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Non-Perturbative Superpotentials and Discrete Torsion[1]

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Abstract—We discuss the non-perturbative superpotential in $E_8 \times E_8$ heterotic string theory on a non-simply connected Calabi–Yau manifold X, as well as on its simply connected covering space $\tilde{X}$. The superpotential is induced by the string wrapping holomorphic, isolated, genus zero curves. We show, in a specific example, that the superpotential is non-zero both on $\tilde{X}$ and on X avoiding the no-go residue theorem of Beasley and Witten. On the non-simply connected manifold X, we explicitly compute the leading contribution to the superpotential from all holomorphic, isolated, genus zero curves with minimal area. The reason for the non-vanishing of the superpotental on X is that the second homology class contains a finite part called discrete torsion. As a result, the curves with the same area are distributed among different torsion classes and their contributions do not cancel each other.

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1. It is well acknowledged that heterotic string (M-) theory on a Calabi–Yau manifold can lead to realistic low-energy physics [1–7]. Even quite advanced questions like proton stability and realistic Yukawa structure can be taken into account [7, 8]. Although these string vacua realise the correct spectrum and interactions of low-energy particle physics, there remains a fundamental problem that the associated manifold and vector bundles have moduli that generically have no potential energy. Therefore, the vacuum values of these fields can be dynamically unstable and, even if time-independent, cannot be uniquely specified, thus rendering explicit predictions of the values of supersymmetry breaking and physical parameters impossible. It follows that the stabilisation of both geometric and vector bundle moduli is one of the most important problem in heterotic string theory.

In this paper we consider heterotic string (M-) theory compactified on a Calabi–Yau manifold X to four dimensions. To specify a four-dimensional vacuum we need to specify a Calabi–Yau manifold X with the Calabi-Yau metric $g_{m\bar{n}}$,[2] a vector bundle V on X with the connection $(A_m, \bar{A}_{\bar{n}})$, and the B-field. To preserve $\mathcal{N} = 1$ supersymmetry in four dimensions the internal gauge field has to satisfy the Hermitian Yang–Mills equations

$$F_{mn} = F_{\bar{m}\bar{n}} = 0, \quad g^{m\bar{n}} F_{m\bar{n}} = 0, \tag{1}$$

and the two-form B has to be closed (modulo the α'-corrections which are not important for our purposes), $dB = 0$. It is well known that the vacuum equations for the metric, the B-field and the gauge field give rise to integration constants called moduli which appear in the low-energy field theory as massless scalar fields. More precisely, the metric and the B-field give rise to Kahler moduli and to complex structure moduli, whereas the gauge field gives rise to vector bundle moduli. All these scalar fields perturbativey do not have potential energy and can have any value.

It was realised back in the 80's that the moduli fields can receive a potential energy from non-perturbative effects. It was shown in [9, 10] that a non-perturbative contribution to the superpotential for the moduli fields can come from worldsheet instantons which are Euclidean strings wrapped on holomorphic curves in X. However, not all holomorphic curves contribute to the superpotential: in addition to being holomorphic they must be isolated and have genus zero. The purpose of this paper is to present an example of explicit calculations of the leading (in the large volume limit) non-perturbative superpotential and show that the result is non-zero. In fact, this is the first example in the literature when the superpotential can rigorously be proven to be non-zero.

2. The general expression for the superpotential induced by a string wrapping a curve C was derived in [11]. It has the following form

$$W(C) = \exp\left[-\frac{A(C)}{2\pi\alpha'} + i\int_C B\right] \frac{\mathrm{Pfaff}(\bar{\partial}_{V_C(-1)})}{[\det(\bar{\partial}_{\mathbb{O}})]^2 \det(\bar{\partial}_{NC})}. \tag{2}$$

[1] The article is published in the original.

[2] As usual, we split the six-dimensional vector index along X into its holomorphic and anti-holomorphic parts.

