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**ELEMENTARY PARTICLES AND FIELDS**  
**Theory**

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**General Principles of Hamiltonian Formulations  
of the Metric Gravity**

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**Abstract**—Principles of successful Hamiltonian approaches, which were developed to describe free gravitational field(s) in the metric gravity, are formulated and discussed. By using the standard  $\Gamma$ – $\Gamma$  Lagrangian  $\mathcal{L}_{\Gamma-\Gamma}$  of the metric GR we properly introduce all momenta of the metric gravitational field and derive the both canonical  $H_C$  and total  $H_t$  Hamiltonians of the metric GR. We also developed an effective method which is used to determine various Poisson brackets between analytical functions of the basic dynamical variables, i.e., generalized coordinates  $g_{\alpha\beta}$  and momenta  $\pi^{\mu\nu}$ . In general, such variables can be chosen either from the straight  $\{g_{\alpha\beta}, \pi^{\mu\nu}\}$ , or dual  $\{g^{\alpha\beta}, \pi_{\mu\nu}\}$  sets of symplectic dynamical variables which always arise (and complete each other) in any Hamiltonian formulation developed for the coupled system of tensor fields. By applying canonical transformation(s) of dynamical variables we reduce the canonical Hamiltonian  $H_C$  to its natural form. The natural form of canonical Hamiltonian provides numerous advantages in actual applications to the metric GR, since the general theory of dynamical systems with such Hamiltonians is well developed. Furthermore, many analytical and numerically exact solutions have been found and described in detail for dynamical systems with the Hamiltonians already reduced to their natural forms. In particular, reduction of the canonical Hamiltonian  $H_C$  to its natural form allows one to derive the Jacobi equation for the free gravitational field(s), which takes a particularly simple form.

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## 1. INTRODUCTION

In 1958 Dirac published his famous Hamiltonian formulation of the metric General Relativity (also the metric gravity, or GR, for short) [1]. Since then and for a very long time that Dirac’s formulation was known as the only correct Hamiltonian approach ever developed for the metric gravity. In particular, only by using this Hamiltonian formulation and all primary and secondary constraints arising in this Dirac’s approach, one is able to restore the complete and correct gauge symmetry (diffeomorphism) of the free gravitational field(s). A different Hamiltonian formulation of the metric gravity published earlier in [2] was overloaded with numerous mistakes, which can easily be found, e.g., in all secondary constraints derived in [2]. Moreover, some important steps of the complete Hamiltonian procedure, developed earlier by Dirac in [3], were missing in [2]. For instance, the closure of Dirac procedure [3] was not demonstrated. In reality, it is impossible to show such a closure with wrong secondary constraints, but after reading [2] one can get an impression that authors did not understand why they need to do this, in principle. The complete and correct version of the Hamiltonian formulation

of the metric gravity, originally proposed in [2], was re-developed and substantially corrected only in 2008 [4] by Kiriushcheva and Kuzmin. Below, to respect this fact we shall call the Hamiltonian formulaion of the metric GR developed in [4] by the K&K approach. This approach also allows one to restore the complete diffeomorphism as a correct and unique gauge symmetry of the free gravitational field.

Note that after publication [4] there were two different and non-contradictory Hamiltonian formulations of the metric gravity. Therefore, it was very interesting to investigate relations between these two approaches. In [5] we have shown that the original (or Dirac) formulation of the metric GR and alternative K&K-formulation are related to each other by a canonical transformation of dynamical variables of the problem, i.e., by a transformation of the generalized ‘coordinates’  $g_{\alpha\beta}$  and corresponding ‘momenta’  $\pi^{\mu\nu}$ . Furthermore, such a canonical transformation in metric GR always has some special and relatively simple form (more details can be found in [5]). After an obvious success of our analysis in [5] the following question has immediately became interesting: is it possible to derive another canonical transformation of dynamical variables in the metric gravity which can reduce the canonical Hamiltonian  $H_C$  of the metric

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GR derived in [4] to some relatively simple forms, e.g., to the forms which are well known in classical mechanics? If the answer is ‘Yes’, then we can use the solutions known for classical Hamiltonian systems to solve new gravitational problems, rigorously predict properties of certain gravitational systems, etc. Below, to answer this question we present the new canonical transformation of dynamical variables, i.e., generalized coordinates and momenta, in the metric General Relativity. This new canonical transformation is also a very special and unique, since it reduces the canonical Hamiltonian  $H_C$  of metric GR to the natural form which is almost identical to the natural form of many ‘regular’ Hamiltonians already known in analytical mechanics of the potential dynamical systems.

In this paper we want to formulate and discuss all essential principles of the Hamiltonian formulation(s) of the metric General Relativity. To achieve this goal, in the next two Sections we introduce the  $\Gamma$ - $\Gamma$  Lagrangian  $L_{\Gamma-\Gamma}$  of the metric gravity. By using this Lagrangian  $L$  we define the corresponding momenta  $\pi^{\alpha\beta}$ . At the next stage we apply the Legendre transformation to exclude velocities, obtain the primary constraints and construct the canonical  $H_C$  and total  $H_t$  Hamiltonians of the metric General Relativity. All formulas and expressions derived in next two Sections and even logic used there are pretty standard for any Hamiltonian formulation of the metric gravity. Moreover, some of these formulas were derived and discussed in a number of earlier studies (see, e.g., [4] and [6]). Nevertheless, the two following Sections are important to make and keep this study completely independent of other publications and united by a central idea to illustrate the power of canonical transformations for Hamiltonian systems. The fundamental Poisson brackets are defined and calculated in Section 3. These brackets are the main working tools to obtain accurate (and even exact) solutions of many gravitational problems and perform research of various Hamiltonian gravitational systems, including gravitational field(s) defined in the metric General Relativity. In particular, our fundamental and secondary Poisson brackets are used to investigate a few fundamental problems currently known in metric GR (see, Section 4). Section 5 is the central part of this study, since here the canonical Hamiltonian  $H_C$  of the metric GR is reduced to its natural form. Here we also illustrate a number of advantages of the normal form of the canonical Hamiltonian  $H_C$  for numerous problems known in the metric GR. A few directions for future development of the Hamiltonian formulations of metric GR are also discussed there. In Section 6 we derive the Jacobi equation for the metric gravity by using our new dynamical and canonical variables  $g_{\alpha\beta}$  and  $P^{\mu\nu}$ . Concluding remarks can be

found in the last Section. This paper also includes three Appendixes. In Appendix A we discuss relations between dynamical variables which are used in our and Dirac formulations of the metric General Relativity, while in the Appendix B we show that dynamical variables of modern geometro-dynamics are not (and cannot be) canonical variables of the metric GR. Appendix C contains explanation of some important ‘technical’ details of our procedure which could not be included in the main text.

Now, we need to introduce a few principal notations which are extensively used below. In particular, in this study the notation  $g_{\alpha\beta}$  stands for the covariant components of the metric tensor (see, e.g., [7]) which are dimensionless values. The determinant of this metric tensor is  $g$  which is the negative value, but  $-g$  is always positive. It is assumed below that an arbitrary Greek index varies between 0 and  $d - 1$ , while an arbitrary Latin index varies between 1 and  $d - 1$ , where  $d$  designates the total dimension of our space-time manifold ( $d \geq 3$ ). The case of two-dimensional space-time, i.e., when  $d = 2$ , differs substantially from the regular cases ( $d \geq 3$ ) and requires a separate consideration (see, e.g., [8] and references therein). In this study we shall not consider the case of two-dimensional metric gravity. The quantities and tensors such as  $B^{((\alpha\beta)\gamma|\mu\nu\lambda)}$ ,  $I_{mnpq}$ , etc, applied below, have been defined in earlier papers [1, 4, 5 and 6]. In this study the definitions of all these quantities and tensors are exactly the same as in the mentioned papers and there is no need to repeat them. The short notations  $g_{\alpha\beta,k}$  and  $g_{\gamma\rho,0}$  are used below for the spatial and temporal derivatives, respectively, of the corresponding components of the metric tensor. Any expression which contains a pair of identical (or repeated) indexes, where one index is covariant and another index is contravariant, means summation over this ‘dummy’ index. This convention is very useful and drastically simplifies many formulas derived in metric GR.

## 2. $\Gamma$ - $\Gamma$ LAGRANGIAN OF THE METRIC GENERAL RELATIVITY

In this Section we introduce the Lagrangian of the metric General Relativity. Formally, such a Lagrangian (or Lagrangian density) should coincide with the integrand in the Einstein-Hilbert integral-action  $L_{EH}$  (see, e.g., [9] and [10]) which equals to the product of scalar curvature of the  $d$  dimensional space  $R = g^{\alpha\beta} R_{\alpha\beta}$  and  $\sqrt{-g}$ , i.e.,  $L_{EH} = \sqrt{-g} g^{\alpha\beta} R_{\alpha\beta}$ . Here  $R_{\alpha\beta}$  is the Ricci tensor (in old books and papers (see, e.g., [7]), it was called the Einstein tensor)

$$R_{\alpha\beta} = \frac{\partial \Gamma_{\alpha\beta}^{\gamma}}{\partial x^{\gamma}} - \frac{\partial \Gamma_{\alpha\gamma}^{\gamma}}{\partial x^{\beta}} + \Gamma_{\alpha\beta}^{\gamma} \Gamma_{\gamma\lambda}^{\lambda} - \Gamma_{\alpha\gamma}^{\lambda} \Gamma_{\beta\lambda}^{\gamma},$$

where  $\Gamma_{\alpha\beta}^{\gamma} = \frac{1}{2}g^{\gamma\nu} \left( \frac{\partial g_{\nu\alpha}}{\partial x^{\beta}} + \frac{\partial g_{\nu\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}} \right)$  are the Cristoffel symbols (see, e.g., [7] and [10]). It is easy to see that this Lagrangian, which is often called the Einstein-Hilbert Lagrangian, contains a few derivatives of the second order  $\frac{\partial^2 g_{\alpha\beta}}{\partial x^{\gamma} \partial x^{\lambda}}$  and cannot be used directly in the principle of least action. However, all these second-order derivatives are included in the Lagrangian  $L_{\text{EH}}$  only as a linear combination with the constant coefficients, which equal +1, or -1. Because of such a linearity the invariant integral  $\int R\sqrt{-g}d\Omega$  (gravitational action) can be transformed by means of Gauss theorem to the integral which does not include any second-order derivatives. Indeed, we can represent this gravitational action integral in the form

$$\begin{aligned} & \int R\sqrt{-g}d\Omega \\ &= \int g^{\alpha\beta} \left( \Gamma_{\alpha\gamma}^{\lambda} \Gamma_{\beta\lambda}^{\gamma} - \Gamma_{\alpha\beta}^{\gamma} \Gamma_{\gamma\lambda}^{\lambda} \right) \sqrt{-g}d\Omega \\ &+ \int \frac{\partial \left[ \sqrt{-g} \left( g^{\alpha\beta} \Gamma_{\alpha\beta}^{\gamma} - g^{\alpha\gamma} \Gamma_{\alpha\beta}^{\beta} \right) \right]}{\partial x^{\gamma}} d\Omega, \end{aligned}$$

where the integrand of the first integral on the right-hand side of this equation contains only components of the metric tensor and their first-order derivatives, while the second integral has the form of a divergence of the vector-like quantity  $\sqrt{-g} \left( g^{\alpha\beta} \Gamma_{\alpha\beta}^{\gamma} - g^{\alpha\gamma} \Gamma_{\alpha\beta}^{\beta} \right)$ . This second integral is transformed by applying Gauss theorem into an integral over a hypersurface surrounding the  $d$  dimensional volume over which the integration is carried out in other two integrals. When we vary the gravitational action, the variation of this (second) term on the the right-hand side of the last equation vanishes, since in respect to the principle of least action, all variations of the (varied) field at the limits of the region of integration must be zero.

Now, we may write for the variations of all terms in the previous equation

$$\begin{aligned} \delta \int R\sqrt{-g}d\Omega &= \delta \int L_{\Gamma-\Gamma}d\Omega \text{ or} \\ \frac{\delta (R\sqrt{-g})}{\delta g_{\mu\nu}} &= \frac{\delta L_{\Gamma-\Gamma}}{\delta g_{\mu\nu}}, \end{aligned} \quad (1)$$

where the notation  $\delta$  means variation, while the notation  $\frac{\delta F}{\delta g_{\mu\nu}}$  means the variational derivative (or Lagrange derivative). Also in this equation the symbol  $L_{\Gamma-\Gamma} = \sqrt{-g}g^{\alpha\beta} \left( \Gamma_{\alpha\gamma}^{\lambda} \Gamma_{\beta\lambda}^{\gamma} - \Gamma_{\alpha\beta}^{\gamma} \Gamma_{\gamma\lambda}^{\lambda} \right)$  stands for the ‘regular’  $\Gamma-\Gamma$  Lagrangian density (or Lagrangian, for short) of the metric gravity which plays a central role in any Hamiltonian approach developed for the metric gravity. As follows from

this equation the variational derivative of the  $L_{\Gamma-\Gamma}$  Lagrangian is a true tensor, while the original  $L_{\Gamma-\Gamma}$  Lagrangian is not a true scalar. Other properties of the  $L_{\Gamma-\Gamma}$  Lagrangian are mentioned below. The equality, Eq. (1), expresses the fact that we have replaced the ‘singular’ Einstein-Hilbert Lagrangian ( $L_{\text{EH}} = \sqrt{-g}R$ ) by the ‘regular’  $\Gamma-\Gamma$  Lagrangian  $L_{\Gamma-\Gamma} = \sqrt{-g}g^{\alpha\beta} \left( \Gamma_{\alpha\gamma}^{\lambda} \Gamma_{\beta\lambda}^{\gamma} - \Gamma_{\alpha\beta}^{\gamma} \Gamma_{\gamma\lambda}^{\lambda} \right)$  which is variationally equivalent to the original Einstein-Hilbert Lagrangian and contains no second-order derivative. There is an obvious logical gap here, since we have replaced the actual Einstein-Hilbert Lagrangian by its variational equivalent, i.e., by the  $\Gamma-\Gamma$  Lagrangian. Briefly, this means that after such a replacement we have to operate only with variations of the both Lagrangian(s) and Hamiltonians. The Hamilton approach for variations of the original Lagrangian is developed in the Appendix C. In earlier papers (see, e.g., [4] and references therein) this Lagrangian was called the  $\Gamma-\Gamma$  part of the Einstein-Hilbert Lagrangian. This  $\Gamma-\Gamma$  Lagrangian is also written in the following form

$$\begin{aligned} L_{\Gamma-\Gamma} &= \frac{1}{4}\sqrt{-g}B^{\alpha\beta\gamma\mu\nu\rho} \left( \frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} \right) \left( \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} \right) \\ &= \frac{1}{4}\sqrt{-g}B^{\alpha\beta\gamma\mu\nu\rho} g_{\alpha\beta,\gamma} g_{\mu\nu,\rho} \end{aligned} \quad (2)$$

where

$$\begin{aligned} B^{\alpha\beta\gamma\mu\nu\rho} &= g^{\alpha\beta} g^{\gamma\rho} g^{\mu\nu} - g^{\alpha\mu} g^{\beta\nu} g^{\gamma\rho} \\ &+ 2g^{\alpha\rho} g^{\beta\nu} g^{\gamma\mu} - 2g^{\alpha\beta} g^{\gamma\mu} g^{\nu\rho} \end{aligned} \quad (3)$$

is a homogeneous cubic polynomial of the contravariant components of the metric tensor  $g^{\alpha\beta}$ . The contravariant components of the metric tensor  $g^{\alpha\beta}$  form a different set of dual coordinates. Note also that in the right-hand side of this formula, Eq. (2), the short notation  $g_{\alpha\beta,\gamma}$  designates the partial derivatives  $\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}$  in respect to the spatial ( $g_{\alpha\beta,k}$ ) and temporal ( $g_{\alpha\beta,0}$ ) coordinates. The partial temporal derivatives  $g_{0\sigma,0}$  ( $= g_{\sigma 0,0}$ ) of the  $g_{0\sigma}$  components are often called the  $\sigma$  velocities.

