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I always knew that sooner or later p-adic numbers
will appear in Physics.
André Weil

Supersymmetric Dynamics and Zeta-Functions[1]

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Abstract—Boson, fermion, and super oscillators and (statistical) mechanism of cosmological constant; finite approximation of the zeta-function and fermion factorization of the bosonic statistical sum concidered.

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Supermathematics unifies discrete and continual aspects of mathematics. Boson oscillator hamiltonian is

$$H_b = \hbar\omega(b^+b + bb^+)/2 = \hbar\omega(b^+b + a), \quad a = 1/2. \quad (1)$$

corresponding energy spectrum E_{bn} and eigenfunctions $|n_b\rangle$ are

$$H_b|n_b\rangle = E_{bn}|n_b\rangle, \quad E_{bn} = \hbar\omega(n_b + a), \quad n_b = 0,1,2,\ldots. \quad (2)$$

Fermion oscillator hamiltonian, eigenvectors and energies are

$$H_f = \hbar\omega(f^+f - ff^+)/2 = \hbar\omega(f^+f - a), \quad H_f = |n_f\rangle = E_{fn}|n_f\rangle, \quad E_{fn} = \hbar\omega(n_f - a), \quad n_f = 0,1. \quad (3)$$

For supersymmetric oscillator we have

$$H = H_b + H_f, \quad H|n_b, n_f\rangle = \hbar\omega(n_b + n_f)|n_b, n_f\rangle, \quad |n_b, n_f\rangle = |n_b\rangle|n_f\rangle, \quad E_{n_b,n_f} = \hbar\omega(n_b + n_f). \quad (4)$$

For background-vacuum $|0,0\rangle$, energy $E_{0,0} = 0$. For higher energy states $|n-1,1\rangle$, $|n,0\rangle$, $E_{n-1,1} = E_{n,0}$. Supersymmetry needs not only the same frequency for boson and fermion oscillators, but also that $2a = 1$.

A minimal realization of the algebra of supersymmetry

$$\{Q, Q^+\} = H, \{Q, Q\} = \{Q^+, Q^+\} = 0, \quad (5)$$

is given by a point particle dynamics in one dimension, [1]

$$Q = f(-iP + W)/\sqrt{2}, \quad Q^+ = f^+(iP + W)/\sqrt{2}, \quad P = -i\partial/\partial x, \quad (6)$$

where the superpotential $W(x)$ is any function of x, and spinor operators f and f^+ obey the anticommuting relations

$$\{f, f^+\} = 1, \quad f^2 = (f^+)^2 = 0. \quad (7)$$

There is a following representation of operators f, f^+ and σ by Pauli spin matrices

$$f = \frac{\sigma_1 - i\sigma_2}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad f^+ = \frac{\sigma_1 + i\sigma_2}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (8)$$

From formulae (5) and (6) then we have

$$H = (P^2 + W^2 + \sigma W_x)/2. \quad (9)$$

The simplest nontrivial case of the superpotential $W = \omega x$ corresponds to the supersimmetric oscillator with Hamiltonian

$$H = H_b + H_f, \quad H_b = (P^2 + \omega^2 x^2)/2, \quad H_f = \omega\sigma/2. \quad (10)$$

The ground state energies of the bosonic and fermionic parts are

$$E_{b0} = \omega/2, \quad E_{f0} = -\omega/2, \quad (11)$$

so the vacuum energy of the supersymmetric oscillator is

$$\langle 0|H|0\rangle = E_0 = E_{b0} + E_{f0} = 0, \quad |0\rangle = |n_b, n_f\rangle = |n_b\rangle|n_f\rangle. \quad (12)$$

Let us see on this toy—solution of the cosmological constant problem from the quantum statistical viewpoint. The statistical sum of the supersymmetric oscillator is

$$Z(\beta) = Z_b Z_f, \quad (13)$$

[1] The article is published in the original.

where

$$Z_b = \sum_n e^{-\beta E_{bn}} = e^{-\beta\omega/2} + e^{-\beta\omega(1+1/2)} + \ldots = e^{-\beta\omega/2}\big/(1 - e^{-\beta\omega}), \tag{14}$$
$$Z_f = \sum_n e^{-\beta E_{fn}} = e^{\beta\omega/2} + e^{-\beta\omega/2}.$$

In the low temperature limit,

$$Z(\beta) = 1 + O(e^{-\beta\omega}) \to 1, \quad \beta = T^{-1}, \tag{15}$$

so cosmological constant $\lambda \sim \ln Z \to 0$. From observable values of β and the cosmological constant we estimate ω.