The expression in the exponent is the classical Euclidean action evaluated on C. In the first term, $A(C)$ is the area of the curve given by

$$A(C) = \int_C \omega, \qquad (3)$$

where ω is the Kahler form on X. Let ω_I be a basis of (1,1)-forms on X, $I = 1,\ldots,h^{1,1}$. Then we can expand $\omega = \sum_{I=1}^{h^{1,1}} t^I \omega_I$, $B = \sum_{I=1}^{h^{1,1}} \phi^I \omega_I$. Let us define the complexified Kahler moduli

$$T^I = \phi^I + i\frac{t^I}{2\pi\alpha'}. \qquad (4)$$

Then the exponential prefactor becomes

$$e^{i\alpha_I(C)T^I}, \quad \alpha_I(C) = \int_C \omega_I. \qquad (5)$$

By construction $\mathrm{Re}(i\alpha_I(C)T^I) < 0$. Pfaff in (2) is the Pfaffian of the Dirac operator which comes from integrating over the right moving fermions in the worldsheet theory. It depends on the connection A on the vector bundle V restricted to the curve C and, hence, on the moduli of the vector bundle V as well as on the complex structure of X. Since the spin bundle on a genus zero curve is $\mathbb{O}_C(-1)$, we additionally tensor V with $\mathbb{O}_C(-1)$ and denote $\mathrm{V}_C(-1) = \mathrm{V}|_C \otimes \mathbb{O}_C(-1)$. In principle, it can be explicitly expressed as a function of the gauge connection A using the WZW model [12]. However, since no explicit solutions to the Hermitian Yang-Mills equations on X are known, it is unclear how to use this in practice. Since right moving worldsheet fermions are Weyl, $\mathrm{Pfaff}(\bar{\partial}_{\mathrm{V}_C(-1)})$ is anomalous. However, this anomaly is cancelled by the variation of the B-field [13]. As the result, the Pfaffian of the Dirac operator is not a function on the moduli space of V but, rather, a section of some line bundle. In the denominator in (2), $\det(\bar{\partial}_{NC})$ comes from integrating over bosonic fluctuations and is the determinant of the $\bar{\partial}$-operator on the normal bundle to the curve C. Finally, $[\det'(\bar{\partial}_{\mathbb{O}})]^2$ is the determinant of the $\bar{\partial}$-operator on the trivial line bundle which is a constant.

In general, a given homology class of X contains more than one holomorphic, isolated, genus zero curve. The number of these curves is referred to as to Gromov–Witten invariant. All such curves in the same homology class have the same area, the same classical action and the same exponential prefactor in (2). However, the 1-loop determinants, in general, are different. Hence, the contribution to the superpotential from all curves C_i in the homology class $[C]$ of the curve C is given by (for simplicity, we remove the constant factor $[\det'(\bar{\partial}_{\mathbb{O}})]^{-2}$)

$$W([C]) = \exp\left[-\frac{A(C)}{2\pi\alpha'} + i\int_C B\right]\sum_{i=1}^{n_{[C]}} \frac{\mathrm{Pfaff}(\bar{\partial}_{\mathrm{V}_{C_i}(-1)})}{[\det \bar{\partial}_{\mathbb{O}_{C_i}(-1)}]^2}, \qquad (6)$$

where $n_{[C]}$ is the number of the holomorphic, isolated, genus zero curves in the homology class $[C]$. To find the complete non-perturbative superpotential W, we then have to sum over all homology classes. That is,