Below, we deal with the  $\Gamma-\Gamma$  Lagrangian only. Therefore, to simplify the following formulas we shall designate this  $L_{\Gamma-\Gamma}$  Lagrangian  $L_{\Gamma-\Gamma}$  by the letter  $L$ , i.e., everywhere below  $L = L_{\Gamma-\Gamma}$ . By using this  $\Gamma-\Gamma$  Lagrangian we need to define the corresponding momenta. First, note that in this study the covariant components of the metric tensor  $g_{\alpha\beta}$  are chosen as the straight set of coordinates for the Hamiltonian formulation(s) of the metric GR. In reality, to derive the closed formula for the Hamiltonian of metric GR we need a slightly different form of the  $\Gamma-\Gamma$  Lagrangian where all temporal derivatives (or time-derivatives)

are explicitly separated from other derivatives (see, e.g., [4])

$$\begin{aligned}
 L &= \frac{1}{4}\sqrt{-g}B^{\alpha\beta 0\mu\nu 0}g_{\alpha\beta,0}g_{\mu\nu,0} \\
 &+ \frac{1}{2}\sqrt{-g}B^{(\alpha\beta 0|\mu\nu k)}g_{\alpha\beta,0}g_{\mu\nu,k} \\
 &+ \frac{1}{4}\sqrt{-g}B^{\alpha\beta k\mu\nu l}g_{\alpha\beta,k}g_{\mu\nu,l}, \tag{4}
 \end{aligned}$$

where the notation  $B^{(\alpha\beta\gamma|\mu\nu\rho)}$  means a ‘symmetrical’  $B^{\alpha\beta\gamma\mu\nu\rho}$  quantity which is symmetrized in respect to the permutation of two groups of indexes, i.e.,

$$\begin{aligned}
 B^{(\alpha\beta\gamma|\mu\nu\rho)} &= \frac{1}{2}\left(B^{\alpha\beta\gamma\mu\nu\rho} + B^{\mu\nu\rho\alpha\beta\gamma}\right) \\
 &= g^{\alpha\beta}g^{\gamma\rho}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu}g^{\gamma\rho} \\
 &+ 2g^{\alpha\rho}g^{\beta\nu}g^{\gamma\mu} - g^{\alpha\beta}g^{\nu\rho}g^{\gamma\mu} - g^{\alpha\rho}g^{\beta\gamma}g^{\mu\nu}. \tag{5}
 \end{aligned}$$

By using the Lagrangian  $L$ , Eq. (4), and standard definition of momentum as a partial derivative of the Lagrangian in respect to the corresponding velocity (see, e.g., [11]), one obtains the explicit formulas for all components of the tensor of momentum  $\pi^{\gamma\sigma}$

$$\begin{aligned}
 \pi^{\gamma\sigma} &= \frac{\partial L}{\partial g_{\gamma\sigma,0}} = \frac{1}{2}\sqrt{-g}B^{((\gamma\sigma)0|\mu\nu 0)}g_{\mu\nu,0} \\
 &+ \frac{1}{2}\sqrt{-g}B^{((\gamma\sigma)0|\mu\nu k)}g_{\mu\nu,k}. \tag{6}
 \end{aligned}$$

The first term in the right-hand side of this equation can be written in the form

$$\begin{aligned}
 &\frac{1}{2}\sqrt{-g}B^{((\gamma\sigma)0|\mu\nu 0)}g_{\mu\nu,0} \\
 &= \frac{1}{2}\sqrt{-g}g^{00}E^{\mu\nu\gamma\sigma}g_{\mu\nu,0}, \tag{7}
 \end{aligned}$$

where the notations  $E^{\mu\nu\gamma\sigma}$  and  $e^{\mu\nu}$  stands for the Dirac tensors, which are

$$\begin{aligned}
 E^{\mu\nu\gamma\rho} &= e^{\mu\nu}e^{\gamma\rho} - e^{\mu\gamma}e^{\nu\rho}, \\
 \text{and } e^{\mu\nu} &= g^{\mu\nu} - \frac{g^{0\mu}g^{0\nu}}{g^{00}} \tag{8}
 \end{aligned}$$

and it is easy to check that  $E^{\mu\nu\gamma\sigma} = E^{\gamma\sigma\mu\nu}$  and  $e^{\mu\nu} = e^{\nu\mu}$ . Also, as follows directly from the formula, Eq. (8), the tensor  $e^{\mu\nu}$  equals zero, if either index  $\mu$ , or index  $\nu$  (or both) equals zero. The same statement is true for the Dirac  $E^{\mu\nu\gamma\sigma}$  tensor, i.e.,  $E^{0\nu\gamma\sigma} = 0$ ,  $E^{\mu 0\gamma\sigma} = 0$ ,  $E^{\mu\nu 0\sigma} = 0$  and  $E^{\mu\nu\gamma 0} = 0$ . The  $E^{pqkl}$  quantity is called the space-like Dirac tensor of the fourth rank. Note that all components of this space-like tensor  $E^{pqkl}$  are not equal zero. Furthermore, the space-like tensor  $E^{pqkl}$  is a positively-defined and invertable tensor. Its inverse space-like tensor  $I_{mnpq}$  is also positively-defined and invertable space-like

tensor of the fourth rank which is written in the form [4]

$$I_{mnpq} = \frac{1}{d-2}g_{mn}g_{pq} - g_{mp}g_{nq}. \tag{9}$$

This tensor plays a very important role in our Hamiltonian analysis (see below). Now we can write  $I_{mnpq}E^{pqkl} = g_m^k g_n^l = \delta_m^k \delta_n^l$ , where the  $g_\beta^\alpha = \delta_\beta^\alpha$  tensor is the substitution tensor [7], while the symbol  $\delta_\beta^\alpha$  denotes the Kroneker delta ( $\delta_\alpha^\alpha = 1$  and  $\delta_\beta^\alpha = 0$ , if  $\alpha \neq \beta$ ).

First, let us consider the ‘regular’ case when in Eq. (6)  $\gamma = p$  and  $\sigma = q$ . In this case one finds the following formula for space-like components of the momentum tensor

$$\begin{aligned}
 \pi^{pq} &= \frac{\partial L}{\partial g_{pq,0}} = \frac{1}{2}\sqrt{-g}B^{((pq)0|\mu\nu 0)}g_{\mu\nu,0} \\
 &+ \frac{1}{2}\sqrt{-g}B^{((pq)0|\mu\nu k)}g_{\mu\nu,k} \tag{10}
 \end{aligned}$$

for each pair of  $(pq)$ -indexes (or  $(mn)$ -indexes). The tensor in the right-hand side of this equation is invertable and the velocity  $g_{mn,0}$  is explicitly expressed as the linear function (or linear combination) of the space-like components  $\pi^{pq}$  of momentum tensor:

$$\begin{aligned}
 g_{mn,0} &= \frac{1}{g^{00}}\left(\frac{2}{\sqrt{-g}}I_{mnpq}\pi^{pq} \right. \\
 &- \left. I_{mnpq}B^{((pq)0|\mu\nu k)}g_{\mu\nu,k}\right) = \frac{1}{g^{00}}I_{mnpq}\left(\frac{2}{\sqrt{-g}}\pi^{pq} \right. \\
 &- \left. B^{((pq)0|\mu\nu k)}g_{\mu\nu,k}\right), \tag{11}
 \end{aligned}$$

where the Dirac tensor  $I_{mnpq}$  is defined by Eq. (9). As follows from Eqs. (10) and (11) for the space-like components of metric tensor  $g_{pq}$  and corresponding momenta  $\pi^{mn}$  one finds no principal difference with the classical dynamical systems, which have Lagrangians written as the quadratic functions of the velocities. Indeed, for such systems the corresponding space-like components of momenta and corresponding velocities are related to each other by a few simple, linear equations. In the metric General Relativity, however, even for spatial components of momenta  $\pi^{pq}$  and velocities  $g_{pq,0}$  such relations take the multi-dimensional, matrix form. This means that one space-like component of momenta  $\pi^{mn}$  depends upon quasi-linear combination of different velocities  $g_{pq,0}$  (and vice versa). Nevertheless, such a matrix definition of momenta works very well and allows one to develop a complete and non-contradictive Hamiltonian approach for the metric GR.

In the second ‘non-regular’ (or singular) case, when  $\gamma = 0$  in Eq. (6), the first term in the right-hand

side of Eq. (6) equals zero and this equation takes the form

$$\pi^{0\sigma} = \frac{\partial L}{\partial g_{0\sigma,0}} = \frac{1}{2} \sqrt{-g} B^{((0\sigma)0|\mu\nu k)} g_{\mu\nu,k}, \quad (12)$$

which contains no velocity. This equation, Eq. (12), determines the momentum  $\pi^{0\sigma}$  as a cubic polynomial of the contravariant components of the metric tensor  $g^{\alpha\beta}$  and a product of the  $\sqrt{-g}$  value and spatial derivative  $g_{\mu\nu,k}$  of the covariant components of metric tensor  $g_{\mu\nu}$ . It is clear that such a situation cannot be found in classical mechanics, or in quantum mechanics of arbitrary systems of particles. However, for actual physical fields similar situations arise quite often. The physical meaning of Eq. (12) is simple and can be expressed in the following words. The functions

$$\phi^{0\sigma} = \pi^{0\sigma} - \frac{1}{2} \sqrt{-g} B^{((0\sigma)0|\mu\nu k)} g_{\mu\nu,k}, \quad (13)$$

where  $\sigma = 0, 1, \dots, d - 1$ , must be equal zero during actual physical motions (or time-evolution) of the gravitational field. In [3] Dirac proposed to write such equalities in the symbolic form  $\phi^{0\sigma} \approx 0$  and called these  $d$  functions  $\phi^{0\sigma}$  (for  $\sigma = 0, 1, \dots, d - 1$ ), Eq. (13), the primary constraints (see, [1, 3, 11]).

### 3. CANONICAL AND TOTAL HAMILTONIANS OF METRIC GENERAL RELATIVITY

Now, by applying the Legendre transformation to the known  $\Gamma$ - $\Gamma$  Lagrangian  $L$ , of the metric GR, Eq. (4), and excluding all space-like field-velocities  $g_{mn,0}$  we can derive the explicit formulas for the total and canonical Hamiltonians of the metric gravity. Note here that the word ‘canonical’ originated from the Greek word  $\kappa\alpha\nu\omicron\nu\nu\iota\kappa\omicron\varsigma$  which means correct, equal, equivalent and true. In particular, the total Hamiltonian  $H_t$  of the gravitational field in metric gravity derived from the  $\Gamma$ - $\Gamma$  Lagrangian  $\mathcal{L}$ , Eq. (2), is written in the form

$$H_t = \pi^{\alpha\beta} g_{\alpha\beta,0} - L = H_C + g_{0\sigma,0} \phi^{0\sigma}, \quad (14)$$

where  $\phi^{0\sigma} = \pi^{0\sigma} - \frac{1}{2} \sqrt{-g} B^{((0\sigma)0|\mu\nu k)} g_{\mu\nu,k}$  are the primary constraints, while  $g_{0\sigma,0}$  are the corresponding  $\sigma$  velocities and  $H_C$  is the canonical Hamiltonian of metric GR

$$\begin{aligned} H_C &= \frac{1}{\sqrt{-g} g_{00}} I_{mnpq} \pi^{mn} \pi^{pq} \\ &- \frac{1}{g_{00}} I_{mnpq} \pi^{mn} B^{(pq)0|\mu\nu k)} g_{\mu\nu,k} \\ &+ \frac{1}{4} \sqrt{-g} \left[ \frac{1}{g_{00}} I_{mnpq} B^{((mn)0|\mu\nu k)} B^{(pq)0|\alpha\beta l)} \right. \end{aligned}$$

$$\left. - B^{\mu\nu k\alpha\beta l} \right] g_{\mu\nu,k} g_{\alpha\beta,l}, \quad (15)$$

which does not contain any primary constraint  $\phi^{0\sigma}$ . In contrast with  $H_C$  the total Hamiltonian  $H_t$ , Eq. (14) includes all  $d$  primary constraints  $\phi^{0\sigma}$ , where  $\sigma = 0, 1, \dots, d - 1$ . It should be emphasized again that these primary constraints arise during our transition from the  $\Gamma$ - $\Gamma$  Lagrangian  $L$ , Eq. (2), to the Hamiltonians  $H_t$  and  $H_C$ , since the  $\Gamma$ - $\Gamma$  Lagrangian  $L$  is a linear (not quadratic!) function of all  $d$  values  $g_{0\sigma,0}$ , or  $\sigma$  velocities. The total and canonical Hamiltonians  $H_t$  and  $H_C$  are the scalar functions defined in the  $d(d + 1)$  dimensional phase space  $\{g_{\alpha\beta}, \pi^{\mu\nu}\}$ , where all components of the metric  $g_{\alpha\beta}$  and momentum  $\pi^{\mu\nu}$  tensors have been chosen as the basic dynamical variables. Such a  $d(d + 1)$  dimensional phase space is, in fact, a symplectic space and the corresponding symplectic structure is determined by the Poisson brackets between its basic dynamical variables, i.e., coordinates  $g_{\alpha\beta}$  and momenta  $\pi^{\mu\nu}$ .

To make the next step we need to define the Poisson brackets (or PB, for short) which are absolutely crucial for creation, development and applications of Hamiltonian approaches for arbitrary, in principle, physical systems of particles and fields, including the metric General Relativity. From now on we shall consider only Hamiltonian approaches (in metric GR) which are canonically related either to the K&K approach [4], or to the Dirac approach [1]. Note again that these two Hamiltonian formulations are canonically related to each other (for more details, see [5]). Therefore, it is possible to obtain and present the basic (or fundamental) set of Poisson brackets only for one of these two Hamiltonian formulations, e.g., for the K&K approach. Analogous Poisson brackets for other Hamiltonian formulations of metric GR can be derived from these basic (or fundamental) values known in the K&K-approach. The basic Poisson brackets between  $\frac{d(d+1)}{2}$  components of the momentum tensor  $\pi^{\mu\nu}$  and  $\frac{d(d+1)}{2}$  ‘coordinates’  $g_{\alpha\beta}$  in the K&K-approach are [4]

$$\begin{aligned} [g_{\alpha\beta}, \pi^{\mu\nu}] &= -[\pi^{\mu\nu}, g_{\alpha\beta}] = g_{\alpha\beta} \pi^{\mu\nu} - \pi^{\mu\nu} g_{\alpha\beta} \\ &= \frac{1}{2} \left( g_{\alpha}^{\mu} g_{\beta}^{\nu} + g_{\alpha}^{\nu} g_{\beta}^{\mu} \right) \\ &= \frac{1}{2} \left( \delta_{\alpha}^{\mu} \delta_{\beta}^{\nu} + \delta_{\alpha}^{\nu} \delta_{\beta}^{\mu} \right) = \Delta_{\alpha\beta}^{\mu\nu}, \end{aligned} \quad (16)$$

where  $g_{\alpha}^{\mu} = \delta_{\alpha}^{\mu}$  is the substitution tensor [7] and symbol  $\delta_{\beta}^{\mu}$  is the Kronecker delta, while the notation  $\Delta_{\alpha\beta}^{\mu\nu}$  stands for the gravitational (or tensor) delta-symbol. All other fundamental Poisson brackets between basic dynamical variables of the metric GR equal zero identically, i.e.,  $[g_{\alpha\beta}, g_{\mu\nu}] = 0$  and

$[\pi^{\alpha\beta}, \pi^{\mu\nu}] = 0$ . Thus, for the  $d$  dimensional metric gravity one finds  $N_p = \frac{d^2(d+1)^2}{2}$  different Poisson brackets, but many of these brackets equal zero identically, if these dynamical variables are canonical. In our case this means that all our momenta  $\pi^{\mu\nu}$  have been defined correctly. The set of  $N_p$  Poisson brackets has a fundamental value, since these PB define the unique symplectic structure directly related to the Riemannian structure of the original  $d(d+1)$  dimensional tensor phase space  $\{g_{\alpha\beta}, \pi^{\mu\nu}\}$  and to the original metric tensor  $g_{\alpha\beta}$ . General properties of Poisson brackets and their symmetries are discussed, e.g., in [12–15]. For any Hamiltonian dynamical system all values and functions must be expressed in terms of the basic dynamical variables, i.e., in terms of generalized coordinates and momenta. Furthermore, all arithmetical, mathematical and other operations between such values and functions must be reduced to the Poisson brackets. Analytical computations of the Poisson brackets is the only actual tool and language of the Hamiltonian theory.

