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Multiparameter Quantum Group and Quantum Minkowski Space-Time[1]

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Abstract—We construct representations of the quantum algebras $U_{q,\mathbf{q}}(gl(n))$ and $U_{q,\mathbf{q}}(sl(n))$ which depend on $n(n-1)/2+1$ deformation parameters q, q_{ij} $(1 \le i < j \le n)$ which is the maximal possible number in the case of $GL(n)$. The representations act on the space of formal power series of $n(n-1)/2$ non-commuting variables which generate quantum flag manifolds of $GL_{q\mathbf{q}}(n)$, $SL_{q\mathbf{q}}(n)$. For $n = 4$ we consider in detail the multiparameter quantum Minkowski space-time.

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1. INTRODUCTION

About 30 years passed since the advent of quantum groups at center-stage of modern mathematical physics. Yet the field is growing stronger every day, cf. a recent review in [1]. In this paper we present briefly representations of multiparameter quantum algebras $U_{q,\mathbf{q}}(gl(n))$ and $U_{q,\mathbf{q}}(sl(n))$ on quantum flag manifolds of quantum $GL(n)$. We consider in detail the case $n = 4$ when the quantum flag manifold contains multiparameter quantum Minkowski space-time.

2. PRELIMINARIES

2.1. Multiparametric Deformation of GL(n)

Here we use the quantum group deformation of $GL(n)$ introduced by Sudbery [2]. That deformation depends on the maximal possible number of parameters: $N = n(n-1)/2+1$. We denote these N parameters by q and q_{ij}, $1 \le i < j \le n$, and also for shortness by the pair $q, \mathbf{q}$. The standard one-parameter deformation is obtained by setting $q_{ij} = q$, $\forall i, j$.

Explicitly, the matrix quantum group $\mathcal{A} \equiv GL_{q\mathbf{q}}(n)$ is generated by the generators a_{ij} $(1 \le i, j \le n)$ with the following commutation relations [2]:

$$a_{ij}a_{i\ell} = pa_{i\ell}a_{ij}, \quad \text{for} \quad j < \ell, \tag{1a}$$

$$a_{ij}a_{kj} = ra_{kj}a_{ij}, \quad \text{for} \quad i < k, \tag{1b}$$

$$pa_{i\ell}a_{kj} = ra_{kj}a_{i\ell}, \quad \text{for} \quad i < k, \quad j < \ell, \tag{1c}$$

$$rqa_{k\ell}a_{ij} - (qp)^{-1}a_{ij}a_{k\ell} = \lambda a_{i\ell}a_{kj}, \quad \text{for} \quad i < k,\ j < \ell, \tag{1d}$$

$$p = q_{j\ell}/q^2, \quad r = 1/q_{ik}, \quad \lambda = q - 1/q. \tag{1e}$$

The comultiplication, counit and antipode are standard [2].

Following the approach of [3] we shall use representations of the dual quantum algebra on suitable quantum flag manifolds of $\mathcal{A}$. For this we first use the triangular decomposition of $\mathcal{A}$ [4]:

$$\begin{gathered} a_{i\ell} = \sum_j Y_{ij}D_{jj}Z_{j\ell}, \quad Y_{ij} = \xi^{1\cdots j-1i}_{1\cdots j}D_j^{-1}, \\ Z_{j\ell} = D_i^{-1}\xi^{1\cdots j}_{1\cdots j-1\ell}, \quad D_{jj} = D_jD_{j-1}^{-1}, \\ D_m = \sum_{\rho\in S_m} \epsilon(\rho)a_{1,\rho(1)}\ldots a_{m,\rho(m)}, \\ \xi^I_J = \sum_{\rho\in S_r}\epsilon(\rho)a_{i_{\rho(1)}j_1}\cdots a_{i_{\rho(r)}j_r}, \\ I = \{i_1 < \cdots < i_r\}, \quad J = \{j_1 < \cdots < j_r\}, \end{gathered} \tag{2}$$