$$W = \sum_{[C]\in H_2(X)} W([C]). \qquad (7)$$

3. In [14] Beasley and Witten showed that, under some rather general assumptions, the sum (6) must vanish for each homology class $[C]$. Let us review their assumptions. Let $\tilde{X}$ be a complete intersection Calabi–Yau threefold in the product of projective spaces[3] $\mathcal{A} = \mathbf{P}^{n_1} \times \ldots \times \mathbf{P}^{n_a}$. That is, $\tilde{X}$ is given by a set of polynomial equations $p_1 = 0, \ldots, p_m = 0$ where $\sum_{i=1}^a n_i - m = 3$. Additionally, assume that the Kahler form $\omega_{\tilde{X}}$ descends from the ambient space, that is, $\omega_{\tilde{X}} = \omega_A|_{\tilde{X}}$, and that the vector bundle $\tilde{V}$ on $\tilde{X}$ is obtained as a restriction of a vector bundle $\mathcal{V}$ on $\mathcal{A}$, $\tilde{V} = \mathcal{V}|_{\tilde{X}}$. Then, it was shown by Beasley and Witten that if these assumptions are satisfied, the sum (6) vanishes for any homology class. This result was proven in [14] and interpreted as a residue theorem. The assumptions of Beasley and Witten are rather general, which means that in a large class of heterotic string models a non-perturbative superpotential cannot be generated. This raises a question of whether moduli in heterotic compactifications can ever be completely stabilised. The aim of this paper is to present explicit examples where the non-perturbative superpotential is indeed non-zero.

As we have said, in the analysis of Beasley and Witten in [14] there is the assumption that $\omega_{\tilde{X}} = \omega_{\mathcal{A}}|_{\tilde{X}}$. It then follows that, in their theorem, the area of all curves in (6), (7) is measured using the Kahler form $\omega_{\mathcal{A}}$ on $\mathcal{A}$ restricted to $\tilde{X}$. However, there are cases when this restriction is not the same as the physical Kahler form on $\tilde{X}$. Indeed, it is possible that $h^{1,1}(\tilde{X})$ is not the same as $h^{1,1}(\mathcal{A})$ because there can be classes in $\tilde{X}$ which do not come as a restriction of classes from the ambient space. Hence, the residue theorem, strictly speaking, is valid only if $h^{1,1}(\tilde{X}) = h^{1,1}(\mathcal{A})$. If $h^{1,1}(\tilde{X}) > h^{1,1}(\mathcal{A})$ the residue theorem cannot be directly applied. In heterotic string theory, the area is measured using the actual Kahler form $\omega_{\tilde{X}}$ on $\tilde{X}$. As a result, the curves which have the same area with respect to $\omega_{\mathcal{A}}|_{\tilde{X}}$ might have different area with respect to $\omega_{\tilde{X}}$ and, hence, might lie in different homology classes. Below, we will give an example where the can-

[3] The results of Beasley and Witten are also expected to be valid for complete intersections in toric spaces.

cellation in the residue theorem cannot happen simply because each curve is unique in its homology class.

4. Our discussion so far has been missing an important ingredient called *discrete torsion*. In general, for an arbitrary complex manifold, X, the second homology group with integer coefficients is of the form

$$H_2(\mathrm{X}, \mathbf{Z}) = \mathbf{Z}^k \oplus \mathrm{G}_{tor}, \quad k > 0, \tag{8}$$

where $\mathbf{Z}^k$ is the free part and G_{tor} is a finite group called discrete torsion. For example, a discrete torsion factor of $H_2(\mathrm{X}, \mathbf{Z})$ can arise when X is a quotient of another Calabi–Yau manifold by a freely acting discrete isometry group K as we will discuss below. The existence of the torsion classes affects the B-field. Since B-field is a closed 2-form, a four-dimensional heterotic vacuum is specified by a choice of its field strength H which must vanish in $H^3(\mathrm{X}, \mathbf{R})$. However, it does not mean that it vanishes in $H^3(\mathrm{X}, \mathbf{Z})$. If H defines a non-trivial torsion element in $H^3(\mathrm{X}, \mathbf{Z})$ its potential B is not globally defined and the exponential prefactor in (2) has to be modified. As was shown in [15, 16] it has to be replaced with

$$e^{i\alpha_I(C)T^I} \to e^{i\alpha_I(C)T^I} \prod_{a=1}^{r} \chi_a^{\beta_a(C)}. \tag{9}$$

Here $\beta_1, \ldots, \beta_r$ are the generators of G_{tor}, $\beta_a(C)$ are their values on the curve C and χ_a are characters of G_{tor}. The choice of characters depends on the choice of H, that is on a (discrete) choice of a four-dimensional vacuum.