#### 4. POISSON BRACKETS

The  $N_p$  Poisson brackets mentioned above are sufficient to operate successfully in any correct Hamiltonian approach developed for the metric GR. However, in many applications it is crucially important to determine other Poisson brackets, which are often called the secondary PB. The secondary PB are calculated between different analytical functions of the basic dynamical variables, i.e., coordinates and momenta, and these PB always appear in actual calculations. In general, it is difficult and time-consuming to derive the explicit formulas for such secondary PB every time when you need them. Furthermore, in actual applications one usually needs to determine a few hundreds of different Poisson brackets. Here we present a number of secondary Poisson brackets which are sufficient for our purposes in this study. The first additional group of secondary Poisson brackets is

$$\begin{aligned} [g^{\alpha\beta}, \pi^{\mu\nu}] &= -\frac{1}{2} \left( g^{\alpha\mu} g^{\beta\nu} + g^{\alpha\nu} g^{\beta\mu} \right) \\ &= -[\pi^{\mu\nu}, g^{\alpha\beta}] \text{ and } [g^{\alpha\beta}, g_{\mu\nu}] = 0. \end{aligned} \quad (17)$$

which include the contravariant components of the metric tensor  $g^{\alpha\beta}$ . These Poisson brackets are of great interest, since our canonical and total Hamiltonians (see above) are overloaded with the contravariant components of the metric tensor.

The second set of additional Poisson brackets arises, if one explicitly introduces the dual system of dynamical variables  $\{g^{\alpha\beta}, \pi_{\mu\nu}\}$  which always exists for any tensor Hamiltonian system. In general, to create the truly correct, covariant and non-contradictory

Hamiltonian formulation for some dynamical system of tensor fields it is much better to deal (instantly) with the two different  $d(d+1)$  dimensional sets of dynamical variables: (a) the straight set  $\{g_{\alpha\beta}, \pi^{\mu\nu}\}$ , and (b) the corresponding dual set  $\{g^{\alpha\beta}, \pi_{\mu\nu}\}$ . Application of the two sets of dynamical variables makes our Hamiltonian formulation complete and physically transparent. The Poisson brackets between all dynamical variables from these two sets must be derived and carefully checked for non-contradictory. The Poisson brackets for the dual set of dynamical variables  $\{g^{\alpha\beta}, \pi_{\mu\nu}\}$

$$\begin{aligned} [g_{\alpha\beta}, \pi_{\mu\nu}] &= \frac{1}{2} (g_{\alpha\mu} g_{\beta\nu} + g_{\alpha\nu} g_{\beta\mu}) \\ \text{and } [g^{\alpha\beta}, \pi_{\mu\nu}] &= -\frac{1}{2} \left( g_{\mu}^{\alpha} g_{\nu}^{\beta} + g_{\nu}^{\alpha} g_{\mu}^{\beta} \right) \\ &= -\Delta_{\mu\nu}^{\alpha\beta} \end{aligned} \quad (18)$$

and also  $[g^{\alpha\beta}, g^{\mu\nu}] = 0$  and  $[g_{\alpha\beta}, g^{\mu\nu}] = 0$ . Another interesting Poisson bracket which we want to present here is

$$\begin{aligned} &[\pi_{\alpha\beta}, \pi^{\mu\nu}] \\ &= \frac{1}{2} \left( \delta_{\alpha}^{\mu} \pi_{\beta}^{\nu} + \delta_{\alpha}^{\nu} \pi_{\beta}^{\mu} + \delta_{\beta}^{\mu} \pi_{\alpha}^{\nu} + \delta_{\beta}^{\nu} \pi_{\alpha}^{\mu} \right), \end{aligned} \quad (19)$$

where  $\pi_{\kappa}^{\rho} = g^{\rho\lambda} \pi_{\lambda\kappa} = g_{\kappa\lambda} \pi^{\lambda\rho}$ . This equality means that the co- and contra-covariant components of the momentum tensor do not commute with each other. If they commuted, then the direct and dual sets of dynamical variables in metric gravity would be equivalent and there would be no real need to apply the two sets of dynamical variables (straight and dual), since at each step of our procedure we can always express one set of dynamic variables in terms of another set and vice versa. However, this is not true for the metric GR.

Let us present the following formula for the fundamental Poisson brackets which unites the both straight and dual sets of dynamical variables

$$[g_{\alpha\beta}, \pi^{\mu\nu}] = \Delta_{\alpha\beta}^{\mu\nu} = [\pi_{\alpha\beta}, g^{\mu\nu}]. \quad (20)$$

This beautiful formula includes two fundamental Poisson bracket(s) and clearly shows the differences which arise during transition from the straight set of canonical variables to analogous dual set. As follows from the formula, Eq. (25) (see below), the truly dual system of dynamical variables (for the original  $\{g_{\alpha\beta}, \pi^{\mu\nu}\}$  system) must be  $\{-g^{\alpha\beta}, \pi_{\mu\nu}\}$  system rather than our dual  $\{g^{\alpha\beta}, \pi_{\mu\nu}\}$  system of variables introduced above. Below, we shall ignore this fact and consider the  $\{g_{\alpha\beta}, \pi^{\mu\nu}\} \rightarrow \{g^{\alpha\beta}, \pi_{\mu\nu}\}$  transition as a canonical transformation of dynamical variables for our Hamiltonian formulation of the metric GR. Therefore, based on the general theory of canonical

transformations in Hamiltonian systems described in [12] we can write the following equality

$$\pi^{\mu\nu} \delta g_{\mu\nu} - H_t \delta t + \delta F = v (\pi_{\mu\nu} \delta g^{\mu\nu} - \overline{H}_t \delta t), \quad (21)$$

where  $v$  is a real, non-zero number which is called the valence of this canonical transformation, while  $F(t, g_{\alpha\beta}, \pi^{\gamma\sigma})$  is its generating function. The notation  $H_t$  and  $\overline{H}_t$  means the total Hamiltonians written in the both systems of dynamical variables, i.e., in the straight  $\{g_{\alpha\beta}, \pi^{\mu\nu}\}$  and dual  $\{g^{\alpha\beta}, \pi_{\mu\nu}\}$  systems of variables, respectively. It is clear that for such a canonical transformation we can use the same time  $t$  (for both systems) and this transformation is univalent which means that  $|v| = 1$ . In our case we have found that  $v = -1$ . Furthermore, it is possible to show that for the  $\{g_{\alpha\beta}, \pi^{\mu\nu}\} \rightarrow \{g^{\alpha\beta}, \pi_{\mu\nu}\}$  canonical transformation the generating function  $F$  can be chosen in a very special form  $F = S(t, g_{\mu\nu}, g^{\alpha\beta})$  which corresponds to the free canonical transformation(s). In this case the previous equation takes the form

$$\begin{aligned} \pi^{\mu\nu} \delta g_{\mu\nu} - H_t \delta t + \delta S(t, g_{\mu\nu}, g^{\alpha\beta}) \\ = v (\pi_{\mu\nu} \delta g^{\mu\nu} - \overline{H}_t \delta t) \end{aligned} \quad (22)$$

and three following equations are also obeyed (for  $v = -1$ )

$$\begin{aligned} \pi^{\mu\nu} = -\frac{\partial S}{\partial g_{\mu\nu}}, \quad \pi_{\mu\nu} = -\frac{\partial S}{\partial g^{\mu\nu}} \\ \text{and } \overline{H}_t + H_t = \frac{\partial S}{\partial t}. \end{aligned} \quad (23)$$

As follows from Eq. (22) the differential form  $dS(t, g_{\mu\nu}, g^{\alpha\beta})$  is the total differential of the potential function  $S(t, g_{\mu\nu}, g^{\alpha\beta})$  which is, in fact, the generating function of the free canonical transformations. This generating function always exists and can explicitly be constructed for an arbitrary Hamiltonian system of tensor fields. Also, these our equations open a short way to the Jacobi equation for the gravitational field in metric GR, but below we shall apply a different approach to solve this interesting problem (see Section 6 below).

The necessity to deal with the two sets of dynamical variables instantaneously is an important difference between Hamiltonian procedures developed for the affine vector spaces and Riemannian tensor spaces. It can be shown that only by dealing with the both straight and dual sets of dynamical variables we can guarantee the internal covariance and self-sustainability of our Hamiltonian approach developed for the metric GR. The fact that we need to operate with the both straight and dual systems of dynamical variables in any Hamiltonian formulation developed for tensor dynamical systems can be illustrated by the following reasoning. To construct the Hamiltonian

formulation, we are free to choose either direct, or dual sets of dynamic variables. For any meaningful physical theory, these Hamiltonian formulations must be equivalent, i.e., they must be connected to each other by a canonical transformation. In other words, the two sets of dynamical variables  $\{g^{\alpha\beta}, \pi^{\rho\sigma}\}$  and  $\{g_{\alpha\beta}, \pi_{\rho\sigma}\}$  are absolutely equivalent in order to develop the new Hamiltonian formulation(s) of the metric GR. The two newly arising Hamiltonian formulations are related to each other by a canonical transformation of variables (see above).

A few following Poisson brackets which are also useful in actual calculations. Let  $g(> 0)$  will be the determinant of the metric tensor  $g_{\alpha\beta}$  and  $F(g)$  is an arbitrary analytical function of  $g$ . In this notation one finds

$$\begin{aligned} [F(g), \pi^{\alpha\beta}] &= \left(\frac{\partial F}{\partial g}\right) g g^{\alpha\beta} \\ \text{and } [\sqrt{-g}, \pi^{\alpha\beta}] &= -\frac{1}{2\sqrt{-g}} g g^{\alpha\beta} \\ &= \frac{1}{2} \sqrt{-g} g^{\alpha\beta} \end{aligned} \quad (24)$$

for  $F(g) = \sqrt{-g}$ , if the determinant  $g$  is negative. Analogously, for the  $\pi_{\alpha\beta}$  momentum we obtain

$$\begin{aligned} [F(g), \pi_{\alpha\beta}] &= \left(\frac{\partial F}{\partial g}\right) g g_{\alpha\beta} \\ \text{and } [\sqrt{-g}, \pi_{\alpha\beta}] &= -\frac{1}{2\sqrt{-g}} g g_{\alpha\beta} \\ &= \frac{1}{2} \sqrt{-g} g_{\alpha\beta}. \end{aligned} \quad (25)$$

These formulas lead to the following expressions

$$\begin{aligned} \left[\frac{1}{\sqrt{-g}}, \pi^{\alpha\beta}\right] &= -\frac{1}{2\sqrt{-g}} g^{\alpha\beta} \\ \text{and } \left[\frac{1}{\sqrt{-g}}, \pi_{\alpha\beta}\right] &= -\frac{1}{2\sqrt{-g}} g_{\alpha\beta} \end{aligned} \quad (26)$$

which are important for our calculations performed in the next Sections. All other Poisson brackets, which are often needed in analytical calculations, can be determined with the use of our PB presented in Eqs. (16)–(26). In general, analytical computation of a large number of Poisson brackets in any Hamiltonian formulation of the metric GR is a very good exercise in tensor calculus.

Another example is slightly more complicated and includes the tensor(s)  $e^{\mu\nu}$  defined above. From the explicit formulas for the components of  $e^{\mu\nu}$  tensor, Eq. (8), one finds that only non-zero elements of this tensor are located in the space-like corner of the total  $e^{\mu\nu}$  tensor. These non-zero elements form the space-like  $e^{pq}$  tensor (or space-like part of the total

$e^{\mu\nu}$  tensor) which is often called the space-like Dirac tensor of the second rank. For this tensor one easily finds the following useful relation

$$\begin{aligned} g_{\alpha\beta}e^{\alpha\beta} &= g_{\alpha\beta}g^{\alpha\beta} - g_{\alpha\beta} \left( \frac{g^{\alpha 0}g^{\beta 0}}{g^{00}} \right) \\ &= d - g_{\beta}^0 \frac{g^{\beta 0}}{g^{00}} = d - \frac{g^{00}}{g^{00}} \\ &= d - 1 = g_{mn}e^{mn}, \end{aligned} \tag{27}$$

where  $g_{\alpha\beta}g^{\alpha\beta} = d$  and  $d$  is the total dimension of our space-time continuum. By using our formulas for the Poisson brackets obtained above we derive the two following expressions

$$\begin{aligned} [e^{pq}, \pi^{\alpha\beta}] &= -\frac{1}{2} \left( g^{p\alpha}g^{q\beta} + g^{p\beta}g^{q\alpha} \right) \\ &+ \frac{1}{2} \left( g^{0\alpha}g^{p\beta} + g^{0\beta}g^{p\alpha} \right) \left( \frac{g^{0q}}{g^{00}} \right) \\ &+ \frac{1}{2} \left( \frac{g^{0p}}{g^{00}} \right) \left( g^{0\alpha}g^{q\beta} + g^{0\beta}g^{q\alpha} \right) \\ &- \frac{g^{0p}g^{q\alpha}g^{0\beta}g^{0q}}{(g^{00})^2} \end{aligned} \tag{28}$$

and

$$\begin{aligned} [e^{pq}, \pi_{\alpha\beta}] &= -\Delta_{\alpha\beta}^{pq} + \Delta_{\alpha\beta}^{0p} \left( \frac{g^{0q}}{g^{00}} \right) \\ &+ \frac{1}{2} \left( \frac{g^{0p}}{g^{00}} \right) \Delta_{\alpha\beta}^{0q} - \Delta_{\alpha\beta}^{00} \frac{g^{0p}g^{0q}}{(g^{00})^2}. \end{aligned} \tag{29}$$

Analytical formulas for these PB are important, since there were some ideas to use components of this space-like tensor  $e^{pq}$  as the new  $\frac{d(d-1)}{2}$  canonical variables, or new coordinates, for another ‘advanced’ Hamiltonian formulation of the metric GR. As follows from Eqs. (28) and (29) the complexity of arising Poisson brackets makes this idea unworkable.

### 5. APPLICATIONS TO ACTUAL PROBLEMS OF METRIC GRAVITY

The knowledge of the fundamental and secondary Poisson brackets allows one to achieve a number of goals in the Hamiltonian formulation(s) of metric General Relativity. In particular, by using these Poisson brackets we can complete the actual Hamiltonian formulation of the metric GR. Another problem which can be solved with the use of our Poisson brackets is explicit derivation of the Hamilton equations of motion for actual gravitational field(s) which are often called the time-evolution equations. Also, by using these Poisson brackets one can find some new canonical transformations which are simplify either the canonical Hamiltonian  $H_C$ , or secondary

constraints  $\chi^{0\sigma}$  (they are defined below). Another important problem is the reduction of the canonical Hamiltonian  $H_C$  to its natural form. The first two problems are briefly discussed in the next two subsections. These two problems were extensively investigated in earlier studies [4, 5 and 6]. Therefore, there is no need for us here to discuss formulation of these problems and repeat all formulas derived in those papers. Here we just want to illustrate how our formulas for Poisson brackets allow one to simplify analytical calculations of many difficult expressions. In contrast with this, the third problem (i.e., reduction of  $H_C$  to its natural form) is one of the central parts of this study and we want to disclose all details of our computations. This problem with all details is described in the next Section. Another aim of this study is to derive Jacobi equation for the free gravitational field(s) in our new Hamiltonian formulation of the metric gravity. This problem is considered in Section 7.

