{MV}^2}{p^2} \right)^{l\varepsilon}, \tag{11}$$

where

$$\Phi_{MV}(m, l, \varepsilon) = \frac{\Gamma(1-(m+1)\varepsilon)\Gamma(1+(m+l)\varepsilon)\Gamma^{2l}(1-\varepsilon)}{\Gamma(1+m\varepsilon)\Gamma(1-(m+l+1)\varepsilon)}. \tag{12}$$

In Eq. (11), the factor $(1-(m+1)\varepsilon)/(1-(m+l+1)\varepsilon)$ has been specially extracted from $\Phi_{MV}(m, l, \varepsilon)$ in order to insure equal transcendental level, i.e., the same value of s for ζ_s at every order of the ε -expansion of $\Phi_{MV}(m, l, \varepsilon)$ (see below).

3.2. MV-Scale

The Γ -function $\Gamma(1+\beta\varepsilon)$ has the following expansion:

$$\Gamma(1+\beta\varepsilon) = \exp\left[-\gamma\beta\varepsilon + \sum_{s=2}^{\infty} (-1)^s \eta_s \beta^s \varepsilon^s\right], \quad \eta_s = \frac{\zeta_s}{s}. \tag{13}$$

Substituting Eq. (13) in Eq. (12), yields for the factor $\Phi_{MV}(m, l, \varepsilon)$:

$$\Phi_{MV}(m, l, \varepsilon) = \exp\left[\sum_{s=2}^{\infty} \eta_s p_s(m, l) \varepsilon^s\right], \tag{14}$$

where

$$p_s(m, l) = (m+1)^s - (m+l+1)^s + 2l + (-1)^s \{(m+l)^s - m^s\}, \tag{15}$$

and, as expected from the MV-scale, we do have:

$$p_1(m, l) = 0, \quad p_2(m, l) = 0. \tag{16}$$

⁴ Notice that the form (10) has been used once to define the MS scheme (see Errata to [22]).

⁵ The results in the case of scalar QED are very similar and can be found in [1].

As can be see from Eq. (14), $\Phi_{MV}(m, l, \varepsilon)$ contains ζ_s -function values of a given weight (or transcendental level) s in factor of ε^s . Such a property strongly constrains the coefficients of the ε -series thereby simplifying our analysis. It is reminiscent of the one earlier found in [24]. When judiciously used, it sometimes allows to derive results without any calculations (as in [25]). In other cases, it simplifies the structure of the results which can then be predicted as an ansatz in a very simple way (see [26, 27]). For a recent application of such property, see the recent papers [28] and references and discussions therein.

4. SOLUTION OF THE RECURRENCE RELATIONS

We now focus on the polynomial $p_s(m, l)$ of Eq. (15) that is conveniently separated in even and odd s values. Then, we see that the following recursion relations hold:

$$p_{2k} = p_{2k-1} + Lp_{2k-2} + p_3, \tag{17}$$

$$p_{2k-1} = p_{2k-2} + Lp_{2k-3} + p_3, \quad L = l(l+1).$$

Specific to the MV-scheme, these relations only depend on L which leads to strong simplifications. Nevertheless, they are difficult to solve for arbitrary k . It is simpler to proceed by explicitly considering the first values of k :

$$p_4 = 2p_3, \quad p_5 = p_4 + Lp_3 + p_3 = (3+L)p_3, \tag{18}$$

$$p_6 = p_5 + Lp_4 + p_3 = (4+3L)p_3,$$

showing that p_s takes the form of a polynomial in L in factor of p_3 . Then, taking the results in (18) together, yields:

$$Lp_3 = p_5 - 3p_3, \quad p_6 = 3p_5 - 5p_3, \tag{19}$$

which reveals that the even polynomial p_6 can be entirely expressed in terms of the lower order odd ones, p_3 and p_5 . We may automate this procedure for higher values of k and express p_{2k} as

$$p_{2k} = \sum_{s=2}^k p_{2s-1} C_{2k, 2s-1} = \sum_{m=1}^{k-1} p_{2k-2m+1} C_{2k, 2k-2m+1}. \tag{20}$$

From these results, it is possible to determine the exact k -dependence of $C_{2k, 2s-1}$, which has the following structure:

$$C_{2k, 2k-2m+1} = b_{2m-1} \frac{(2k)!}{(2m-1)!(2k-2m+1)!}, \tag{21}$$

with the first coefficients b_{2m-1} taking the values:

$$b_1 = \frac{1}{2}, \quad b_3 = -\frac{1}{4}, \quad b_5 = \frac{1}{2}, \quad b_7 = -\frac{17}{2}, \quad b_9 = \frac{31}{2},$$

$$b_{11} = -\frac{691}{4}, \quad b_{13} = \frac{5461}{2}, \quad b_{15} = -\frac{929569}{16},$$

$$\begin{aligned} b_{17} &= \frac{3202291}{2}, & b_{19} &= -\frac{221930581}{4}, \\ b_{21} &= \frac{4722116521}{2}, & b_{23} &= -\frac{968383680827}{8}. \end{aligned} \quad (22)$$