Let us now refine eq. (2) in the presence of discrete torsion. Let $[C]$ be the homology class of the curve C in $H_2(\mathrm{X}, \mathbf{R}) = \mathbf{R}^k$. As we have just discussed, the curves in $[C]$ do not necessarily lie in the same homology class in $H_2(\mathrm{X}, \mathbf{Z})$ because they might belong to different torsion classes. Curves belonging to different torsion classes pick up different characters. Hence, equation (2) is modified to become

$$W([C]) = e^{i\alpha_I(C)T^I} \sum_{i=1}^{n_{[C]}} \frac{\mathrm{Pfaff}(\bar{\partial}_{\mathrm{V}_{C_i}(-1)})}{[\det \bar{\partial}_{\mathbb{O}_{C_i}(-1)}]^2} \prod_{a=1}^{r} \chi_a^{\beta_a(C_i)}, \quad [C] \in H_2(X, \mathbf{R}). \tag{10}$$

To find the complete non-perturbative superpotential, we have to sum over all homology classes $[C] \in H_2(\mathrm{X}, \mathbf{R})$.

5. We will now consider a specific Calabi–Yau manifold $\tilde{X}$ called the Schoen manifold. It is a complete intersection in $\mathcal{A} = \mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2$ given by the following equations

$$\begin{aligned} F_1 &= t_0(x_0^2 + x_1^3 + x_2^3) + t_1(x_0 x_1 x_2) = 0, \\ F_2 &= (\lambda_1 t_0 + t_1)(y_0^2 + y_1^3 + y_2^3) \\ &\quad + (\lambda t_0 + \lambda_3 t_1)(y_0 y_1 y_2) = 0. \end{aligned} \tag{11}$$

Here $[t_0, t_1]$, $[x_0, x_1, x_2]$, $[y_0, y_1, y_2]$ are the homogeneous coordinates on $\mathbf{P}^1$, $\mathbf{P}^2$ and $\mathbf{P}^2$ respectively. Let us state some relevant mathematical properties of the Schoen manifold. First, $\pi_1(\tilde{X}) = 0$, that is $\tilde{X}$ is simply connected. Second, $h^{1,1}(\tilde{X}) = 19 > 3 = h^{1,1}(\mathcal{A})$, which means that there are 16 (1,1) classes on $\tilde{X}$ that do not come from the ambient space. Third, $H_2(\tilde{X}, \mathbf{Z}) = \mathbf{Z}^{19}$ which means that there is no torsion. The Schoen manifold (11) admits an action of a discrete $\mathbf{Z}_3 \times \mathbf{Z}_3$ symmetry whose generators g_1 and g_2 act as follows

$$\begin{aligned} g_1 &: [x_0, x_1, x_2] \to [x_0, e^{2\pi i/3} x_1, e^{4\pi i/3} x_2], \\ &\quad [y_0, y_1, y_2] \to [y_0, e^{2\pi i/3} y_1, e^{4\pi i/3} y_2], \\ g_2 &: [x_0, x_1, x_2] \to [x_1, x_2, x_0], \\ &\quad [y_0, y_1, y_2] \to [y_1, y_2, y_3]. \end{aligned} \tag{12}$$