S_n is the permutation group of n elements. Note that $Y_{i\ell} = 0$ for $i < \ell$, $Y_{ii} = 1_{\mathcal{A}}$, $Z_{i\ell} = 0$ for $i > \ell$, $Z_{ii} = 1_{\mathcal{A}}$, $D_0 \equiv 1_{\mathcal{A}}$, $\xi^{1\cdots i}_{1\cdots i}D_i$. Then $\mathcal{G}_{q,\mathbf{q}} \equiv \{Y_{j\ell}, j > \ell\}$, may be regarded as a quantum analogue of the flag manifold $GL(n)/DZ$, $\mathfrak{Z}_{q,\mathbf{q}} \equiv \{Z_{ij}, i < j\}$, may be regarded as a quantum analogue of the flag manifold $B\backslash GL(n)$.

We give the commutation relation between the generators Y_{ji} since we shall build our representations on $\mathcal{G}_{q,\mathbf{q}}$. The indices used below obey $1 \le i < j < k < l \le n$. We also use the notation:

$$p_{ij} \equiv \frac{q_{ij}}{q^2}, \quad p'_{ij} \equiv \frac{q'_{ij}}{q'^2}, \quad q' \equiv 1/q, \quad q'_{ij} \equiv q_{ij}/q^2. \tag{3}$$

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[1] The article is published in the original.

We have:

$$Y_{kj}Y_{ki} = \frac{q_{ij}q_{jk}}{q_{ik}} Y_{ki}Y_{kj}, \quad Y_{ki}Y_{ji} = \frac{q_{ij}q_{jk}}{q_{ik}} Y_{ji}Y_{ki}, \tag{4a}$$

$$Y_{kj}Y_{ji} = \frac{p_{ij}p_{jk}}{p_{ik}} Y_{ji}Y_{kj} + u^{-1}(u-u^{-1})Y_{ki}, \tag{4b}$$

$$Y_{li}Y_{kj} = \frac{q_{ik}q_{kl}}{q_{ij}q_{jl}} Y_{kj}Y_{li}, \quad Y_{lk}Y_{ji} = \frac{q_{ik}q_{jl}}{q_{il}q_{jk}} Y_{ji}Y_{lk}, \tag{4c}$$

$$\frac{q_{jl}}{q_{jk}q_{kl}} Y_{lj}Y_{ki} = \frac{p_{ij}p_{jl}}{p_{il}} Y_{ki}Y_{lj} + u^{-1}(u-u^{-1})Y_{kj}Y_{li}. \tag{4d}$$

2.2. Multiparameter Dual algebra

In [5] we have found the dual to $\mathcal{A}$ algebra $\mathcal{U}_g \equiv U_{q,\mathbf{q}}(gl(n))$. We fix the standard decomposition $gl(n) = sl(n) \oplus \mathcal{Z}$, where $\mathcal{Z}$ is the central subalgebra of $gl(n)$.

The Drinfeld–Jimbo form of the dual commutation algebra $\mathcal{U}_g$ in terms of the $sl(n)$ generators $H_i, X_i^{\pm}$ and the $\mathcal{Z}$ generator K is given as follows:

$$[H_i, X_j^{\pm}] = \pm c_{ij} X_j^{+}, \tag{5a}$$

$$[X_i^+, X_i^-] = \lambda^{-1}\left(q^{H_i} - q^{-H_i}\right) \equiv [H_i]_q, \tag{5b}$$

$$[K, Y] = 0, \quad \forall Y \in sl(n), \tag{5c}$$

where $\lambda \equiv q - q^{-1}$, c_{ij} is the standard Cartan matrix of $gl(n, \mathbb{C})$).

Thus as a *commutation algebra* we have the splitting $\mathcal{U}_{q\mathbf{q}} \cong U_q(sl(n,\mathbb{C})) \otimes U_q(\mathcal{Z})$, and dependence only on the parameter q.