#### 5.1. Secondary constraints. Dirac closure

Let us complete the Hamiltonian formulation of the metric GR described above, by using the space-like momenta  $\pi^{mn}$ , its temporal components  $\pi^{0\sigma}$  (or primary constraints  $\phi^{0\sigma}$ ) and canonical Hamiltonian  $H_C$  defined in Eq. (10), Eq. (12) and Eq. (15), respectively. First, we need to determine PB between the canonical Hamiltonian  $H_C$ , Eq. (15), and primary constraints  $\phi^{0\sigma}$ , Eq. (13). This directly leads to the secondary constraints  $\chi^{0\sigma} = [\phi^{0\sigma}, H_C]$ , where  $\sigma = 0, 1, \dots, d - 1$  (see discussion in [4]), since these secondary constraints  $\chi^{0\sigma}$  are not equal zero identically. In Dirac procedure these  $d$  secondary constraints  $\chi^{0\sigma}$  become an integral part of the Hamiltonian formulation. The explicit formulas for the secondary constraints  $\chi^{0\sigma}$  are [4] (see also [6]):

$$\begin{aligned} \chi^{0\sigma} &= -\frac{g^{0\sigma}}{2\sqrt{-g}g^{00}} I_{mnpq} \pi^{mn} \pi^{pq} \\ &+ \frac{g^{0\sigma}}{2g^{00}} I_{mnpq} \pi^{mn} U^{(pq0|\mu\nu k)} g_{\mu\nu, k} \\ &+ \left[ \pi_{,k}^{\sigma k} + \left( \pi^{pk} e^{q\sigma} - \frac{1}{2} \pi^{pq} e^{k\sigma} \right) g_{pq, k} \right] \\ &- \frac{\sqrt{-g}}{8} \left( \frac{g^{0\sigma}}{g^{00}} I_{mnpq} B^{((mn)0|\mu\nu k)} B^{(pq0|\alpha\beta t)} \right. \\ &\quad \left. - g^{0\sigma} B^{\mu\nu k\alpha\beta t} \right) g_{\mu\nu, k} g_{\alpha\beta, t} \\ &+ \frac{\sqrt{-g}}{4g^{00}} I_{mnpq} B^{((mn)0|\mu\nu k)} g_{\mu\nu, k} g_{\alpha\beta, t} \left[ g^{\sigma t} (g^{00} g^{p\alpha} g^{q\beta} \right. \\ &\quad \left. + g^{pq} g^{0\alpha} g^{0\beta} - 2g^{\alpha q} g^{0p} g^{0\beta}) - g^{\sigma p} (2g^{00} g^{q\alpha} g^{t\beta} \right. \end{aligned}$$

$$\begin{aligned}
& -g^{00}g^{\alpha\beta}g^{qt} + g^{\alpha\beta}g^{0q}g^{0t} - 2g^{q\alpha}g^{0\beta}g^{0t} - 2g^{t\alpha}g^{0\beta}g^{0q} \\
& \quad + 2g^{qt}g^{0\alpha}g^{0\beta}) + g^{0\sigma}(2g^{\beta t}g^{\alpha p}g^{0q} \\
& \quad - 2g^{p\alpha}g^{q\beta}g^{0t} - 2g^{pq}g^{t\beta}g^{0\alpha} + 2g^{pt}g^{q\beta}g^{0\alpha} \\
& \quad + g^{pq}g^{\alpha\beta}g^{0t} - g^{tp}g^{\alpha\beta}g^{0q}) \\
& - \frac{\sqrt{-g}}{4}g_{\mu\nu,k}g_{\alpha\beta,t} \left[ g^{\sigma t}(g^{\alpha\mu}g^{\beta\nu}g^{0k} + g^{\mu\nu}g^{\alpha t}g^{0\beta} \right. \\
& - 2g^{\mu\alpha}g^{k\nu}g^{0\beta}) + g^{0\sigma}(2g^{\alpha t}g^{\beta\mu}g^{\nu k} - 3g^{t\mu}g^{\nu k}g^{\alpha\beta} \\
& - 2g^{\mu\alpha}g^{\nu\beta}g^{kt} + g^{\mu\nu}g^{kt}g^{\alpha\beta} + 2g^{\mu t}g^{\nu\beta}g^{k\alpha}) \\
& + g^{\sigma\mu}((g^{\alpha\beta}g^{\nu t} - 2g^{\nu\alpha}g^{t\beta})g^{0k} + 2(g^{\beta\nu}g^{kt} \\
& - g^{\beta k}g^{t\nu})g^{0\alpha} + (2g^{k\beta}g^{\alpha t} - g^{\alpha\beta}g^{kt})g^{0\nu}) \\
& \left. - \frac{\sqrt{-g}g^{00}}{2}E^{pqt\sigma} \left( \frac{1}{g^{00}}I_{mnpq}B^{((mn)0|\mu\nu k)}g_{\mu\nu,k} \right),_t \right. \\
& \left. - \frac{\sqrt{-g}}{2}B^{((\sigma 0)k|\alpha\beta t)}g_{\alpha\beta,kt}, \right. \quad (30)
\end{aligned}$$

where  $U^{(pq)0|\mu\nu k}$  is the symmetrized form of the following expression

$$\begin{aligned}
U^{\alpha\beta 0\mu\nu k} &= B^{(\alpha\beta 0|\mu\nu k)} \\
&- g^{0k}E^{\alpha\beta\mu\nu} + 2g^{0\mu}E^{\alpha\beta k\nu} \quad (31)
\end{aligned}$$

and  $\sigma = 0, 1, \dots, d-1$ . The total number of primary and secondary constraints for this Hamilton formulation equals  $d + d = 2d$ . Note also that all these primary and secondary constraints  $\phi^{0\sigma}$  and  $\chi^{0\sigma}$ , where  $\sigma = 0, 1, \dots, d-1$ , are the first-class constraints [11]. In general, our formulas for Poisson brackets (or PB, for short) substantially simplify the whole process of derivation of the explicit formulas for the primary and secondary constraints and for PB between them. In particular, by using our Poisson brackets one can show that all Poisson brackets between primary constraints equal zero identically, i.e.,  $[\phi^{0\lambda}, \phi^{0\sigma}] = 0$ , while  $[\phi^{0\lambda}, \chi^{0\sigma}] = \frac{1}{2}g^{\lambda\sigma}\chi^{00}$ . The Poisson brackets between canonical Hamiltonian  $H_C$  and secondary constraints  $\chi^{0\sigma}$  are expressed as ‘quasi-linear’ combinations of the same secondary constraints  $\chi^{0\sigma}$ , i.e., we obtain

$$\begin{aligned}
[\chi^{0\sigma}, H_C] &= -\frac{2}{\sqrt{-g}}I_{mnpq}\pi^{mn} \left( \frac{g^{\sigma q}}{g^{00}} \right) \chi^{0p} \\
&+ \frac{1}{2}g^{\sigma k}g_{00,k}\chi^{00} + \delta_0^\sigma \chi_{,k}^{0k} \\
&+ \left( -2\frac{1}{\sqrt{-g}}I_{mnpk}\pi^{mn}\frac{g^{\sigma p}}{g^{00}} \right. \\
&+ \left. I_{mkpq}g_{\mu\nu,l}\frac{g^{\sigma m}}{g^{00}}U^{(pq)0|\mu\nu l} \right) \chi^{0k} \\
&- \left( g^{0\sigma}g_{00,k} + 2g^{n\sigma}g_{0n,k} \right.
\end{aligned}$$

where  $U^{(pq)0|\mu\nu k}$  is the quantity  $U^{\alpha\beta 0\mu\nu k}$  from Eq. (31) which is symmetrized upon all  $p \leftrightarrow q$  permutations. The combination of secondary constraints in the right-hand side is field-dependent. Here and below such combinations are called quasi-linear. The Poisson bracket, Eq. (32), indicates that the Hamilton procedure developed for the metric GR in [4] and [5] is closed (Dirac closure), i.e., the Poisson bracket  $[\chi^{0\sigma}, H_C]$  does not lead to any tertiary, or other constraints of higher order(s). Analogously, the Poisson brackets between secondary constraints  $[\chi^{0\sigma}, \chi^{0\gamma}]$ , where  $\sigma \neq \gamma$  (if  $\sigma = \gamma$ , then this PB equals zero identically), are

$$\begin{aligned}
[\chi^{0\sigma}, \chi^{0\gamma}] &= [\chi^{0\sigma}, [\phi^{0\gamma}, H_C]] \\
&= -[\phi^{0\gamma}, [H_C, \chi^{0\sigma}]] - [H_C, [\chi^{0\sigma}, \phi^{0\gamma}]] \\
&= [\phi^{0\gamma}, [\chi^{0\sigma}, H_C]] - \frac{1}{2}[g^{\sigma\gamma}, H_C], \quad (33)
\end{aligned}$$

where the Poisson bracket  $[\chi^{0\sigma}, H_C]$  is given by the formula, Eq. (32). This formula also does not lead to any constraint of higher order (see discussion in [11]). This proves that the Hamiltonian system which includes the canonical Hamiltonian  $H_C$  and all primary  $\phi^{0\lambda}$  and secondary  $\chi^{0\sigma}$  constraints is closed (here  $\lambda = 0, 1, \dots, d-1$  and  $\sigma = 0, 1, \dots, d-1$ ). The actual closure of the Dirac procedure [3] for the Hamiltonian formulation of the metric GR was shown for the first time in [4]. Formally, the explicit demonstration of closure of the whole Dirac procedure [3] is the last and most important step for any Hamiltonian formulation of the metric GR [11, 16]. However, in reality one needs to check one more condition which appears to be crucial for separation of the actual Hamiltonian formulations of the metric GR from numerous quasi-Hamiltonian constructions developed in this area of gravitational research, since the end of 1950’s.

## 5.2. Reconstruction of the Complete Diffeomorphism Invariance of Metric Gravity

There is a simple and physically transparent criterion which can be used to separate all correct Hamiltonian formulations of metric gravity, from similar, but wrong formulations. This criterion (or condition) follows from rigorous conservation of the gauge invariance (or symmetry) of the metric GR during transformations from the original  $\Gamma - \Gamma$  Lagrangian to the Hamiltonian formulation. In other words, we cannot reduce (or increase) the gauge symmetry known for the free gravitational field(s) which obeys the original Einstein equations. Disappearance (or reduction) of the gauge invariance of the original problem simply

means that our transformations to the Hamiltonian formulation are wrong and non-equivalent, or simply that they are not canonical. The formulas for the Hamiltonians  $H_t, H_C$  presented above and explicit expressions for all primary and secondary constraints [4, 6] allow one to derive (with the use of Castellani procedure [17]) the correct generators of gauge transformations, which directly and unambiguously lead to the diffeomorphism invariance [4]. The diffeomorphism invariance is well known gauge symmetry (or gauge, for short) of the free gravitational field(s) which was discovered in early years of the metric GR (see, e.g., [10] and references therein).

The Castellani procedure is based on the explicit derivation of generators of gauge transformations which are unambiguously defined by the chain of first-class constraints [4]. In general, we start from the primary first-class constraints and then construct the complete set of generators of gauge transformation(s). These primary constraints play the central role in the Castellani procedure, since each of these constraints generates a separate chain of gauge generators. Furthermore, during the actual motion of any constrained dynamical system all primary constraints always equal zero. This allows us to introduce the corresponding zero-surface (or shell) of primary constraints  $S_p$ . For the metric gravity we have  $d$  primary constraints  $\phi^{0\lambda}$  (see above). By following [4] let us consider the case when all chains of gauge transformations are of length two, i.e., the Castellani generators are the linear combination of the two  $C_1^\lambda(x)$  and  $C_0^\lambda(x)$  functions, where

$$C_1^\lambda(x) = \phi^{0\lambda} \text{ and } C_0^\lambda(x) = [H_t, \phi^{0\lambda}] + \int A_\mu^\lambda(x, y) \phi^{0\mu}(y) d^3y, \quad (34)$$

where  $\lambda = 0, 1, \dots, d - 1$  is the index of the chain (or index of the generating primary constraint), while the lower index  $k$  is used to numerate all gauge generators in one chain. The  $A_\mu^\lambda(x, y)$  are the functions which are chosen from the fact that these chains of generators must be finished on the surface of primary constraints  $S_p$ . This leads to the following condition for the Poisson bracket

$$[C_0^\lambda(x), H_t] = \sum_\nu L_\nu^\lambda(x) \phi^{0\nu}(x) = \text{linear combination of primary constraints}, \quad (35)$$

where  $L_\nu^\lambda(x)$  are some continuous functions. Now, the Castellani generators take the following general form  $G(\xi_\lambda) = \xi_\lambda C_0^\lambda(x) + \xi_{\lambda;0} C_1^\lambda(x)$ , where  $\xi_\lambda$  is the  $\lambda$ -th gauge parameter, while  $\xi_{\lambda;0}$  is its time-derivative of the first order. The gauge parameter is a function of the spatial coordinates and time only, but it cannot

depend upon the field itself, or upon any component of the metric gravitational field in our case. Moreover, in applications to the metric GR such gauge parameters can be used only in a completely covariant form. Segregation of some 'selected' components of these parameters is strictly prohibited, since it devaluates the original Castellani procedure and leads to the results which are fundamentally wrong.

Now, by using the criterion, Eq. (35), we obtain the following equation

$$[C_0^\lambda(x), H_t] = -[\chi^{0\lambda}, H_t] + \int [A_\mu^\lambda(x, y), H_t] \phi^{0\mu}(y) d^3y + \int A_\mu^\lambda(x, y) [\phi^{0\mu}(y), H_t] d^3y, \quad (36)$$

which can be used to determine the unknown function  $A_\mu^\lambda(x, y)$ . Formally, the second term in the right-hand side of this equation is already written as the linear combination of the primary constraints only. On the surface of primary constraints  $S_p$  this term vanishes. For now this (second) term can be neglected. This allows us to derive the following explicit formula for the  $C_0^\lambda(x)$  function [4]

$$C_0^\lambda(x) = -\chi^{0\lambda} - \left( \frac{1}{2} g_{00,0} g^{0\lambda} + g_{0m,0} g^{\lambda m} - \frac{1}{2} g^{\lambda m} g_{00,m} \right) \phi^{00} - \delta_0^\lambda \phi_{,k}^{0k} - \left( \frac{2}{\sqrt{-g}} I_{nmpk} \pi^{mn} \frac{g^{\lambda p}}{g^{00}} - I_{mkpq} g_{\alpha\beta,l} A^{(pq)0\alpha\beta l} \frac{g^{\lambda m}}{g^{00}} \right) \phi^{0k} - \left[ g^{0\lambda} g_{00,k} + 2g^{n\lambda} g_{0n,k} + g^{n\lambda} \frac{g^{0m}}{g^{00}} (g_{mn,k} + g_{km,n} - g_{kn,m}) \right] \phi^{0k}.$$

The Castellani generator is now determined by the relation  $G(\xi_\lambda) = \xi_\lambda C_0^\lambda(x) + \xi_{\lambda;0} C_1^\lambda(x)$  mentioned above. This generator can be applied to obtain the transformation of the metric tensor  $g_{\alpha\beta}$ , i.e.,

$$\delta g_{\alpha\beta} = [G(\xi_\lambda), g_{\alpha\beta}] = [\xi_\lambda C_0^\lambda + \xi_{\lambda;0} \phi^{0\lambda}, g_{\alpha\beta}]. \quad (37)$$

The following transformations (see, Section 5 in [4]) directly and unambiguously lead to the diffeomorphism invariance, which is the well known gauge invariance of the free gravitational field(s). Briefly, we can say that the Einstein equations of motion  $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$  are transformed into a linear combination of the same equations. Note that such an invariance is true only on-shell of the primary constraints (see,

discussions in [4] and [36]). Formally, the Dirac closure of the metric GR is needed to be checked on the same zero-surface of primary constraints only [4].