This allows us to define another Calabi–Yau manifold $X = \tilde{X}/\mathbf{Z}_3 \times \mathbf{Z}_3$ with properties [17]: $\pi_1(X) = \mathbf{Z}_3 \times \mathbf{Z}_3$, $H_2(X, \mathbf{Z}) = \mathbf{Z}^3 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_3$. The later means that X has $\mathbf{Z}_3 \oplus \mathbf{Z}_3$ torsion. The classes which descend from $\tilde{X}$ to X are the invariant classes in $\tilde{X}$ represented by invariant differential forms. One can show [16] that they are given by a restriction of the Kahler forms on $\mathbf{P}^1$, $\mathbf{P}^2$ and $\mathbf{P}^2$, $J_I = \mathcal{J}_I|_{\tilde{X}}$. The Kahler form on $\tilde{X}$ is given by

$$\omega_{\tilde{X}} = \omega_{\mathcal{A}}|_{\tilde{X}} + \Delta\omega_{\tilde{X}} = 2\pi\alpha' \sum_{i=1}^{3} \mathrm{Im}(T^I) J_I + \Delta\omega_{\tilde{X}}, \tag{13}$$

where $\Delta\omega_{\tilde{X}}$ is the contribution from the additional 16 classes, and the Kahler form on X is given in terms of the invariant classes

$$\omega_X = 2\pi\alpha' \sum_{i=1}^{3} \mathrm{Im}(T^I) J_I, \tag{14}$$

where by slightly abusing notation we denote the basis of (1,1)-forms on X also by J_I and the Kahler parameters by T^I.

The Gromov–Witten invariant of X can be computed using the type II prepotential [17]. Let $[C_1], [C_2], [C_3]$ be co-invariant homology classes dual to J_1, J_2, J_3. Let us denote

$$p = e^{iT^1}, \quad q = e^{iT^2}, \quad r = e^{iT^3}, \tag{15}$$

where the exponents are just the integral of the complexified Kahler form on X evaluated on the basis classes $[C_1], [C_2], [C_3]$. In addition, we have two torsion generators β_1, β_2 and, hence, any curve C is labeled by the integers

$$\beta_1(C) = m_1, \quad \beta_2(C) = m_2, \quad m_1, m_2 = 0, 1, 2. \tag{16}$$

Thus, any curve in the homology class $[C] \in H_2(X, \mathbf{Z})$ is labeled by the set of integers $(n_1, n_2, n_3, m_1, m_2) \in \mathbf{Z}^3 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_3$. The type II prepotential is then given by the following expression

$$\mathcal{F}_X = \sum_{[C] \in H_2(X,\mathbf{Z})} n_{[C]} \mathrm{Li}_3(p^{n_1} q^{n_2} r^{n_3} \chi_1^{n_1} \chi_2^{n_1}). \tag{17}$$

Knowing $\mathcal{F}_X$ and expanding the poly-logarithm we can read off the Gromov–Witten invariants $n_{[C]}$. The prepotential $\mathcal{F}_X$ was computed in [17] and here we will just quote the result to the lowest order in p, q, r which corresponds to the leading order in the large volume limit:

$$\mathcal{F}_X = p(1 + \chi_1 + \chi_1^2)(1 + \chi_2 + \chi_2^2)(1 + \mathbb{O}(q) + \mathbb{O}(r)) + \mathbb{O}(p^2). \tag{18}$$

From this result it follows that the leading superpotential behaves as e^{iT^1} and that there are 9 curves which contribute to it. All these curves have the same (complexified) area iT^1 but lie in 9 different homology classes once torsion is taken into account.

Due to the $\mathbf{Z}_3 \times \mathbf{Z}_3$ symmetry the above 9 curves in X originate from 81 curves in $\tilde{X}$. These 81 curves can be constructed very explicitly. Since all of them originate from $\mathbf{P}^1 \in \mathcal{A}$ parameterised by $[t_0, t_1]$, to find them we simply solve $F_1 = 0$, $F_2 = 0$ for arbitrary t_0, t_1 which gives

$$x_0 x_1 x_2 = 0, \quad x_0^3 + x_1^3 + x_2^3 = 0, \qquad y_0 y_1 y_2 = 0, \quad y_0^3 + y_1^3 + y_2^3 = 0. \tag{19}$$