This splitting is preserved also by the co-unit and the antipode:

$$\varepsilon_{\mathcal{U}}(Y) = 0, \quad Y = X_i^{\pm}, H_i, K, \tag{6a}$$

$$\gamma_{\mathcal{U}}(X_i^{\pm}) = -q^{\pm 1}(X_i^{\pm}), \quad \gamma_{\mathcal{U}}(Y) = -Y, \quad Y = H_i, \ K, \tag{6b}$$

and by the coproducts of H_i, K:

$$\delta_{\mathcal{U}}(Y) = Y \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes Y, \quad Y = H_i, \ K. \tag{7}$$

However, for the coproducts of the Chevalley generators $X_i^{\pm}$ we have:

$$\delta_{\mathcal{U}}(X_i^+) = X_i^+ \otimes \mathcal{P}_i^{1/2} + \mathcal{P}_i^{-1/2} \otimes X_i^+, \tag{8a}$$

$$\delta_{\mathcal{U}}(X_i^-) = X_i^- \otimes \mathcal{Q}_i^{1/2} + \mathcal{Q}_i^{-1/2} \otimes X_i^-, \tag{8b}$$

$$\mathcal{P}_i = \left(\prod_{s=1}^{i-1}\left(\frac{q_{si}}{q_{s,i+1}}\right)^{\hat{H}_s}\right)\left(\frac{q^2}{q_{i,i+1}}\right)^{\hat{H}_i} \times \left(\frac{1}{q_{i,i+1}}\right)^{\hat{H}_{i+1}} \prod_{t=i+2}^{n-1}\left(\frac{q_{i+1,t}}{q_{it}}\right)^{\hat{H}_t}, \quad \mathcal{Q}_i = q^{2H_i}P_i, \quad \hat{H}_i \equiv \sum_{j=i}^{n-1} H_j. \tag{9}$$

Thus, the coproduct structure is not split and depends on all parameters. Yet for a special choice of $n-1$ of the parameters (e.g., $q_{i,i+1}$) $\mathcal{U}_g$ can be split as a direct product of two Hopf subalgebras: $\mathcal{U} \equiv U_{q,\mathbf{q}}(sl(n))$ and $U_q(\mathcal{Z})$, where $\mathcal{U}$ depends only on $(n^2 - 3n + 4)/2$ parameters [5].

2.3. Representations of the Dual Algebra

We shall work with representation spaces of $\mathcal{U}$ parametrized by $n-1$ numbers r_i which will be integers initially. The elements of these spaces will be formal power series:

$$\tilde{\varphi}(\bar{Y}, \bar{D}) = \sum_{\bar{m}\in\mathbb{Z}_+} \mu_{\bar{m}} (Y_{21})^{m_{21}} \ldots (Y_{n,n-1})^{m_{n,n-1}} \times (D_1)^{r_1} \ldots (D_{n-1})^{r_{n-1}} = \tilde{\varphi}(\bar{Y})(D_1)^{r_1} \ldots (D_{n-1})^{r_{n-1}}, \tag{10}$$

where $\bar{Y}, \bar{D}$ denote the variables Y_{il},, $i > \ell$, D_i, $i < n$.

First we shall give the left representation action π of $\mathcal{U}$ on $\hat{\varphi}$. Besides the action of the 'Chevalley' generators $K_i \equiv q^{H_i}, X_i^{\pm}$ we shall give for the readers convenience also the action of $\mathcal{P}_i, \mathcal{Q}_i$ though it follows from that of K_i. We have:

$$\pi(K_i)Y_{lj} = q^{(\delta_{i+1,l} - \delta_{i+1,j} - \delta_{il} + \delta_{ij})/2} Y_{lj}, \tag{11a}$$

$$\pi(X_i^+)Y_{lj} = -qQ_{i,i+1}^{-1/2}Q_{ij}^{-1/2}\delta_{il}Y_{l+1,j}) + qQ_{i,i+1}^{-1/2}Q_{il}^{-1/2} \times \left(\frac{q_{j,j+1}q_{j+1,l}}{q_{jl}}\right)^{(1-\delta_{l,j+1})} \delta_{ij}Y_{j+1,j}Y_{lj} + qQ_{i,i+1}^{-1/2}Q_{il}^{1/2}Q_{i,j-1}^{-1/2} \times Q_{ij}^{-1/2}\delta_{i+1,j}\left\{\frac{q_{j-1,l}}{q_{j-1,j}q_{jl}}Y_{l,j-1} - Y_{j,j-1}Y_{lj}\right\}, \tag{11b}$$