Examining the numerators of b_{2m-1} , one can see that they are proportional to the numerators of Bernoulli numbers. Indeed, a closer inspection reveals that, accurate to a sign, the coefficients b_{2m-1} coincide with the zero values of Euler polynomials $E_n(x)$:

$$b_{2m-1} = -E_{2m-1}(x=0), \quad (23)$$

and therefore to Bernoulli and Genocchi numbers, B_m and G_m , respectively, because

$$E_{2m-1}(x=0) = \frac{G_{2m}}{2m}, \quad G_{2m} = -\frac{(2^{2m}-1)}{m} B_{2m}. \quad (24)$$

Hence, the compact formula for the coefficients b_{2m-1} , expressed through the well known Bernoulli numbers B_m , reads:

$$b_{2m-1} = \frac{(2^{2m}-1)}{m} B_{2m}. \quad (25)$$

Together with (21), Eq. (25) provides an exact analytic expression for p_{2k} , Eq. (20), for arbitrary values of k .

5. HATTED ζ -VALUES

At this point, it is convenient to represent the argument of the exponential in the r.h.s. of (14) as follows:

$$\sum_{s=3}^{\infty} \eta_s p_s \varepsilon^s = \sum_{k=2}^{\infty} \eta_{2k} p_{2k} \varepsilon^{2k} + \sum_{k=2}^{\infty} \eta_{2k-1} p_{2k-1} \varepsilon^{2k-1}. \quad (26)$$

With the help of Eq. (20), the first term in the r.h.s. of Eq. (26) may be expressed as:

$$\begin{aligned} \sum_{k=2}^{\infty} \eta_{2k} p_{2k} \varepsilon^{2k} &= \sum_{k=2}^{\infty} \eta_{2k} \varepsilon^{2k} \sum_{s=2}^k p_{2s-1} C_{2k,2s-1} \\ &= \sum_{s=2}^{\infty} p_{2s-1} \sum_{k=s}^{\infty} \eta_{2k} C_{2k,2s-1} \varepsilon^{2k}. \end{aligned} \quad (27)$$

Then, Eq. (26) can be written as $\sum_{s=2}^{\infty} \hat{\eta}_{2s-1} p_{2s-1} \varepsilon^{2s-1}$ where

$$\begin{aligned} \hat{\eta}_{2s-1} &= \eta_{2s-1} + \sum_{k=s}^{\infty} \eta_{2k} C_{2k,2s-1} \varepsilon^{2(k-s)+1}, \\ C_{2k,2s-1} &= b_{2k-2s+1} \frac{(2k)!}{(2s-1)(2k-2s+1)!}. \end{aligned} \quad (28)$$

Thus, Eq. (14) can be represented as:

$$\begin{aligned} \Phi_{\text{MV}}(m, l, \varepsilon) &= \exp \left[\sum_{s=2}^{\infty} \hat{\eta}_{2s-1} p_{2s-1} \varepsilon^{2s-1} \right] \\ &= \exp \left[\sum_{s=2}^{\infty} \frac{\zeta_{2s-1}}{2s-1} p_{2s-1} \varepsilon^{2s-1} \right], \end{aligned} \quad (29)$$

where

$$\zeta_{2s-1} = \zeta_{2s-1} + \sum_{k=s}^{\infty} \zeta_{2k} \hat{C}_{2k,2s-1} \varepsilon^{2(k-s)+1} \quad (30)$$

with

$$\begin{aligned} \hat{C}_{2k,2s-1} &= \frac{2s-1}{2k} C_{2k,2s-1} \\ &= b_{2k-2s+1} \frac{(2k-1)!}{(2s-2)(2k-2s+1)!}. \end{aligned} \quad (31)$$

Together with (31) and (25), Eq. (30) provides an exact expression for the hatted ζ -values in terms of the standard ones valid for all ε .

6. SUMMARY

From the result (11) corresponding to the LKF transformation of the fermion propagator we have found peculiar recursion relations (17) between even and odd values of the polynomial associated to the uniformly transcendental factor $\Phi_{\text{MV}}(m, l, \varepsilon)$ (12). These relations are simple in the MV-scheme that we have introduced in Eq. (10). They relate the even and odd parts in a rather simple way (see (20)) which reveals the possibility (29) to express all results for $\Phi_{\text{MV}}(m, l, \varepsilon)$ in terms of hatted ζ -values. Our careful study of the recursion relations (17) allowed us to derive exact formulas, Eqs. (28) and (30), relating hatted and standard ζ -values to all orders of perturbation theory. The coefficients of the relations are expressed through the well-known Bernoulli numbers, B_{2m} (see (31) and (25)). Our results provide stringent constraints on multi-loop calculations at any order in perturbation theory.

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