Note that currently we have only two known Hamiltonian formulations [1] and [4] of the metric GR which reproduce the actual diffeomorphism invariance directly and transparently. In contrast with this, numerous Hamiltonian formulations of metric GR based on the ADM dynamical variables [18] and [19] fail at this crucial point (see Conclusion and Appendix B). Note that for all approaches which are directly based on the  $\Gamma$ - $\Gamma$  Lagrangian of the metric GR, such a reconstruction of the diffeomorphism invariance is a relatively simple problem (see, e.g., [20]). In contrast with this, for any Hamiltonian-based formulation the complete solution of similar problem requires a substantial work. On the other hand, analytical derivation of the diffeomorphism invariance is a very good test for the total  $H_t$  and canonical  $H_C$  Hamiltonians as well as for all primary  $\phi^{0\sigma}$  and secondary  $\chi^{0\sigma}$  constraints derived in any new Hamiltonian formulation of the metric GR. Any mistake made either in the  $H_t, H_C$  Hamiltonians, or in the  $\phi^{0\lambda}$  and  $\chi^{0\sigma}$  constraints leads to the loss of true diffeomorphism invariance for the free gravitational field.

### 5.3. Hamilton Equations of Motion for the Free Gravitational Field

In general, if we know the total  $H_t$  Hamiltonian, Eq. (14), then we can derive the Hamilton equations of motion which describe the time-evolution of all essential dynamical variables in the metric GR, i.e., time-evolution of each component of the metric tensor  $g_{\alpha\beta}$  and momentum tensor  $\pi^{\gamma\rho}$ . These equations are [6]

$$\frac{dg_{\alpha\beta}}{dx_0} = [g_{\alpha\beta}, H_t] \quad \text{and} \quad \frac{d\pi^{\gamma\rho}}{dx_0} = [\pi^{\gamma\rho}, H_t], \quad (38)$$

where the notation  $x_0$  denotes the temporal variable. In particular, for the spatial components  $g_{ij}$  of the metric tensor  $g_{\alpha\beta}$  one finds the following equations

$$\begin{aligned} \frac{dg_{ij}}{dx_0} &= [g_{ij}, H_t] = [g_{ij}, H_C] = \frac{2}{\sqrt{-gg^{00}}} I_{(ij)pq} \pi^{pq} \\ &- \frac{1}{g^{00}} I_{(ij)pq} B^{(pq0|\mu\nu k)} g_{\mu\nu, k} = \frac{2}{\sqrt{-gg^{00}}} I_{(ij)pq} \\ &\times \left[ \pi^{pq} - \frac{1}{2} \sqrt{-g} B^{(pq0|\mu\nu k)} g_{\mu\nu, k} \right], \quad (39) \end{aligned}$$

where the notation  $I_{(ij)pq}$  stands for the  $(ij)$  symmetrized values of the  $I_{ijpq}$  tensor defined in Eq. (9), i.e.,

$$I_{(ij)pq} = \frac{1}{2} (I_{ijpq} + I_{jipq})$$

$$= \frac{1}{d-2} g_{ij} g_{pq} - \frac{1}{2} (g_{ip} g_{jq} + g_{iq} g_{jp}). \quad (40)$$

Analogously, for the  $g_{0\sigma}$  components of the metric tensor one finds the following equations of time-evolution

$$\frac{dg_{0\sigma}}{dx_0} = [g_{0\sigma}, H_t] = g_{0\sigma, 0}, \quad (41)$$

since all  $g_{0\sigma}$  components commute with the canonical Hamiltonian  $H_C$ , Eq. (15), while all  $g_{ij}$  commute with the primary constraints  $\phi^{0\sigma}$ . This result could be expected, since the equation, Eq. (41), is, in fact, a definition of the  $\sigma$  velocities (or  $g_{0\sigma, 0}$  derivatives), where  $\sigma = 0, 1, \dots, d-1$ .

The Hamilton equations for the tensor components of momentum  $\pi^{\alpha\beta}$ , Eq. (38), are substantially more complicated. They are derived by calculating the Poisson brackets between each term in  $H_t$  and  $\pi^{\gamma\rho}$ . This general formula takes the form

$$\begin{aligned} \frac{d\pi^{\alpha\beta}}{dx_0} &= -[H_t, \pi^{\alpha\beta}] = - \left[ \frac{I_{mnpq}}{\sqrt{-gg^{00}}}, \pi^{\alpha\beta} \right] \pi^{mn} \pi^{pq} \\ &+ \left[ \frac{I_{mnpq}}{g^{00}}, \pi^{\alpha\beta} \right] \pi^{mn} B^{(pq0|\mu\nu k)} g_{\mu\nu, k} \\ &+ \frac{1}{g^{00}} I_{mnpq} \pi^{mn} \left[ B^{(pq0|\mu\nu k)}, \pi^{\alpha\beta} \right] g_{\mu\nu, k} + \dots \quad (42) \end{aligned}$$

Let us determine the first Poisson bracket in this formula (other terms in Eq. (42) are considered analogously, i.e., term-by-term). The explicit expression for this term is

$$\begin{aligned} &- \left[ \frac{I_{mnpq}}{\sqrt{-gg^{00}}}, \pi^{\alpha\beta} \right] \pi^{mn} \pi^{pq} \\ &= - \frac{[I_{mnpq}, \pi^{\alpha\beta}]}{\sqrt{-gg^{00}}} \pi^{mn} \pi^{pq} \\ &- \left[ \frac{1}{\sqrt{-gg^{00}}}, \pi^{\alpha\beta} \right] I_{mnpq} \pi^{mn} \pi^{pq}. \quad (43) \end{aligned}$$

Thus, we have the three following cases: (1) for a pair of space-like indexes, i.e., for  $(\alpha\beta) = (ab)$ , where one finds

$$\begin{aligned} \left( \frac{d\pi^{ab}}{dx_0} \right)_1 &= - \frac{2}{d-2} g_{mn} \pi^{mn} \pi^{ab} \\ &+ 2g_{mp} \pi^{ma} \pi^{pb} + \frac{I_{mnpq}}{2\sqrt{-gg^{00}}} g^{ab} \pi^{mn} \pi^{pq}, \quad (44) \end{aligned}$$

while for the mixed pair of indexes  $(\alpha\beta) = (0a)$  the analogous expression is

$$\left( \frac{d\pi^{0a}}{dx_0} \right)_1 = \frac{I_{mnpq}}{2\sqrt{-gg^{00}}} g^{0a} \pi^{mn} \pi^{pq}. \quad (45)$$

Finally, for the temporal pair of indexes  $(\alpha\beta) = (00)$  pair one finds

$$\left(\frac{d\pi^{00}}{dx_0}\right)_1 = \frac{I_{mnpq}}{2\sqrt{-g}} \left(1 + \frac{2}{(g^{00})^2}\right) \pi^{mn} \pi^{pq}. \quad (46)$$

In general, analytical calculations of other Poisson brackets in the formula, Eq. (42), is a straightforward task, but the final formula contains more than one hundred terms. This drastically complicates all operations with the formula, Eq. (42), for the  $\frac{d\pi^{\gamma\rho}}{dx_0}$  derivative. Actual analytical and numerical computations of the time-evolution of the free gravitational field(s) can be performed by using modern packages of computer algebra. Nevertheless, the complete Hamilton equations of motion for the free gravitational field(s) in metric GR have been derived and written explicitly in a closed analytical forms.

#### 5.4. On the General Form of Canonical Transformations in Metric Gravity

As is well known all canonical transformations for an arbitrary Hamiltonian system form a closed algebraic group. This means that in any Hamiltonian system: (1) consequence of the two canonical transformations is the new canonical transformation, (2) identical transformation of dynamical variables is the canonical transformation, (3) any canonical transformation has its inverse transformation which is also canonical and unique. In general, there are quite a few canonical transformations in the metric General Relativity, and some of them can be used to simplify either Hamiltonian(s), or secondary constraints, or some other crucial quantities, including a few important Poisson brackets. As is well known (see, e.g., [9, 10]) the metric General Relativity is a non-linear theory which cannot rigorously be linearized even in second-order approximation. Therefore, the linear canonical transformations of dynamical variables are no interest for the Hamiltonian formulations which have been developed for the metric GR. Furthermore, it can be shown that among all possible non-linear canonical transformations the following ‘special’ transformations play a great role in derivation of all new Hamiltonian formulations of the metric GR. These special canonical transformations can be written in the form:  $\{g_{\alpha\beta}, \pi^{\mu\nu}\} \rightarrow \{g_{\alpha\beta}, \Pi^{\rho\sigma}\}$ , where the new momenta  $\Pi^{\rho\sigma}$  are the linear (or quasi-linear) combinations of old momenta  $\pi^{\mu\nu}$  and all spatial derivatives of the components of metric tensor. In general, such a combination is written in the form

$$\Pi^{\gamma\rho} = A_{\lambda\sigma}^{\gamma\rho} \pi^{\lambda\sigma} - \frac{1}{2} \sqrt{-g} C_{\lambda\sigma}^{\gamma\rho} D^{\lambda\sigma 0\mu\nu k} g_{\mu\nu, k}, \quad (47)$$

where  $A_{\lambda\sigma}^{\gamma\rho}$  and  $C_{\lambda\sigma}^{\gamma\rho}$  are the constant (or numerical) tensors, while the tensor-functions  $D^{\gamma\rho 0\mu\nu k}$  are the

cubic polynomial of all contravariant components of the metric tensor  $g^{\alpha\beta}$ . When the both  $A_{\lambda\sigma}^{\gamma\rho}$  and  $C_{\lambda\sigma}^{\gamma\rho}$  tensors are the diadas of the two substitution tensors, i.e., each of them equals to the product  $\delta_\lambda^\gamma \delta_\sigma^\rho$ , then from Eq. (47) one finds

$$\Pi^{\gamma\rho} = \pi^{\gamma\rho} - \frac{1}{2} \sqrt{-g} D^{\gamma\rho 0\mu\nu k} g_{\mu\nu, k}. \quad (48)$$

At this moment all known canonical transformations which arise in the metric GR are represented in the form of Eq. (47), or Eq. (48). In particular, the canonical transformation which relates the two correct Hamiltonian formulations currently known in the metric GR, i.e., Hamiltonian formulations developed in [1] and [4]. Our new transformation of Hamiltonian dynamical variables of metric GR, which is described below, is also written in the form of Eq. (48), i.e., it is canonical transformation. Therefore, we can expect that our Hamiltonian formulation is correct. Furthermore, it can be shown that transformations of the dynamical variables of metric GR, constructed in the form of Eq. (48), will preserve the complete diffeomorphism as a gauge symmetry of the free gravitational field. It is clear that such ‘special’ form of canonical transformations in the metric General relativity is essentially determined by the  $\Gamma$ - $\Gamma$  Lagrangian written in Eq. (2). Indeed, each term in this  $\Gamma$ - $\Gamma$  Lagrangian, Eq. (2), is a cubic polynomial of the contravariant  $g^{\alpha\beta}$  components of metric tensor. On the other hand, the same term is a quadratic polynomial in the first-order derivatives of the covariant components of the metric tensor  $g_{\mu\nu, k}$ . It is highly likely that this conclusion can be reversed, i.e., we can say that any transformation of Hamiltonian variables of the metric gravity, which cannot be written in the form of Eq. (47) and/or Eq. (48), is not canonical. It is clear that similar transformations cannot be used to relate two different (but canonical) Hamiltonian formulations of the metric gravity.

In order to understand a special role of the canonical transformations chosen in the form of Eq. (48), consider the difference between the two following differential forms

$$\begin{aligned} & \pi^{\alpha\beta} dg_{\alpha\beta} - \Pi^{\alpha\beta} dg_{\alpha\beta} \\ &= \frac{1}{2} \sqrt{-g} D^{\alpha\beta 0\mu\nu k} g_{\mu\nu, k} dg_{\alpha\beta}. \end{aligned} \quad (49)$$

This form is, in fact, the difference between the two relative integral invariants, or Poincaré integral invariants. The well known Poincaré theorem (see, e.g., [12]) states that the transformation  $\{g_{\alpha\beta}, \pi^{\mu\nu}\} \rightarrow \{g_{\alpha\beta}, \Pi^{\mu\nu}\}$  of dynamical variables will be canonical if (and only if) such a difference of the two relative integral invariants, i.e., the expression on the right-hand side of the last equation, will be a total differential. This means that canonicity of the new

variables will be obeyed only in those cases, when the expression  $\frac{1}{2}\sqrt{-g}D^{\alpha\beta 0\mu\nu k}g_{\mu\nu,k}dg_{\alpha\beta}$  is the total differential of some function. Now, the integrability conditions are written in the following forms

$$\frac{\partial [\sqrt{-g}D^{\alpha\beta 0\mu\nu k}]}{\partial g_{\lambda\sigma}} = \frac{\partial [\sqrt{-g}D^{\lambda\sigma 0\mu\nu k}]}{\partial g_{\alpha\beta}}, \quad (50)$$

which are completely equivalent to the conditions  $[\Pi^{\alpha\beta}, \Pi^{\lambda\sigma}] = 0$  for the fundamental Poisson brackets of the new dynamical variables  $\{g_{\alpha\beta}, \Pi^{\lambda\sigma}\}$ . Indeed, for these Poisson brackets we can write

$$\begin{aligned} 0 &= [\Pi^{\alpha\beta}, \Pi^{\lambda\sigma}] \\ &= -\frac{1}{2}[\pi^{\alpha\beta}, \sqrt{-g}D^{\lambda\sigma 0\mu\nu k}]g_{\mu\nu,k} \\ &\quad + \frac{1}{2}[\pi^{\lambda\sigma}, \sqrt{-g}D^{\alpha\beta 0\mu\nu k}]g_{\mu\nu,k}, \end{aligned} \quad (51)$$

which is easily reduced to the form of Eq. (50). It is interesting to compare this equation to Eq. (29) from [5] which also has the form of Poisson brackets. However, it is also a set of integrability conditions [21] for some differential 1-form. If Eq. (50) are obeyed, then the both 1-forms  $\pi^{\alpha\beta}dg_{\alpha\beta}$  and  $\Pi^{\alpha\beta}dg_{\alpha\beta}$  in Eq. (49) are the Poincaré intergal invariants. As follows from this discussion all fundamental Poisson brackets between basic dynamical variables of an arbitrary Hamiltonian system can be split into three different groups and two of these groups can be considered as the systems of integrability conditions. In general, the Poisson brackets from the third group, i.e.  $[p_i, p_j] = 0$ , are, in fact, the integrability conditions for some first-order differential form [21] which is written in the coordinate space. Analogously, the Poisson brackets from the second group, i.e.,  $[q_i, q_j] = 0$ , can be considered as the integrability conditions for same first-order differential form written in momentum representation.