$$\pi(X_i^-)Y_{lj} = -q^{-2}Q_{ii}^{1/2}Q_{ij}^{1/2}q^{-\delta_{ij}}\delta_{i+1,l}Y_{l-1,j}, \tag{11c}$$

$$\pi(\mathcal{P}_i^{1/2})Y_{lj} = Q_{il}^{-1/2}Q_{ij}^{1/2}Y_{lj}, \tag{11d}$$

$$\pi(\mathcal{Q}_i^{1/2})Y_{lj} = q^{(\delta_{i+1,l} - \delta_{i+1,j} - \delta_{il} + \delta_{ij})}Q_{il}^{1/2}Q_{ij}^{-1/2}Y_{lj}, \tag{11e}$$

where

$$Q_{is} = \begin{cases} \dfrac{q_{si}}{q_{s,i+1}}, & s \le i-1 \\ \dfrac{q^2}{q_{i,i+1}}, & s = i \\ \dfrac{1}{q_{i,i+1}}, & s = i+1 \\ \dfrac{q_{i+1,s}}{q_{is}}, & s \ge i+2 \end{cases}. \quad (12)$$

The above is supplemented with the following action on the unit element of $\mathcal{A}$:

$$\pi(K_i)1_{\mathcal{A}} = 1_{\mathcal{A}}, \quad \pi(X_i^{\pm})1_{\mathcal{A}} = 0. \quad (13)$$

In order to derive the action of $\pi(y)$ on arbitrary elements of the basis (10), we use the twisted derivation rule consistent with the coproduct and the representation structure, namely, we take: $\pi(y)\varphi\psi = \pi(\delta'_{\mathcal{U}}(y))(\varphi \otimes \psi)$, where $\delta'_{\mathcal{U}} = \sigma \circ \delta_{\mathcal{U}}$ is the opposite coproduct (σ is the permutation operator). Thus, we have:

$$\pi(K_i)\varphi\psi = \pi(K_i)\varphi\pi(K_i)\psi, \quad (14a)$$

$$\pi(X_i^+)\varphi\psi = \pi(X_i^+)\varphi\pi(\mathcal{P}_i^{-1/2})\psi + \pi(\mathcal{P}_i^{1/2})\varphi\pi(X_i^+)\psi, \quad (14b)$$

$$\pi(X_i^-)\varphi\psi = \pi(X_i^-)\varphi\pi(\mathcal{Q}_i^{-1/2})\psi + \pi(\mathcal{Q}_i^{1/2})\varphi\pi(X_i^-)\psi. \quad (14c)$$

From now on we suppose that none of the deformation parameters q, q_{ij} is a nontrivial root of unity.

Applying (14) to (11) we have:

$$\pi(K_i)(Y_{lj})^k = q^{k(\delta_{i+1,l}-\delta_{i+1,j}-\delta_{il}+\delta_{ij})/2}(Y_{lj})^k, \quad (15a)$$

$$\begin{aligned}\pi(X_i^+)(Y_{lj})^k &= -qQ_{i,i+1}^{-1/2}Q_{ij}^{(k-2)/2}c_l\delta_{il}(Y_{lj})^{k-1}Y_{l+1,j} \\ &+ qQ_{i,i+1}^{-1/2}Q_{il}^{(k-2)/2}c_j\left(\frac{q_{j,j+1}q_{j+1,l}}{q_{jl}}\right)^{(1-\delta_{l,j+1})}\delta_{ij}Y_{j+1,j}(Y_{lj})^k \\ &+ qQ_{i,i+1}^{-1/2}Q_{il}^{k/2}\left(\frac{q_{j-1,j}}{q}\right)^k\tilde{c}_{j-1}\delta_{i+1,j} \\ &\times\left\{\frac{q_{j-1,l}}{q_{j-1,j}q_{jl}}Y_{l,j-1}(Y_{lj})^{k-1} - Y_{j,j-1}(Y_{lj})^k\right\},\end{aligned} \quad (15b)$$