This system has 81 solutions, each defining a curve in $\tilde{X}$ of the form $\mathbf{P}^1 \times s$, where s is a point in $\mathbf{P}^2 \times \mathbf{P}^2$. All these curves are holomorphic, isolated, genus zero curves. Furthermore, they all have the same area with respect to the Kahler form restricted from the ambient space $\omega_{\mathcal{A}}|_{\tilde{X}}$. In fact, they form the full set of curves whose (complexified) area is iT^1. However, using the mathematical properties of the Schoen manifold one can prove that they all lie in different homology classes and, hence, have different area with respect to the actual Kahler form $\omega|_{\tilde{X}}$ on $\tilde{X}$.

Since each of these curves is unique in its homology class it follows that the non-perturbative superpotential in heterotic string theory compactified on $\tilde{X}$ is non-zero as long as $\mathrm{Pfaff}(\bar{\partial}_{V_{C_i}(-1)}) \neq 0$ at least for one of these curves.

6. Now we will consider a specific model for a vector bundle. We will start with a vector bundle $\mathcal{V}$ on $\mathcal{A}$ which we then restrict to $\tilde{X}$ to define $\tilde{V} = \mathcal{V}|_{\tilde{X}}$. We will also choose $\mathcal{V}$ and $\tilde{V}$ to be equivariant under the $\mathbf{Z}_3 \times \mathbf{Z}_3$ action to define a bundle $V = \tilde{V}/\mathbf{Z}_3 \times \mathbf{Z}_3$ on X.

We will choose the vector bundle to have structure group $SU(3)$ which corresponds to low-energy field theory with gauge group E_6 and construct it as follows

$$0 \to \mathcal{L}_1 \to \mathcal{W} \to \mathcal{L}_2 \to 0, \qquad 0 \to \mathcal{W} \to \mathcal{V} \to \mathcal{L}_3 \to 0, \tag{20}$$

where

$$\mathcal{L}_1 = \mathbb{O}_{\mathcal{A}}(-2, 2, 1), \quad \mathcal{L}_2 = \mathbb{O}_{\mathcal{A}}(0, 1, -1), \quad \mathcal{L}_3 = \mathbb{O}_{\mathcal{A}}(2, -3, 0). \tag{21}$$

Knowing $\mathcal{W}$ and $\mathcal{V}$ we then define $\tilde{W}$, $\tilde{V}$ and W, V as discussed above. One can show [16] that the moduli space of thus constructed bundle V is $\mathbf{P}^1 \times \mathbf{P}^{12}$. The first factor corresponds to the moduli space of W and will not play any role in our analysis below, so we will ignore it. A point $v \in \mathbf{P}^{12}$ can be parameterised explicitly in terms of the invariant polynomials on the ambient space:

$$v = t_0^2 f_1(\mathbf{x}, \mathbf{y}) + t_0 t_1 f_2(\mathbf{x}, \mathbf{y}) + t_1^2 f_3(\mathbf{x}, \mathbf{y}). \tag{22}$$

Here f_1, f_2, f_3 are homogeneous polynomials on $\mathbf{P}^2 \times \mathbf{P}^2$ of degree (5, 1)

$$f_1 = \sum_{\alpha=1}^{7} a_\alpha E_\alpha, \quad f_2 = \sum_{\alpha=1}^{7} b_\alpha E_\alpha, \quad f_3 = \sum_{\alpha=1}^{7} c_\alpha E_\alpha, \tag{23}$$

with E_α being a convenient basis of such polynomials and $a_\alpha, b_\alpha, c_\alpha$ being the bundle moduli. The 21 moduli just introduced are not independent, in fact, they satisfy 8 linear relations which can be found in [16]. This gives 13 homogeneous coordinates parameterising $\mathbf{P}^{12}$.