## 6. REDUCTION OF THE CANONICAL HAMILTONIAN TO ITS NATURAL FORM

In this Section we reduce the canonical Hamiltonian  $H_C$  to its natural form, which will play a significant role in numerous applications to the metric gravity. We perform such a reduction of  $H_C$  by using some new canonical transformation of the dynamical variables  $g_{\alpha\beta}$  and  $\pi^{\rho\sigma}$  defined above. First, let us write the canonical Hamiltonian, Eq. (15), in the form

$$\begin{aligned} H_C &= \frac{I_{mnpq}}{\sqrt{-g}g^{00}} \\ &\times \left[ \pi^{mn}\pi^{pq} - \sqrt{-g}\pi^{mn}B^{(pq0|\mu\nu k)}g_{\mu\nu,k} \right. \end{aligned}$$

$$\begin{aligned} &\left. + \frac{1}{4}(-g)B^{(mn0|\mu\nu k)}B^{(pq0|\alpha\beta l)}g_{\mu\nu,k}g_{\alpha\beta,l} \right] \\ &+ \frac{1}{4}\sqrt{-g}\left\{ \frac{1}{g^{00}}I_{mnpq}B^{([mn]0|\mu\nu k)}B^{(pq0|\alpha\beta l)} \right. \\ &\quad \left. - B^{\mu\nu k\alpha\beta l} \right\}g_{\mu\nu,k}g_{\alpha\beta,l}, \end{aligned} \quad (52)$$

which is more appropriate for our purposes in this study. In Eq. (52) the notation  $B^{([mn]0|\mu\nu k)}$  stands for the  $B^{(mn0|\mu\nu k)}$  cubic polynomial of the contravariant components of the metric tensor which is completely anti-symmetric in respect to all permutations of the  $m$  and  $n$  indexes. The explicit formula for the  $B^{([mn]0|\mu\nu k)}$  function is

$$\begin{aligned} B^{([mn]0|\mu\nu k)} &= g^{mk}g^{n\nu}g^{\nu 0} - g^{nk}g^{m\nu}g^{\nu 0} \\ &+ \frac{1}{2}(g^{n\mu}g^{m\nu}g^{k0} + g^{nk}g^{\mu\nu}g^{m0} - g^{m\mu}g^{n\nu}g^{k0} \\ &\quad - g^{mk}g^{\mu\nu}g^{m0}). \end{aligned} \quad (53)$$

Now, we can see that the first term in [...] brackets in Eq. (52) can be written as a pure quadratic function of the new  $P^{mn} = \pi^{mn} - \frac{1}{2}\sqrt{-g}B^{(mn0|\mu\nu k)}g_{\mu\nu,k}$  variables (spatial momenta), i.e.,

$$\begin{aligned} H_C &= \frac{I_{mnpq}}{\sqrt{-g}g^{00}} \left( \pi^{mn} - \frac{1}{2}\sqrt{-g}B^{(mn0|\mu\nu k)}g_{\mu\nu,k} \right) \\ &\quad \times \left( \pi^{pq} - \frac{1}{2}\sqrt{-g}B^{(pq0|\alpha\beta l)}g_{\alpha\beta,l} \right) \\ &+ \frac{1}{4}\sqrt{-g}\left\{ \frac{1}{g^{00}}I_{mnpq}B^{([mn]0|\mu\nu k)}B^{(pq0|\alpha\beta l)} \right. \\ &\quad \left. - B^{\mu\nu k\alpha\beta l} \right\}g_{\mu\nu,k}g_{\alpha\beta,l} + T_1 + T_2, \end{aligned} \quad (54)$$

where the two additional terms  $T_1$  and  $T_2$  are:

$$\begin{aligned} T_1 &= \frac{I_{mnpq}}{2\sqrt{-g}g^{00}}[\pi^{mn}, \sqrt{-g}]B^{(pq0|\alpha\beta l)}g_{\alpha\beta,l} \\ &= -\frac{I_{mnpq}g^{mn}}{2g^{00}}B^{(pq0|\alpha\beta l)}g_{\alpha\beta,l} \end{aligned} \quad (55)$$

and

$$\begin{aligned} T_2 &= -\frac{I_{mnpq}}{2g^{00}}[B^{(mn0|\mu\nu k)}, \pi^{pq}]g_{\mu\nu,k} \\ &= -\frac{I_{mnpq}}{2g^{00}}\left[ \frac{1}{2}(g^{\mu p}g^{mq} + g^{\mu q}g^{mp})g^{n\nu}g^{k0} \right. \\ &\quad + \frac{1}{2}g^{\mu m}(g^{np}g^{\nu q} + g^{nq}g^{\nu p})g^{k0} \\ &\quad + \frac{1}{2}g^{\mu m}g^{n\nu}(g^{kp}g^{0q} + g^{kq}g^{0p}) \\ &\quad \left. - \frac{1}{2}(g^{mp}g^{nq} + g^{mq}g^{np})g^{k0}g^{\mu\nu} \right] \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{2}g^{mn} (g^{pk}g^{q0} + g^{p0}g^{qk}) \\
 & -\frac{1}{2}g^{mn}g^{k0} (g^{\mu p}g^{\nu q} + g^{\mu q}g^{\nu p}) \\
 & - (g^{mp}g^{kq} + g^{mq}g^{kp}) g^{\nu\mu}g^{\mu 0} \\
 & - g^{mk} (g^{np}g^{\nu q} + g^{nq}g^{\nu p}) g^{\mu 0} \\
 & -\frac{1}{2}g^{mk}g^{\nu\mu} (g^{\mu p}g^{0q} + g^{\mu q}g^{0p}) \\
 & +\frac{1}{2}(g^{mp}g^{nq} + g^{mq}g^{np}) g^{\nu k}g^{0\mu} \\
 & +\frac{1}{2}g^{mn} (g^{\nu p}g^{kq} + g^{\nu q}g^{kp}) g^{\mu 0} \\
 & +\frac{1}{2}g^{mn}g^{\nu k} (g^{p0}g^{\mu q} + g^{0q}g^{\mu p}) \\
 & +\frac{1}{2}(g^{kp}g^{mq} + g^{kq}g^{mp}) g^{\nu k}g^{0\mu} \\
 & +\frac{1}{2}g^{km} (g^{\mu p}g^{\nu q} + g^{p\nu}g^{\mu q}) g^{n0} \\
 & +\frac{1}{2}g^{km}g^{\mu\nu} (g^{np}g^{0q} + g^{nq}g^{0p}) \Big] g_{\mu\nu,k}, \tag{56}
 \end{aligned}$$

respectively.

Now, we can explicitly introduce the new momenta  $P^{\gamma\rho}$  which are written in the following form

$$P^{\gamma\rho} = \pi^{\gamma\rho} - \frac{1}{2}\sqrt{-g}B^{(\gamma\rho 0|\mu\nu k)}g_{\mu\nu,k}, \tag{57}$$

where  $\pi^{\gamma\rho}$  are the ‘old’ momenta used in [4]. These new momenta can be considered as the contravariant components of the tensor of one ‘united’ momentum  $P$  of the metric gravitational field. Note that the explicit expressions for the old velocities written in terms of new momenta  $P^{ab}$  are even simpler  $g_{mn,0} = \frac{1}{\sqrt{-g}g^{00}}I_{mnpq}P^{pq}$ , than the expression given by Eq. (11). The explicit formulas for the primary constraints are also simpler:  $P^{0\gamma} \approx 0$  for  $\gamma = 0, 1, \dots, d-1$ . The generalized coordinates are chosen in the old (or traditional) form, i.e., they coincide with the covariant components of the metric tensor  $g_{\alpha\beta}$ . Such a choice of the generalized coordinates provides a number of additional advantages in applications to metric gravity. For instance, by using the metric tensor one can raise and lower indexes in arbitrary vectors and tensors. Also, all tensor derivatives of the metric tensor always equal zero, i.e., this tensor behaves as a constant during similar operations. More unique and remarkable properties of the metric tensor are discussed, e.g., in [7]. For the purposes of this study it is important to note that our new system of dynamical variables contains the same coordinates  $g_{\alpha\beta}$  and new momenta  $P^{\gamma\rho}$ .

The Poisson brackets between our new dynamical variables can easily be determined by using the known

values of Poisson brackets written in the old dynamical variables  $\{g_{\alpha\beta}, \pi^{\gamma\rho}\}$  defined above. Indeed, for the corresponding Poisson brackets one finds:  $[g_{\alpha\beta}, P^{\gamma\rho}] = [g_{\alpha\beta}, \pi^{\gamma\rho}] = \Delta_{\alpha\beta}^{\gamma\rho} = \frac{1}{2}(\delta_{\alpha}^{\gamma}\delta_{\beta}^{\rho} + \delta_{\alpha}^{\rho}\delta_{\beta}^{\gamma})$ ,  $[g_{\alpha\beta}, g_{\gamma\rho}] = 0$  and  $[P^{\alpha\beta}, P^{\gamma\rho}] = 0$ . The last equality we consider in detail

$$\begin{aligned}
 [P^{\alpha\beta}, P^{\gamma\rho}] &= [\pi^{\alpha\beta}, \pi^{\gamma\rho}] \\
 &- \frac{1}{2}[\sqrt{-g}B^{(\alpha\beta 0|\mu\nu k)}, \pi^{\gamma\rho}]g_{\mu\nu,k} \\
 &+ \frac{1}{2}[\sqrt{-g}B^{(\alpha\beta 0|\lambda\sigma l)}, \pi^{\gamma\rho}]g_{\lambda\sigma,l} \\
 &+ [\sqrt{-g}B^{(\alpha\beta 0|\mu\nu k)}g_{\mu\nu,k}, \sqrt{-g}B^{(\gamma\rho 0|\lambda\sigma l)}g_{\lambda\sigma,l}], \tag{58}
 \end{aligned}$$

where the first and last terms equal zero identically, since the variables  $g_{\alpha\beta}$  and  $\pi^{\mu\nu}$  are canonical. This directly leads to the formula

$$\begin{aligned}
 [P^{\alpha\beta}, P^{\gamma\rho}] &= -\frac{1}{2}[\sqrt{-g}B^{(\alpha\beta 0|\mu\nu k)}, \pi^{\gamma\rho}]g_{\mu\nu,k} \\
 &+ \frac{1}{2}[\sqrt{-g}B^{(\alpha\beta 0|\lambda\sigma l)}, \pi^{\gamma\rho}]g_{\lambda\sigma,l}. \tag{59}
 \end{aligned}$$

Now, we can replace the dummy indexes in the second term of this equation by the values which coincide with the corresponding dummy indexes in the first term, i.e.,  $\lambda \rightarrow \mu, \sigma \rightarrow \nu$  and  $l \rightarrow k$ . This substitution reduces Eq. (59) to the form (compare with Eq. (51) from above)

$$\begin{aligned}
 [P^{\alpha\beta}, P^{\gamma\rho}] &= -\frac{1}{2}[\sqrt{-g}B^{(\alpha\beta 0|\mu\nu k)}, \pi^{\gamma\rho}]g_{\mu\nu,k} \\
 &+ \frac{1}{2}[\sqrt{-g}B^{(\alpha\beta 0|\mu\nu,k)}, \pi^{\gamma\rho}]g_{\mu\nu,k} = 0, \tag{60}
 \end{aligned}$$

since it is the difference of the two identical expressions. This shows that the ‘new’ dynamical variables  $\{g_{\alpha\beta}, P^{\mu\nu}\}$  are also canonical, and they can be used in the metric gravity, since they are canonically related to the ‘old’ set of K&K variables  $\{g_{\alpha\beta}, \pi^{\mu\nu}\}$  [4].

As follows from the formulas derived above the canonical Hamiltonian  $H_C$  is reduced to the following final form

$$\begin{aligned}
 H_C &= \frac{I_{mnpq}}{\sqrt{-g}g^{00}}P^{mn}P^{pq} \\
 &+ \frac{1}{4}\sqrt{-g}\left[\frac{I_{mnpq}}{g^{00}}B^{([mn]0|\mu\nu k)}B^{(pq0|\alpha\beta l)}\right. \\
 &\quad \left.- B^{\mu\nu k\alpha\beta l}\right]g_{\mu\nu,k}g_{\alpha\beta,l} \\
 &- \frac{I_{mnpq}}{2g^{00}}g^{mn}B^{(pq0|\alpha\beta l)}g_{\alpha\beta,l} + T_2, \tag{61}
 \end{aligned}$$

which can be re-written in the following symbolic form

$$H_C = \frac{1}{2} \sum_{i,j=1}^n \hat{M}_{ij}(q_1, q_2, \dots, q_n) p_i p_j + \sum_{i,j=1}^n \hat{V}_{mn}(q_1, q_2, \dots, q_n), \quad (62)$$

where  $\hat{M}$  is a positively defined and invertable  $n \times n$  (symmetric) matrix which is often called the matrix of inverse masses. The  $\hat{V}$  matrix in this equation is an arbitrary, in principle, symmetric  $n \times n$  matrix which is called the potential matrix (or matrix of the potential energy). Here  $n$  is the total number of generalized coordinates  $q_1, q_2, \dots, q_n$ . Each matrix element of the potential matrix  $\hat{V}$  in Eq. (62) is a polynomial which depends upon these generalized coordinates. Also, in Eq. (62) the notation  $p_i$  and  $p_j$  designate the momenta conjugate to the corresponding generalized coordinates  $q_i$  and  $q_j$ , respectively, i.e.,  $[q_k, p_l] = \delta_{kl}$ . In classical mechanics the phase space is flat, and, therefore, the both covariant and contravariant components of any vector coincide with each other. The form of the Hamiltonian  $H_C$ , Eq. (62), is called normal, and it is well known in classical mechanics of Hamiltonian systems. Furthermore, more than 90% of all problems ever solved in classical Hamiltonian mechanics have Hamiltonians which are already written in the normal form, or their Hamiltonians can easily be reduced to their normal forms by some canonical transformation(s) of variables. The idea of reducing the Hamiltonians to their normal forms goes back to Poincaré [22]. A separate area of modern mathematical physics is the study of the normal forms of different Hamiltonians in the vicinities of equilibrium positions [23] (see also discussion and references in the Appendix 7 from [24]).

To improve the overall quality of our analogy between metric GR and classical Hamiltonian mechanics let us introduce the new set of dynamical variables which include the total momentum of the free gravitational field  $P = g_{\alpha\beta} P^{\alpha\beta}$  (it is a tensor invariant) and its tensor ‘projections’  $P_\alpha^\beta = g_{\alpha\gamma} P^{\gamma\beta}$ . The corresponding space-like quantities  $P = g_{mn} P^{mn}$  and  $P_m^n = g_{mp} P^{pn}$  are included in our canonical Hamiltonian  $H_C$ , Eq. (61). By using our formulas presented above one easily finds a few following Poisson brackets:

$$\begin{aligned} [P, P^{ab}] &= [g_{mn}, P^{ab}] P^{mn} = \Delta_{mn}^{ab} P^{mn} = P^{ab}, \\ [g_{cd}, P] &= g_{mn} [g_{cd}, P^{mn}] = g_{cd}, \\ [g_{\alpha\beta}, P_\sigma^\gamma] &= \frac{1}{2} (g_{\beta\sigma} \delta_\alpha^\gamma + g_{\alpha\sigma} \delta_\beta^\gamma), \\ [g^{\alpha\beta}, P] &= g^{\alpha\beta} \end{aligned}$$

and others. By using the total momentum  $P$  and its tensor projections (i.e.,  $P^{\alpha\beta}, P_\sigma^\gamma$ , etc) one can write the Hamilton equations in the form which almost coincides with analogous equations known for Hamiltonian systems in classical mechanics. This is another interesting direction for future development of the Hamiltonian formulation(s) of metric GR. Other applications of our new canonical variables  $\{g_{\lambda\kappa}, P^{\alpha\beta}\}$  to some interesting problems in metric GR will be considered elsewhere. Relations between our dynamical variables  $\{g_{\lambda\kappa}, P^{\alpha\beta}\}$  and analogous variables used in Dirac formulation of the metric General Relativity  $\{g_{\lambda\kappa}, p^{\alpha\beta}\}$  are discussed in the Appendix A.