$$\pi(X_i^-)(Y_{lj})^k = -q^{-2}Q_{ii}^{1/2}Q_{ij}^{k/2}q^{-k\delta_{ij}}c_{l-1}\delta_{i+1,l}Y_{l-1,j}(Y_{lj})^{k-1}, \quad (15c)$$

$$\pi(\mathcal{P}_i^{1/2})(Y_{lj})^k = Q_{il}^{-k/2}Q_{ij}^{k/2}(Y_{lj})^k, \quad (15d)$$

$$\pi(\mathcal{Q}_i^{1/2})(Y_{lj})^k = q^{k(\delta_{i+1,l}-\delta_{i+1,j}-\delta_{il}+\delta_{ij})}Q_{il}^{k/2}Q_{ij}^{-k/2}(Y_{lj})^k, \quad (15e)$$

$$c_i = (q_{i,i+1})^{(k-1)/2}[k]_q, \quad \tilde{c}_i = (q_{i,i+1})^{(1-k)/2}[k]_q, \quad [k]_q = (q^k - q^{-k})/\lambda. \quad (16)$$

$$\pi(K_i)(D_j)^k = q^{-k\delta_{ij}/2}(D_j)^k, \quad (17a)$$

$$\pi(X_i^+)(D_j)^k = -qQ_{i,i+1}^{-1/2}\left(\prod_{s=1}^{j-1}Q_{is}^{k/2}\right)\tilde{c}_j\delta_{ij}Y_{j+1,j}(D_j)^k, \quad (17b)$$

$$\pi(X_i^-)(D_j)^k = 0, \quad (17c)$$

$$\pi(\mathcal{P}_i^{1/2})(D_j)^k = \left(\prod_{s=1}^{j}Q_{is}^{-k/2}\right)(D_j)^k, \quad (17d)$$

$$\pi(\mathcal{Q}_i^{1/2})(D_j)^k = q^{-k\delta_{ij}}\left(\prod_{s=1}^{j}Q_{is}^{k/2}\right)(D_j)^k. \quad (17e)$$

The action of $\mathcal{U}$ on arbitrary elements $\tilde{\varphi}, \hat{\varphi}$ is found by combining the formulae (15), (17) via (14).

3. MULTIPARAMETER QUANTUM MINKOWSKI SPACE-TIME

We consider now the case of $GL(4)$ which has a flag manifold $\mathcal{G}^4 = GL(4)/\tilde{B} = SL(4)/B$, where $\tilde{B}, B$ are the Borel subgroups of $GL(4), SL(4)$, respectively, consisting of all upper diagonal matrices. Under a natural conjugation (cf. also below) this is also a flag manifold of the conformal group $SU(2,2)$.

In this case there are six coordinates Y_{ij} of $\mathcal{G}^4_{q,\mathbf{q}}$. In [6] we have found the following correspondence with variables that are standard in conformal invariant theories:

$$Y_{31} \leftrightarrow v \equiv x_1 - ix_2, \quad Y_{42} \leftrightarrow \bar{v} \equiv x_1 + ix_2, \quad (18)$$

$$Y_{41} \leftrightarrow x_+ \equiv x_0 + x_3, \quad Y_{32} \leftrightarrow x_-x_\pm \equiv x_0 - x_3, \quad (19)$$

$$Y_{21} \leftrightarrow z, \quad Y_{43} \leftrightarrow \bar{z}, \quad (20)$$

where x_μ ($\mu = 0,1,2,3$) are the standard coordinates of 4d Minkowski space-time, while $z, \bar{z}$ are the so-called spin variables carrying the Lorenz representations.