Let us now compute the Pfaffians in this model for our 81 curves in $\tilde{X}$ and 9 curves in X.[4] Since no explicit solutions to the Hermitian Yang–Mills equations are know we will use the algebraic approach developed in [12, 13, 16]. Since $\mathrm{Pfaff}(\bar{\partial}_{V_C(-1)})$ is a holomorphic section on $\mathbf{P}^{12}$ it is a homogeneous polynomial in $a_\alpha, b_\alpha, c_\alpha$ which is uniquely determined (up to a coefficient) by its zeros. Hence, we have to derive the polynomial equation in the moduli space which governs the existence of zero mode of $\bar{\partial}_{V_C(-1)}$. The details of the derivation can be found in [16] and here we will just state the results.

[4] For simplicity, we will work for a fixed complex structure which makes the Pfaffians to be the only non-trivial one-loop determinants.

In the theory defined by $(\tilde{X},\tilde{V})$ the leading superpotential comes from 81 curves of the form $[t_0,t_1]\times s_i$, where s_i are the 81 points solving the system (19). Due to the $\mathbf{Z}_3\times\mathbf{Z}_3$ these curves break into 9 orbits with 9 curves in each orbit. After we mod out by $\mathbf{Z}_3\times\mathbf{Z}_3$ the entire orbit becomes a single curve in X. The symmetry also implies that all curves in the same orbit have the same Pfaffian, that is we have to compute the Pfaffians only for 9 curves representing the orbits. It also follows from the symmetry that

$$\mathrm{Pfaff}_X(\bar{\partial}_{V_{C_i}(-1)}) = \mathrm{Pfaff}_{\tilde{X}}(\bar{\partial}_{\tilde{V}_{C_i}(-1)}), \tag{24}$$

which means that all calculations in the theory on X can be done on the covering manifold. Aalysing the zeros of the Pfaffians one can show that [16]

$$\begin{aligned}\mathrm{Pfaff}_{\tilde{X}}(\bar{\partial}_{V_C(-1)}) &\sim (f_1f_3 - f_2^2)(\mathbf{x},\mathbf{y})\\ &= \sum_{\alpha,\beta=1}^{7}(a_\alpha c_\beta - b_\alpha b_\beta)E_\alpha E_\beta(\mathbf{x},\mathbf{y}),\end{aligned} \tag{25}$$

up to a numerical coefficient which we are not able to compute by our method. Denoting

$$\mathcal{R}_{\tilde{X},i} = (f_1f_3 - f_2^2)(s_i), \quad i = 1,\ldots,9, \tag{26}$$

we obtain the following 9 polynomials $\mathcal{R}_i$ corresponding to 9 independent (that is unrelated by the $\mathbf{Z}_3\times\mathbf{Z}_3$ action) solutions to (19)

$$\begin{aligned}
\mathcal{R}_{\tilde{X},1} &= -(2b_1 - b_2 - b_3)^2 + (2a_1 - a_2 - a_3)(2c_1 - c_2 - c_3),\\
\mathcal{R}_{\tilde{X},2} &= -(b_2 + b_3\zeta^2 + b_1\zeta)^2 + (a_2 + a_3\zeta^2 + a_1\zeta)(c_2 + c_3\zeta^2 + c_1\zeta),\\
\mathcal{R}_{\tilde{X},3} &= -(b_2 + b_3\zeta + b_1\zeta^2)^2 + (a_2 + a_3\zeta + a_1\zeta^2)(c_2 + c_3\zeta^2 + c_1\zeta),\\
\mathcal{R}_{\tilde{X},4} &= -(-b_1 + b_3 + b_5 - b_6)^2 + (-a_1 + a_3 + a_5 - a_6)(-c_1 + c_3 + c_5 - c_6),\\
\mathcal{R}_{\tilde{X},5} &= -(-b_1 + b_2 - b_5 + b_6)^2 + (-a_1 + a_2 - a_5 + a_6)(-c_1 + c_2 - c_5 + c_6),\\
\mathcal{R}_{\tilde{X},6} &= -(-b_1 + b_3 + (b_5 - b_6)\zeta^2)^2 + (-a_1 + a_3 + (a_5 - a_6)\zeta^2)(-c_1 + c_3 + (c_5 - c_6)\zeta^2),\\
\mathcal{R}_{\tilde{X},7} &= -(-b_1 + b_2 - (b_5 - b_6)\zeta^2)^2 + (-a_1 + a_2 - (a_5 - a_6)\zeta^2)(-c_1 + c_2 - (c_5 - c_6)\zeta^2),\\
\mathcal{R}_{\tilde{X},8} &= -(-b_1 + b_2 - (b_5 - b_6)\zeta)^2 + (-a_1 + a_2 - (a_5 - a_6)\zeta)(-c_1 + c_2 - (c_5 - c_6)\zeta),\\
\mathcal{R}_{\tilde{X},9} &= -(-b_1 + b_3 + (b_5 - b_6)\zeta)^2 + (-a_1 + a_3 + (a_5 - a_6)\zeta)(-c_1 + c_3 + (c_5 - c_6)\zeta),
\end{aligned} \tag{27}$$