## 7. JACOBI EQUATION FOR THE FREE GRAVITATIONAL FIELD IN OUR HAMILTONIAN FORMULATION

In general, if we have found the canonical Hamiltonian  $H_C$  and properly defined all essential momenta  $\pi^{mn}$  of the free gravitational field, then it is possible to derive the famous Jacobi equation which governs propagation and time-evolution of the free gravitational field. For the first time the Jacobi equation for the free gravitational field has been derived in our earlier paper [6], which was based on the Hamiltonian formulation of metric gravity developed in [4]. In this study to describe the free gravitational field we apply different dynamical variables  $\{g_{\lambda\kappa}, P^{\alpha\beta}\}$  which are canonically related to the  $\{g_{\lambda\kappa}, \pi^{\alpha\beta}\}$  dynamical variables from [4]. Let us derive analogous Jacobi equation written in our new dynamical variables  $\{g_{\lambda\kappa}, P^{\alpha\beta}\}$ . Complete derivation of the Jacobi equation(s) for the free gravitational field in the metric gravity is complex and requires many pages of additional text. To simplify the process, below restrict ourselves to a brief derivation of the Jacobi equation by varying the gravitational action  $S$  written in the form of temporal integral of the  $\Gamma$ – $\Gamma$  Lagrangian  $L$ . This can be written in the form

$$\begin{aligned} &\delta S(g_{\alpha\beta}(t), t) \\ &= \delta \int_\gamma L_{\Gamma-\Gamma}(\tau, g_{\alpha\beta}(\tau), g_{\alpha\beta;0}(\tau)) d\tau, \quad (63) \end{aligned}$$

where the integral is taken along the extremal  $\gamma$  which connects the initial  $A$  point (or  $t_0$ -point) and the final  $B$  point (or  $t_f = t$ -point). The final point is a free point which is varied when we make a choice between extremals. Note that the variation of any action, including the gravitational action  $\delta S(g_{\alpha\beta}(t), t)$  between two actual extremals is always the first (and total) differential of the function  $S$  (or action  $S$ ) [15]. Furthermore, for any Hamiltonian system the differential

$dS$  is written in the form  $dS = \sum_{i=1}^n p_i dy_i - H dt$ , where  $H$  is the Hamiltonian, while  $p_i$  are the momenta. From the expression for the first differential  $dS$  one finds the two following equations  $H = -\frac{\partial S}{\partial t}$  and  $p_i = \frac{\partial S}{\partial y_i}$ . From these equations we find that  $S$ , as a function of the coordinates of the final point  $B(t = t_f)$ , satisfies the following equation

$$\frac{\partial S}{\partial t} + H\left(t; y_1, \dots, y_n; \frac{\partial S}{\partial y_1}, \dots, \frac{\partial S}{\partial y_n}\right) = 0, \quad (64)$$

which is called the Jacobi equation. Here we have assumed that all extremals of our problem, which begin at the given initial point do not intersect each other, but form a central field of extremals [15]. In other words, our extremal is imbedded in a central field of extremals which start at a given initial point  $t_0$ . It appears that the canonical Hamilton equations:  $\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$  and  $\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$  are the system of characteristic equations for the Jacobi equation which is non-linear (see, e.g., [25, 26 and [27]). This is the shortest way to the Jacobi equation from the canonical Hamilton equations. The direct physical meaning of the Jacobi equation and its solutions (for Hamiltonian systems only!) was formulated in our earlier paper [6] as follows: *the real trajectory of the system propagates in time from the end point of one (old) Lagrange extremal to the end point located at the new Lagrange extremal, if it satisfies the Jacobi equation*. In other words, the Hamilton-Jacobi equation is the necessary condition for some currently known system's trajectory, which propagates in the nearest future, to reach (at fixed time) the end point of the 'new' extremal.

For the free gravitational field in metric gravity there are a few additional complications, since the field itself is a tensor and metric gravity is a dynamical system with constraints. Nevertheless, it is possible to write two similar differential 1-forms  $dS = \pi^{mn} dg_{mn} - H_C dt$  and  $dS = \pi^{\alpha\beta} dg_{\alpha\beta} - H_t dt$  in multi-dimensional phase spaces. The first  $dS = \pi^{mn} dg_{mn} - H_C dt$  form is defined in the  $[d(d - 1) + 1]$  dimensional phase space, while the second form  $dS = \pi^{\alpha\beta} dg_{\alpha\beta} - H_t dt$  is defined in the  $[d(d + 1) + 1]$  dimensional phase space. The both these spaces are odd-dimensional and, therefore, it is correct to introduce the differential operators such as *curl* (or *rotor*), which are defined in each of these spaces, and derive the closed system of the canonical Hamilton equations (applicability of the Stoke's theorem is discussed, e.g., in [21] and references therein). Then, by using this system of the canonical Hamilton equations we need to restore the original (non-linear) Jacobi equation for which these Hamilton equations are the system of characteristic equations [15] and [25].

In our dynamical variables we can write the two differential forms  $dS = P^{mn} dg_{mn} - H_C dt$  and  $dS = P^{\alpha\beta} dg_{\alpha\beta} - H_t dt$ , where the dynamical variables are  $\{g_{\lambda\kappa}, P^{\alpha\beta}\}$ , while the canonical  $H_C$  and total  $H_t$  Hamiltonians have been defined in the previous Section. These differential forms are included (as integrands) in the main integral invariant of mechanics, which is also known as the Poincaré-Cartan integral invariant [12]. Now, by using these forms and procedure described in [12] (see also [6]) one can derive (or restore) the original Jacobi equation. To simplify this (Jacobi) equation from the very beginning we introduce the new (local) temporal variable  $dx_0 = \sqrt{-g} g^{00} dy_0$ . In this variable the Jacobi equation takes the following form

$$\begin{aligned} -\left(\frac{\partial S}{\partial y_0}\right) &= I_{mnpq} \left(\frac{\partial S}{\partial g_{mn}}\right) \left(\frac{\partial S}{\partial g_{pq}}\right) \\ &+ \frac{1}{4}(-g)[I_{mnpq} B^{([mn]0|\mu\nu k)} B^{(pq0|\alpha\beta l)} \\ &\quad - g^{00} B^{\mu\nu k\alpha\beta l}] g_{\mu\nu, k} g_{\alpha\beta, l} \\ &- \frac{1}{2}\sqrt{-g} I_{mnpq} g^{mn} B^{(pq0|\alpha\beta l)} g_{\alpha\beta, l} \\ &\quad + \sqrt{-g} g^{00} T_2, \end{aligned} \quad (65)$$

where the explicit formula for the  $T_2$  term is given by Eq. (56). This equation is the actual Jacobi equation (also called the Hamilton-Jacobi equation) for the free gravitational field in the metric GR, which has been derived in our new Hamiltonian formulation. As expected this Jacobi equation does not contain terms which are linear upon the partial  $\frac{\partial S}{\partial g_{mn}}$  derivatives of the gravitational action  $S$ .

As we have mentioned in [6] in analytical mechanics (see, e.g., [12, 24]) the methods based on the Jacobi equation are considered as the most effective procedures ever created to analyze the motion of an arbitrary, in principle, Hamiltonian system. It is also clear that all methods based on the Jacobi equation are usually very effective for dynamical systems with Hamiltonians which contain only a few relatively small powers of all essential momenta. This obviously includes the free gravitational field in metric GR, where the canonical and total Hamiltonians  $H_C$  and  $H_t$  are the quadratic functions of space-like momenta  $\pi^{mn}$ , Eq. (15). The total Hamiltonian is also a linear function of  $\sigma$  momenta  $\pi^{0\sigma}$ , or primary constraints  $\phi^{0\sigma}$ . To conclude this brief Section, let us note that if we draw an analogy with optics, then the momenta of the gravitational field ( $\pi^{\alpha\beta}$ , or  $P^{\alpha\beta}$ ) should be called the tensor components of normal slowness tensor, while the gravitational action should be called the

gravitational path length. In this language our Jacobi equation, Eq. (65), plays the role of Huygens' principle for the free gravitational field. Although the physical meaning of similar analogies is quite limited.

## 8. DISCUSSIONS AND CONCLUSION

In this study we have developed the new, physically transparent and logically self-consistent Hamiltonian formulation of the metric gravity. We have determined all essential (fundamental and secondary) Poisson brackets which can now be used to perform a large amount of analytical and numerical calculations. The fundamental Poisson brackets are defined between all components of the gravitational field and corresponding momenta (or components of the momentum tensor). The secondary Poisson brackets define commutation relations between arbitrary, in principle, analytical functions of coordinates (components of the gravitational field) and momenta. These Poisson brackets become the main working tools of the metric gravity, which can now be considered as an actual Hamiltonian system. The Poisson brackets defined in this paper can be used to solve various problems in metric gravity, e.g., obtain trajectories, derive and confirm new conservation laws, find integrals of motion, derive and investigate the laws of time-evolution for different quantities, vectors and tensors.

Our approach allows one to determine the Poisson brackets from the two sets of basic dynamical variables: (a) set of straight dynamical variables, e.g.,  $\{g_{\alpha\beta}, \pi^{\gamma\rho}\}$  (or  $\{g_{\alpha\beta}, P^{\gamma\rho}\}$ ), and (b) dual set of basic dynamical variables  $\{g^{\alpha\beta}, \pi_{\gamma\rho}\}$  (or  $\{g^{\alpha\beta}, P_{\gamma\rho}\}$ ). We have found that these two sets of dynamical variables are always needed to construct the truly covariant and correct Hamiltonian formulations of the metric gravity. The fundamental relation between these two sets of dynamical variables is given by the Poisson bracket, Eq. (20). In our new dynamical variables the same relation takes the form  $[g_{\alpha\beta}, P^{\mu\nu}] = \Delta_{\alpha\beta}^{\mu\nu} = [P_{\alpha\beta}, g^{\mu\nu}]$ . The straight and dual sets of canonical (tensor) variables complement each other and they are crucially important to develop any non-contradictory Hamiltonian approach to a system of interacting tensor fields, including the metric gravity.

Another remarkable result obtained in this study should be emphasized here again: the canonical Hamiltonian  $H_C$ , which describes time-evolution of relativistic gravitational fields, can be reduced to its natural form, and this form is quadratic upon all essential momenta and coincides with the Hamiltonian of the non-relativistic system of  $N(=d)$  interacting particles. Physical meaning of dynamical variables is obviously very different in both these cases, but almost identical forms of their Hamiltonians were

absolutely unexpected. Briefly, the canonical Hamiltonian  $H_C$  of the free gravitational field(s), Eq. (52), is reduced to the natural form, Eq. (61), which includes a pure quadratic function of the space-like momenta  $P^{mn}$  with a positive coefficient in front of it. Indeed, the factor, which is located in front of the  $P^{mn}P^{pq}$  product in the  $H_C$  Hamiltonian, is the positively defined space-like tensor of the fourth rank  $I_{mnpq}$  (or  $\frac{1}{\sqrt{-g}}I_{mnpq}$ ). This factor can be considered as an effective inverse 'quasi-mass' tensor of the free gravitational field in metric GR. Also, as directly follows from the explicit form of the canonical Hamiltonian  $H_C$ , Eq. (61), each of the remaining terms in this canonical Hamiltonian  $H_C$  is a finite polynomial function of contravariant components  $g^{\alpha\beta}$  of the metric tensor. The maximal power of such polynomials upon  $g^{\alpha\beta}$  does not exceed eight. Some terms in the  $H_C$  Hamiltonian also include the factors  $\sqrt{-g}$  (or  $\frac{1}{\sqrt{-g}}$ ) and  $g^{00}$  (or  $\frac{1}{g^{00}}$ ) component of the metric tensor.

Note also that during our investigations we have constructed the set of new canonical  $\{g_{\alpha\beta}, P^{\gamma\rho}\}$  variables for the metric gravity. The total number of canonical variables equals  $2d$ . The Poisson brackets between these variables are:  $[g_{\alpha\beta}, P^{\gamma\rho}] = \Delta_{\alpha\beta}^{\gamma\rho} = \frac{1}{2}(\delta_{\alpha}^{\gamma}\delta_{\beta}^{\rho} + \delta_{\alpha}^{\rho}\delta_{\beta}^{\gamma}) = [P_{\gamma\rho}, g^{\alpha\beta}]$ ,  $[g_{\alpha\beta}, g_{\gamma\sigma}] = 0$  and  $[P^{\alpha\beta}, P_{\gamma\rho}] = 0$ . This indicates clearly that these new dynamical variables are truly canonical and can be used in the new Hamiltonian formulation of the metric gravity together with analogous set of dynamical variables  $\{g^{\alpha\beta}, P_{\gamma\rho}\}$  which is the dual set of canonical variables.

To conclude our analysis let us formulate explicitly all essential, basic principles of the Hamiltonian formulations of metric gravity, which must be fulfilled during construction of any working, physically significant and consistent Hamiltonian theory of the coupled tensor fields, including the free gravitational field(s). For simplicity, these principles can be separated into three following groups: (1) the general classical Hamilton-Jacobi principles (generalized to the coupled tensor fields (see Section 3 above)) which are applied to any regular Hamiltonian approach, (2) Dirac rules developed for dynamical systems with constraints, which include the Dirac closure, and (3) conservation of the actual gauge symmetry during transition to the Hamiltonian approach (the Kiriushcheva-Kuzmin criterion of gauge conservation). Note that the principles from these three groups have extensively been discussed in this study. Nevertheless, here we repeat them to emphasize the crucial aspects of their definitions. First of all, we assume that there is the original non-singular Lagrangian which is written as an explicit

function of all generalized coordinates and corresponding velocities. The first group of fundamental principles (or rules) tell us that transition from the original Lagrangian to the final Hamiltonian must be performed properly and unambiguously, e.g., by using the Legendre transformation. The arising Hamiltonian must be an explicit function of the momenta, which are conjugate to the corresponding velocities, and cannot include any of the essential velocities (or velocities of non-constraint motions). After transition to the final Hamiltonian(s) any introduction and/or injection of the new dynamical variables into these Hamiltonian(s) is strictly prohibited. Ignoring this simple rule leads to fundamental errors that are fatal for any Hamiltonian theory (examples are discussed in [28] and in our Appendix B).

These three principles of Hamiltonian formulations must be applied in the natural order, i.e., from step 1 to step 3. This can be illustrated by the following example. Suppose that the new set of phase variables was introduced to describe the time-evolution of metric gravitational fields. The main question is to show that your new momenta and coordinates are the canonical variables for the metric gravity. At the first stage one needs to check all essential Poisson brackets between all your variables in the both straight and dual phase spaces (see Section 3 above). If the new variables passed this step, then it is necessary to derive all first-class constraints and check the actual closure of the constraint chain (Dirac closure). In addition to this, all PB between the first-class constraints must be expressed as quasi-linear combinations of these constraints and canonical  $H_C$  Hamiltonian. If this step has also been performed with no contradiction, then we have to go to the last step of the procedure and check the conservation of gauge symmetry originally known for the given dynamical system with constraints. At this step by using all first-class (or essential) constraints (i.e., all primary, secondary, etc, constraints) which have been derived at the previous step, one needs to show that the corresponding gauge generators, which are constructed with the help of Castellani procedure [17], allows one to restore the correct and complete gauge invariance (diffeomorphism) of the original system. In general, the last step is the most difficult step for actual checking, since all analytical computations here are very complex and require your constant and substantial attention.

In the case of metric gravity this three step procedure works very well, and it allows one restore the complete diffeomorphism as the correct gauge symmetry of the free gravitational field. The diffeomorphism plays a central role in the Hamiltonian metric gravity, since it guarantees that the connection between components of metric tensor  $g_{\alpha\beta}$  (or

$g^{\alpha\beta}$ ) and coordinates  $x^\mu$  is analytical and uniform at any time (see Appendix C). Note also that all currently known Hamiltonian formulations of the metric General Relativity are very complicated procedures, which are in dozens of times more complicated than analogous Hamiltonian formulations of the Maxwell electrodynamics. To operate successfully with the different Hamiltonian formulations of metric GR one needs to be familiar with the classical Hamiltonian procedures and tensor calculus. On the other hand, it is absolutely necessary to know well the both Dirac approach to the constrained dynamical systems and Castellani procedure which allows one to determine all generators of the actual gauge transformations. The only, but very substantial, indulgence is the fact that the checking of Dirac's closure and derivation of the diffeomorphism with the use of Castellani procedure can be performed on the zero-surface of primary constraints (on-shell).

General principles of Hamiltonian formulation(s) are formulated here in the form which can be generalized to many other dynamical systems with constraints. For the metric gravity we have another crucial restriction, which follows from the fact that the structure and numbers of all essential, first-class constraints arising in the metric gravity are well known. Indeed, any new Hamiltonian formulation of the metric GR must lead to the  $d$  primary constraints and  $d$  secondary constraints [6] where  $d$  is the dimension of our space-time manifold (time is always assumed to be scalar, one-dimensional variable). There is no way around this fact in the metric GR based on the  $\Gamma$ - $\Gamma$  Lagrangian, but the explicit forms of all these constraints in the new variables can be substantially different. As follows from our discussion of the fundamental principles of Hamiltonian formulation(s), currently, there are only three different, true canonical Hamiltonian formulations of the metric gravity: Dirac formulation [1], K&K formulation [4] and formulation based on the use of dynamical variables mentioned in Section 5. Numerous quasi-Hamiltonian constructions created in this area of science since the end of 1950's are not canonical Hamiltonian formulations of the metric gravity. Therefore, it is useless to discuss that someone could 'quantize' metric gravity by using similar quasi-Hamiltonian constructions. In addition to this, it is clear that quantization of the co- and contravariant components of any tensor fields require a completely new procedure, since we have at least two different uncertainty relations. This follows from the fact that the two fundamental Poisson brackets  $[\pi^{\alpha\beta}, g_{\mu\nu}]$  and  $[\pi^{\alpha\beta}, g^{\mu\nu}]$  which must be obeyed simultaneously. There are some other similar facts which must be explained well before such a quantization of metric GR can be completed.