As discussed Section 2.1 we start from the multiparameter deformation $GL_{q,\mathbf{q}}(n)$ of $GL(n)$ which depends on $(n^2 - n + 2)/2$ parameters q, q_{ij}, $1 \le i < j \le n$. Thus, the flag manifold $\mathcal{G}_{q,\mathbf{q}} = GL_{q,\mathbf{q}}(n)/\tilde{B}_{q,\mathbf{q}}(n)$ depends on the same number of parameters. For $n = 4$ the explicit relations are [6]:

$$\begin{aligned} x_+v &= \frac{q_{23}q_{34}}{q_{24}}vx_+, \quad \bar{v}x_+ = \frac{q_{14}}{q_{12}q_{24}}x_+\bar{v}, \\ x_-v &= \frac{q_{13}}{q_{12}q_{23}}vx_-, \quad \bar{v}x_- = \frac{q_{13}q_{34}}{q_{14}}x_-\bar{v}, \\ \bar{v}v &= \frac{q_{13}q_{34}}{q_{12}q_{24}}v\bar{v}, \quad \frac{qq_{24}}{q_{23}q_{34}}x_+x_- = \frac{q_{12}q_{24}}{qq_{14}}x_-x_+ + \lambda v\bar{v}, \end{aligned} \quad (21)$$

$$\bar{z}z = \frac{q_{13}q_{24}}{q_{14}q_{23}} z\bar{z}, \quad \bar{z}x_+ = \frac{q_{13}q_{34}}{q_{14}} x_+\bar{z},$$
$$\bar{z}x_- = \frac{q_{23}q_{34}}{q^2 q_{24}} x_-\bar{z} + \lambda\bar{v}, \quad \bar{z}\bar{v} = \frac{q_{23}q_{34}}{q_{24}} \bar{v}\bar{z},$$
$$\bar{z}v = \frac{q_{13}q_{34}}{q^2 q_{14}} v\bar{z} + \lambda x_+, \tag{22}$$
$$x_+z = \frac{q_{14}}{q_{12}q_{24}} zx_+, \quad x_-z = \frac{q^2 q_{13}}{q_{12}q_{23}} zx_- - \lambda v,$$
$$vz = \frac{q_{13}}{q_{12}q_{23}} zv, \quad \bar{v}z = \frac{q^2 q_{14}}{q_{12}q_{24}} z\bar{v} - \lambda x_+.$$

Thus, in (21) we have a seven-parameter quantum Minkowski space-time.

We note that when all deformation parameter are phases, i.e., $|q| = 1$, $|q_{ij}| = 1$, and in addition holds the following relations:

$$q_{13} = \frac{q_{12}q_{24}}{q_{34}}, \quad q_{14} = \frac{q_{12}q_{24}^2}{q_{23}q_{34}}, \tag{23}$$

then the commutation relations (21) and (11) are preserved by an anti-linear anti-involution ω acting as:

$$\omega(x_\pm) = x_\pm, \quad \omega(v) = \bar{v}, \quad \omega(z) = \bar{z}. \tag{24}$$

Further, we recall from [5] that the dual quantum algebra $U_{q,\mathbf{q}}(gl(n))$ has the quantum algebra $U_{q,\mathbf{q}}(sl(n))$ as a commutation subalgebra, but not as a co-subalgebra. In order to achieve the complete splitting of $U_{q,\mathbf{q}}(sl(n))$ we have to impose some relations between the parameters, thus the genuine multiparameter deformation $U_{q,\mathbf{q}}(sl(n))$ depends on $(n^2 - 3n + 4)/2$ parameters. Thus, in the case of $n = 4$ for the genuine $U_{q,\mathbf{q}}(sl(4))$ we have four parameters. Explicitly, we achieve this by imposing that the parameters $q_{i,i+1}$ are expressed through the rest as:

$$q_{12} = \frac{q^3}{q_{13}q_{14}}, \quad q_{23} = \frac{q^4}{q_{13}q_{14}q_{24}}, \quad q_{34} = \frac{q^3}{q_{14}q_{24}}. \tag{25}$$

Thus, the four-parameter quantum Minkowski space-time and the embedding quantum flag manifold $\mathcal{G}^4_{q,\mathbf{q}}$ are given by (21) and (11) with (25) enforced.