where $\zeta = e^{2\pi i/3}$. Let us now introduce the proportionality coefficient into (25). For each of our 9 curves, we denote it by $A_{\tilde{X},i}$ where $i = 1,\ldots,9$. That is, we have

$$\mathrm{Pfaf}f_{\tilde{X}}(\bar{\partial}_{\tilde{V}_{C_i}(-1)}) = A_{\tilde{X},i}R_{\tilde{X},i}. \tag{28}$$

We now conclude that in the theory on X the non-perturbative superpotential is non-zero because each curve is in its own homology class and the Pfaffians are not identically zero.

Though we are not able to compute the coefficients $A_{\tilde{X},i}$ we can constrain them. It follows from the residue theorem that if measure the area of all curves using the Kahler form restricted from the ambient space the total result must vanish. It then follows that

$$\sum_{i=1}^{9} A_{\tilde{X},i}\mathcal{R}_{\tilde{X},i} = 0. \tag{29}$$

Let us stress that eq. (29) does not imply that the superpotentail in the heterotic string theory on $\tilde{X}$ vanishes because in (29) we are summing the Pfaffians of curves lying in different homology classes and having different area with respect to the proper Kahler form $\omega_{\tilde{X}}$. Hence, in the superpotential these Pfaffians will be weighted with different exponential prefactors and cannot cancel each other. Eq. (29) constrains the coefficients $A_{\tilde{X},i}$. It is possible to satisfy eq. (29) if and only if the polynomials $\mathcal{R}_{\tilde{X},i}$ in (27) are linearly dependent which is a non-trivial consistency check of our calculations. It is possible to check that these polynomials are indeed linearly dependent and it possible to adjust the parameters $A_{\tilde{X},i}$ so that the sum in (29) vanishes. The precise constraints on the coefficient $A_{\tilde{X},i}$ can be found in [16].

In the theory defined by (X,V) the leading superpotential is given by

$$W_X = e^{iT^1}\sum_{i=1}^{9} A_{\tilde{X},i}\mathcal{R}_{\tilde{X},i}\chi_1^{m_1}\chi_2^{m_1}, \tag{30}$$

where we used the relation (24) and the characters depend on the choice of the characteristic class of the H-field. In the absence of torsion the superpotential would be zero by the residue theorem as explained above. Choosing the vacuum such that at least one of the characters is non-trivial the sum (30) is no longer zero because

$$\sum_{i=1}^{9} A_{\tilde{X},i}\mathcal{R}_{\tilde{X},i} = 0 \Rightarrow \sum_{i=1}^{9} A_{\tilde{X},i}\mathcal{R}_{\tilde{X},i}\chi_1^{m_1}\chi_2^{m_1} \neq 0. \tag{31}$$

This gives an explicit proof that the leading superpotential is non-zero. Let us emphasise that the key reason in the proof is discrete torsion.

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