The Riemann zeta function (RZF) can be interpreted in thermodynamic terms as a statistical sum of a system with energy spectrum: $E_n = \ln n, \quad n = 1, 2, \ldots$:

$$\zeta(s) = \sum_{n\geq 1} n^{-s} = Z(\beta) = \sum_{n\geq 1} \exp(-\beta E_n), \tag{16}$$
$$\beta = s, \quad E_n = \ln n, \quad n = 1, 2, \ldots.$$

Let us consider the following finite approximation of RZF

$$\zeta_N(s) = \sum_{n=1}^{N} n^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \frac{e^{-t} - e^{-(N+1)t}}{1 - e^{-t}} = \zeta(s) - \Delta_N(s), \quad \mathrm{Re}\, s > 1, \tag{17}$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1}}{e^t - 1}, \quad \Delta_N(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1} e^{-Nt}}{e^t - 1}.$$

Another formula, which can be used on critical line, is

$$\zeta(s) = (1 - 2^{1-s})^{-1} \sum_{n\geq 1} (-1)^{n+1} n^{-s} = \frac{1}{1 - 2^{1-s}} \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} dt}{e^t + 1}, \quad \mathrm{Re}\, s > 0. \tag{18}$$

Corresponding finite approximation of RZF is

$$\zeta_N(s) = (1 - 2^{1-s})^{-1} \sum_{n=1}^{N} (-1)^{n-1} n^{-s} = \frac{1}{1 - 2^{1-s}} \frac{1}{\Gamma(s)} \times \int_0^\infty \frac{t^{s-1}(1 - (-e^{-t})^N) dt}{e^t + 1} = \zeta(s) - \Delta_N(s), \tag{19}$$

$$\Delta_N(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1}(-e^{-t})^N}{e^t + 1} \sim \pm N^{-s}$$

at a (nontrivial) zero of RZF, s_0, $\zeta_N(s_0) = -\Delta_N(s_0)$. In the integral form, dependence on N is analytic and we can consider any complex valued N. It is interesting to see dependence (evolution) of zeros with N. For the simplest nontrivial integer $N = 2$,

$$\zeta_2(s) = (1 - 2^{1-s})^{-1}(1 - 2^{-s}) = \frac{1 - 2^{-s}}{1 - 2^{1-s}} = \frac{2^s - 1}{2^s - 2} = \frac{2^{s-1/2} - 1/\sqrt{2}}{2^{s-1/2} - \sqrt{2}}, \tag{20}$$

we have zeros at $s = 2\pi i n/\ln 2, \quad n = 0, \pm 1, \pm 2, \ldots$.

Let as consider the following formula (Qvelementar particles)

$$\frac{1}{1 - q} = (1 + q)(1 + q^2)(1 + q^4)\ldots, \quad |q| < 1, \tag{21}$$

which can be proved as

$$p_k \equiv (1 + q)(1 + q^2)(1 + q^4)\ldots(1 + q^{2^k}) = \frac{1 - q^{2^{(k+1)}}}{1 - q}, \quad c(1 - |q|^{2^{(k+1)}}) < |p_k| < c(1 + |q|^{2^{(k+1)}}), \tag{22}$$
$$\lim_{k\to\infty} |p_k| = c = 1/\|1 - q\|, \quad \lim_{k\to\infty} p_k = 1/(1 - q).$$

The formula (21) reminds us the boson and fermion statsums

$$Z_b = \frac{q^a}{1 - q}, \quad Z_f = \frac{1 + q}{q^a}, \quad q = \exp(-\beta\hbar\omega), \quad a = 1/2, \quad \beta = 1/T, \tag{23}$$

and can be transformed in the following relation

$$Z_b(\omega) = Z_f(\omega) Z_f(2\omega) Z_f(4\omega)\ldots. \tag{24}$$

Indeed,

$$Z_b(\omega) = \frac{q^a}{1 - q} = q^b Z_f(\omega) Z_f(2\omega) Z_f(4\omega)\ldots, \quad b = 2a + 2a(1 + 2 + 2^2 + \ldots) = 2a\left(1 + \frac{1}{1 - 2}\right) = 0, \quad |2|_2 = 1/2, \tag{25}$$

where $|n|_p = 1/p^k$, $n = p^k m$, is p-adic norm of n, k is the number of p-prime factors of n.

Bytheway we have an extra bonus! We see that the fermion content of the boson wears the p-adic sense [2]. The prime $p = 2$, in this case. Also, the vacuum energy of the oscillators wear p-adic sense.

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