If we would like to enforce also the conjugation (24) then there are more relations between the deformation parameters, namely, we get:

$$q_{12} = q_{23} = q_{34} = \frac{q^2}{q_{14}}, \quad q_{13} = q_{24} = q. \tag{26}$$

Thus, in this case we have a two-parameter deformation and using the above relations (21) and (11) simplify as follows:

$$x_+v = pvx_+, \quad \bar{v}x_+ = p^{-1}x_+\bar{v},$$
$$x_-v = p^{-1}vx_-, \quad \bar{v}x_- = px_-\bar{v}, \tag{27}$$
$$\bar{v}v = v\bar{v}, \quad \frac{q}{p}x_+x_- = \frac{p}{q}x_-x_+ + \lambda v\bar{v},$$

$$\bar{z}z = z\bar{z}, \quad \bar{z}x_+ = px_+\bar{z}, \quad \bar{z}x_- = \frac{p}{q^2}x_-\bar{z} + \lambda\bar{v},$$
$$\bar{z}\bar{v} = p\bar{v}\bar{z}, \quad \bar{z}v = \frac{p}{q^2}v\bar{z} + \lambda x_+,$$
$$x_+z = p^{-1}zx_+, \quad x_-z = \frac{q^2}{p}zx_- - \lambda v, \tag{28}$$
$$vz = p^{-1}zv, \quad \bar{v}z = \frac{q^2}{p}z\bar{v} - \lambda x_+,$$

where $p \equiv q^3/q_{14}^2$.

Another question is the quantum Minkowski length. In the one-parameter case it is given by [6]:

$$\mathcal{L}_q = x_-x_+ - q^{-1}v\bar{v}. \tag{29}$$

It commutes with the q-Minkowski coordinates and has the correct classical limit $\mathcal{L}_{q=1} = \mathcal{L} = x_0^2 - \vec{x}^2$. In the multiparameter case we try a similar Ansatz:

$$\mathcal{L}_{q,\mathbf{q}} = x_-x_+ - \beta(q,\mathbf{q})v\bar{v}. \tag{30}$$

In the general seven-parameter case this quantum Minkowski length commutes with the quantum Minkowski coordinates if the following conditions hold:

$$\beta(q,\mathbf{q}) = \frac{q_{14}}{q_{12}q_{24}}, \quad q_{23} = \frac{q^2 q_{14}}{q_{12}q_{34}}, \quad q_{13} = \frac{q_{12}q_{24}}{q_{34}}. \tag{31}$$

Thus, it becomes five-parameter case.

In the split four-parameter case commutativity of quantum Minkowski length (30) occurs when in addition to (25) hold also:

$$\beta(q,\mathbf{q}) = \frac{q_{14}}{qq_{24}}, \quad q_{13} = \frac{q^2}{q_{14}}, \quad q_{24}^2 = \frac{q^4}{q_{14}^2}. \tag{32}$$

Thus, it becomes a two-parameter case (up to a phase).

In the case all deformation parameter are phases, i.e., $|q| = 1$, $|q_{ij}| = 1$, commutativity of quantum Minkowski length (30) occurs when in addition to (23) hold also:

$$\beta(q,\mathbf{q}) = \frac{1}{\sqrt{q^2}}, \quad q_{23}^2 q_{34}^2 = q^2 q_{24}^2. \tag{33}$$

Thus, it becomes a four-parameter case (up to a phase).

Finally, in the split and phase case commutativity of quantum Minkowski length (30) occurs when in addition to (26) hold also:

$$\beta(q,\mathbf{q}) = \frac{q_{14}^2}{q^3}, \quad q_{14}^4 = q^4. \tag{34}$$

Thus, it becomes a one-parameter case (up to a phase).

4. ACTION ON THE QUANTUM MINKOWSKI FLAG MANIFOLD

The action of $\mathcal{U}$ on the elements $\hat{\varphi}_{ijk\ell mn} \equiv z^i v^j x_-^k x_+^\ell \bar{v}^m \bar{z}^n$ of the quantum Minkowski flag manifold $\mathcal{G}_{q,\mathbf{q}}^4$ is found by combining formulae (15) and (11). For the lack of space we show only the action of X_s^-, $s = 1,2,3$:

$$\pi(X_1^-)\hat{\varphi}_{ijk\ell mn} = -[i]_q \left(\frac{q^{k+m}}{qq_{12}}\right)\left(\frac{q_{13}}{q_{12}q_{23}}\right)^{\frac{j}{2}+\frac{k}{2}}\left(\frac{q_{14}}{q_{12}q_{24}}\right)^{\frac{\ell}{2}+\frac{m}{2}}\left(\frac{q_{14}q_{23}}{q_{13}q_{24}}\right)^{\frac{n}{2}}\hat{\varphi}_{i-1,jk\ell mn}, \tag{35}$$

$$\begin{aligned}\pi(X_2^-)\hat{\varphi}_{ijk\ell mn} &= -[j]_q \left(\frac{q^{n-k-1}}{q_{23}}\right)\left(\frac{q_{13}}{q_{12}q_{23}}\right)^{\frac{i}{2}-\frac{j}{2}}\left(\frac{q_{12}q_{24}}{q_{13}q_{34}}\right)^{\ell/2}\left(\frac{q_{24}}{q_{23}q_{34}}\right)^{\frac{m}{2}+\frac{n}{2}}\hat{\varphi}_{i+1,j-1,k\ell mn}\\ &- [k]_q \left(\frac{q^{j+n-1}}{qq_{23}}\right)\left(\frac{q_{13}}{q_{12}q_{23}}\right)^{\frac{i}{2}+\frac{j}{2}}\left(\frac{q_{12}q_{24}}{q_{13}q_{34}}\right)^{\ell/2}\left(\frac{q_{24}}{q_{23}q_{34}}\right)^{\frac{m}{2}+\frac{n}{2}}\hat{\varphi}_{ij,k-1,\ell mn},\end{aligned} \tag{36}$$

$$\begin{aligned}\pi(X_3^-)\hat{\varphi}_{ijk\ell mn} = &-\left(\frac{1}{q_{34}}\right)\left(\frac{q_{14}q_{23}}{q_{13}q_{24}}\right)^{\frac{i}{2}}\left(\frac{q_{14}}{q_{13}q_{34}}\right)^{\frac{j}{2}}\left(\frac{q_{24}}{q_{23}q_{34}}\right)^{\frac{k}{2}}\left(\frac{q_{13}q_{34}}{q_{14}}\right)^{\ell/2}\left(\frac{q_{23}q_{34}}{q_{24}}\right)^{\frac{m}{2}}\left\{[\ell]_q\left(\frac{q_{13}}{q_{12}q_{23}}\right)^k q^{-m-n-1-2k}\right.\\ &\times \hat{\varphi}_{i,j+1,k,k\ell-1,mn} + q^{-\ell-n-1}[m]_q\left(\frac{q_{12}q_{23}}{q_{13}}\right)^\ell \hat{\varphi}_{ij,k+1,k\ell,m-1,n} + q^{\ell+m-1+k}[n]_q\left(\frac{q_{14}}{q_{13}q_{34}}\right)^\ell\left(\frac{q_{24}}{q_{23}q_{34}}\right)^m\hat{\varphi}_{ijk\ell m,n-1}\\ &\left. + \lambda[\ell]_q[m]_q q^{-n-1+i-k}\left(\frac{q_{14}q_{23}}{q_{13}q_{24}}\right)^\ell\left(\frac{q_{23}q_{34}}{q_{24}}\right)^{m-1}\hat{\varphi}_{ij,k+1,\ell-1,m-1,n+1}\right\}.\end{aligned} \tag{37}$$

Note that unlike other deformations ours is nontrivial as the last term of (37) contains the factor λ which becomes zero for $q = 1$.

The action of the other generators will be given elsewhere [7].

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