Finally, as we all know many physicists called and considered the General Relativity (or metric GR in our words) as “the most beautiful of all existing physical theories” (see, e.g., [9], page 228). Here we wish to note that the correct Hamiltonian formulation of the metric General Relativity (or, gravity, for short) is also very beautiful physical theory. Furthermore, the truly covariant, very powerful and explicitly beautiful apparatus of this theory corrects everybody (even its authors), if they make one step away from the unique, truly covariant and correct direction of actual theory. No comparison can be made with an ugly form of the original geometrodynamics [18, 19, 29–33] (see, also Appendix) and other similar Hamiltonian-like creations (see discussion and references in [28]), which were declared to be canonically related with the geometrodynamics. Note that Hawking in [34] called this ‘super-advanced’ geometrodynamics as the theory which “contradicts to the whole spirit of General Relativity”. As is shown in the Appendix B all these theories are based on the use of non-canonical variables. Therefore, all these theories and constructions have nothing to do with the actual Hamiltonian (metric) gravity. This explains the current catastrophe of Western gravity in application to many actual problems of metric General Relativity. Since 1959 more than 2000 papers were published in numerous journals about Hamiltonian formulations of the metric gravity and their applications where their authors tried to predict and describe various gravitational phenomena. This also includes the famous paper by Isham and Kuchar [33] where authors suddenly discovered that ADM-formulation of metric gravity cannot reproduce, in principle, the well known diffeomorphism invariance of the free gravitational field. All these theories were based on the ADM variables, and on other similar sets of variables which are canonically related to ADM variables. This includes so-called Ashtykar dynamical variables, variables used in loop quantum gravity, etc (more details can be found in [28]). Similar ‘advanced’ variables are not canonical variables for the metric gravity and cannot be transformed (canonically) in such variables. In addition to the use of non-canonical variables, ADM gravity and closely related theories (see, e.g., [29–33]) have a large number of troubling spots. The loss of complete diffeomorphism as a known gauge symmetry of the free gravitational field in metric gravity [33] is only one of them. The well established fact is that after 60 years of development and applications of geometrodynamics and other similar theories, which must be considered as different parts of one united approach to the Hamiltonian (metric) gravity, turned out to be incorrect and incomplete for solving actual problems of the metric gravity (for more details, see discussion in [28]).

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## Appendix A

### RELATIONS BETWEEN DIFFERENT SETS OF DYNAMICAL VARIABLES IN METRIC GRAVITY

In this Appendix we discuss relations between dynamical variables which are used in our and Dirac formulations of the metric General Relativity (or metric gravity). In earlier paper [5] we have shown that dynamical variables  $\{g_{\lambda\kappa}, \pi^{\alpha\beta}\}$ , which are used in the K&K formulation of the metric GR, and analogous Dirac dynamical variables  $\{g_{\lambda\kappa}, p^{\alpha\beta}\}$  [1] are related to each other by some canonical transformation. That canonical transformation was written in the form [5]

$$g_{\lambda\kappa} = g_{\lambda\kappa} \quad \text{and} \\ p^{\alpha\beta} = \pi^{\alpha\beta} - \frac{1}{2}\sqrt{-g}A^{(\alpha\beta)0\mu\nu k}g_{\mu\nu, k}, \quad (\text{A.1})$$

where the quantity  $A^{(\alpha\beta)0\mu\nu k}$  is

$$A^{(\alpha\beta)0\mu\nu k} = B^{((\alpha\beta)0|\mu\nu k)} - g^{0k}E^{(\alpha\beta)\mu\nu} + 2g^{0\mu}E^{(\alpha\beta)k\nu}, \quad (\text{A.2})$$

where  $B^{((\alpha\beta)0|\mu\nu k)}$  is the  $B^{(\alpha\beta 0|\mu\nu k)}$  quantity (see, Eq. (3)) symmetrized in terms of all  $\alpha \leftrightarrow \beta$  permutations. Analogously, the  $E^{(\alpha\beta)\mu\nu}$  and  $E^{(\alpha\beta)k\nu}$  are the two symmetrized quantities (in respect to the  $\alpha \leftrightarrow \beta$  permutations), i.e.,

$$E^{(\alpha\beta)\mu\nu} = e^{\alpha\beta}e^{\mu\nu} - \frac{1}{2}(e^{\alpha\mu}e^{\beta\nu} + e^{\alpha\nu}e^{\beta\mu})$$

$$\text{and } E^{(\alpha\beta)k\nu} = e^{\alpha\beta}e^{k\nu} - \frac{1}{2}(e^{\alpha k}e^{\beta\nu} + e^{\alpha\nu}e^{\beta k}),$$

respectively, and  $e^{\sigma\rho}$  are the Dirac tensors defined in Eq. (8).

As is shown in the main text the relation between our dynamical variables and dynamical K&K variables introduced in [4] takes the form  $g_{\lambda\kappa} \rightarrow g_{\lambda\kappa}$  and  $P^{\alpha\beta} \rightarrow \pi^{\alpha\beta}$ , where

$$P^{\alpha\beta} = \pi^{\alpha\beta} - \frac{1}{2}\sqrt{-g}B^{(\alpha\beta 0|\mu\nu k)}g_{\mu\nu, k}. \quad (\text{A.3})$$

From the last equation it is easy to obtain the following expression for our momenta  $P^{\alpha\beta}$  written in terms of the Dirac momenta  $p^{\alpha\beta}$

$$P^{\alpha\beta} = p^{\alpha\beta} - \frac{1}{2}\sqrt{-g}[B^{([\alpha\beta]0|\mu\nu k)} - g^{0k}E^{(\alpha\beta)\mu\nu} + 2g^{0\mu}E^{(\alpha\beta)k\nu}], \quad (\text{A.4})$$

where the quantity  $B^{([\alpha\beta]0|\mu\nu k)}$  is the  $B^{(\alpha\beta 0|\mu\nu k)}$  coefficient, Eq. (3), anti-symmetrized in respect to all permutations of the  $\alpha$  and  $\beta$  indexes. The transformation of dynamical variables  $g_{\lambda\kappa} \rightarrow g_{\lambda\kappa}$  and  $P^{\alpha\beta} \rightarrow p^{\alpha\beta}$ , Eq. (A.4), is the canonical transformation (this can be shown in the same way as it is done in the main text). Its inverse transformation is also canonical. This means that currently we have three different sets of dynamical variables which can be applied for the known and new Hamiltonian formulations of the metric gravity: (a) Dirac variables [1], (b) K&K variables [4], and (c) our variables defined in Section 4 of this study. These three different sets of dynamical variables are related to each other by simple canonical transformations.

The canonicity of transformation of one set of dynamical variables into another such set is a necessary and sufficient condition for ordinary Hamiltonian systems. For Hamiltonian systems with constraints, this condition alone is no longer sufficient. An additional condition is formulated in the form that each constraint must be transferred into a similar constraint and vice versa, i.e., each primary constraint goes into the primary, while each secondary constraint goes into the secondary (for more detail, see, e.g., [5] and [6]). Currently, these two conditions should be considered as independent and completely sufficient for the transformation of dynamical variables to be canonical. This implies, in particular, that any correct Hamiltonian formulation of the metric gravity must have  $d$  primary and  $d$  secondary constraints. Other Hamiltonian formulations of the metric General Relativity with different numbers of constraints, including formulations with tertiary constraints, are wrong *a priori*. In addition to this, one can show that all possible Hamiltonian formulations with the even total number(s) of essential constraints are also wrong [6].

### Appendix B

#### ON CANONICITY OF THE ADM VARIABLES

In this Appendix we want to show that dynamical variables which are used in geometrodynamics [29] are not canonical. Therefore, this theory cannot be considered as the regular Hamiltonian formulation(s) of the metric GR. Furthermore, this theory, or geometrodynamics, cannot canonically be related to any of the correct Hamiltonian formulations currently known in the metric General Relativity. On the other hand, all similar ‘theories’ which are canonically related to the geometrodynamics are equally wrong quasi-Hamiltonian constructions which cannot help anybody to solve problems arising in the metric General Relativity.

The history of creation of geometrodynamics, which is also often called the ADM gravity, is straightforward. After an obvious success of Dirac paper [1] a small group of young authors, which included Arnowitt, Deser, and Misner [19], decided to create some alternative (but Dirac-like!) formulation of the metric GR. Dynamical variables in this ADM approach were chosen as follows. The generalized six coordinates coincide with the corresponding space-space components  $g_{pq}$  of the metric tensor  $g_{\alpha\beta}$  defined in the four-dimensional space-time (or  $(3 + 1)$ -dimensional space-time, if we want to be historically precise). Four remaining coordinates were chosen in the form: the “lapse”  $N = \frac{1}{\sqrt{-g^{00}}}$  and three “shifts”

$N^k = -\frac{g^{0k}}{g^{00}}$ , where  $k = 1, 2, 3$  (very likely, the idea to use these four coordinates was proposed by Wheeler). The corresponding space-like components of momenta  $\Pi^{mn}$  were simply taken from Dirac paper [1] (see also our Appendix A), i.e., they coincide with the  $p^{mn}$  momenta introduced by Dirac (see Appendix A). The four remaining momenta were not defined in the original ADM paper [29]. Probably, this was done on purpose, since these four momenta lead to the primary constraints anyway, but ADM group has developed some special methods to operate with such ‘constraints’ which included, in particular, the two important steps known as ‘constraints reshuffling’ and ‘constraints solving’ (like algebraic equations!). Right now, it is very hard to describe and discuss the internal logic of this quasi-theory, but we have to note that geometrodynamics was carefully analyzed earlier in [28] with many details and references.

In fact, we do not need to dive into a deep discussion of ADM formulation, since we already have their ten generalized coordinates (one laps  $N$ , three shifts  $N^k$  and six components of the metric tensor  $g_{pq}$ ) and six momenta  $\Pi^{mn}$  which coincide with the momenta  $p^{mn}$  defined in Dirac’s paper. By using only these dynamic variables of ADM gravity we can prove that these dynamical variables are not canonical. To prove this statement we have to calculate the two following Poisson brackets: (1) PB between “laps”  $N$  and  $\Pi^{mn}$  (or  $p^{mn}$ ) momenta, and (2) PB between “shifts” and the same  $\Pi^{mn}$  (or  $p^{mn}$ ) momenta. If this theory is a truly Hamiltonian, then all these Poisson brackets must be equal zero identically. Let us check this simple fact. The first Poisson bracket is

$$\begin{aligned}
 [N, \Pi^{mn}] &= \left[ \frac{1}{\sqrt{-g^{00}}}, p^{mn} \right] \\
 &= -\frac{1}{\sqrt{(-g^{00})^3}} [g^{00}, p^{mn}]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\sqrt{(-g^{00})^3}} \frac{1}{2} (g^{0m} g^{0n} + g^{0n} g^{0m}) \\
 &= \frac{1}{\sqrt{(-g^{00})^3}} g^{0m} g^{0n} \neq 0, \tag{B.1}
 \end{aligned}$$

while for the second bracket one finds

$$\begin{aligned}
 [N^k, \Pi^{mn}] &= \left[ -\frac{g^{0k}}{g^{00}}, p^{mn} \right] \\
 &= \frac{1}{2g^{00}} (g^{0m} g^{0n} + g^{0n} g^{0m}) \\
 -\frac{1}{(g^{00})^2} g^{0k} g^{0m} g^{0n} &= \frac{1}{2(g^{00})^2} (g^{00} g^{0m} g^{kn} \\
 + g^{00} g^{0n} g^{km} - 2g^{0k} g^{0m} g^{0n}) &\neq 0, \tag{B.2}
 \end{aligned}$$

where  $k = 1, 2, 3$ . As follows from these equations none of these four Poisson brackets equal zero identically. Therefore, these dynamical variables are not canonical and theory which uses these variables is not a Hamiltonian theory of anything. Furthermore, it cannot be transformed into the correct Hamiltonian theory of metric GR, e.g., by applying some canonical transformation. Now, we can only guess that P.A.M. Dirac calculated the four Poisson brackets mentioned here in the end of 1950's (for him it took, probably, a few minutes). Even then Dirac could predict the sad failure of ADM approach (and other similar approaches) to the Hamiltonian formulation of metric GR in the future. Now that future has become our present.

### Appendix C

#### ON TRANSITIONS FROM LAGRANGIANS TO HAMILTONIANS. DIFFEOMORPHISM AND INTEGRALS OVER COMPONENTS OF THE METRIC TENSOR

In this Appendix we discuss some important technical details which are crucially important to construct different Hamiltonian approaches to the metric gravity, but we could not include them in the main text. First, by using some help from Lanczos [35] we develop the Hamilton approach for those cases when we have to operate with variations of the original Lagrangian only and cannot use (for some reasons) the Lagrangian function itself (see, e.g., Section 2 in the main text). To achieve this goal we represent the variation of the original Lagrangian  $\delta L$  in the form

$$\delta L = \delta(v_i p_i - H) = p_i \delta v_i + v_i \delta p_i - \delta H, \tag{C.1}$$

where  $L$  is the function of the generalized coordinates  $q_i$  and velocities  $v_i = \frac{dq_i}{dt}$  only, while  $H$  is the function of the same coordinates  $q_i$  and new variables (momenta)  $p_i$  only. This means that:

(1) an arbitrary variation of the momentum  $p_i$  has no influence on the variation of  $L$ .

(2) an arbitrary variation of the velocity  $v_i$  has no influence on the variation of  $H$ .

At this point all momenta must be considered as some unidentified functions.

First, we re-write Eq. (C.1) to the form

$$\delta L - p_i \delta v_i = v_i \delta p_i - \delta H, \tag{C.2}$$

or

$$\begin{aligned}
 &\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial v_i} \delta v_i - p_i \delta v_i \\
 &= v_i \delta p_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i. \tag{C.3}
 \end{aligned}$$

From here one finds

$$\begin{aligned}
 &\frac{\partial L}{\partial q_i} \delta q_i + \left( \frac{\partial L}{\partial v_i} - p_i \right) \delta v_i \\
 &= \left( v_i - \frac{\partial H}{\partial p_i} \right) \delta p_i - \frac{\partial H}{\partial q_i} \delta q_i. \tag{C.4}
 \end{aligned}$$

Now, the point (1) leads to the equation  $p_i = \frac{\partial L}{\partial v_i}$  which is, in fact, the definition of momenta. Analogously, based on the point (2) one finds another equation  $v_i = \frac{\partial H}{\partial p_i}$  which is the definition of the velocities in the phase space. After these steps we also obtain the third equation  $\frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i}$ . If the both functions  $L$  and  $H$  explicitly depend upon time  $t$  and an additional parameter  $\alpha$ , then here we can write analogous equations  $\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$  and  $\frac{\partial L}{\partial \alpha} = -\frac{\partial H}{\partial \alpha}$ . These three groups of equations are the basic equations of the Hamilton procedure. To make this system of basic equations a complete system of dynamical equation one has to add the system of Lagrange equations of actual motion  $\frac{d}{dt} \frac{\partial L}{\partial v_i} = \frac{\partial L}{\partial q_i}$  which in Hamiltonian variables is written in the form  $\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$ . Finally, we can write all actual dynamical equations in the Hamilton method  $\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$ ,  $\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$ . These equations are logically closed and contain neither the original Lagrangian  $L$ , nor the velocities  $v_i$ . Now, we can call and consider the function  $H$ , which appears in these equations, as the Hamiltonian of the system.

Another crucial thing which we need to discuss here is the chain and multi-chain integrals over different components of the metric tensor. First, consider the following one-dimensional metric integral

$$\begin{aligned}
 &\int F^{\alpha\beta} dg_{\beta\gamma} = \int F^{\alpha\beta} \frac{\partial g_{\beta\gamma}}{\partial x^\mu} dx^\mu \\
 &= \int F^{\alpha\beta} (\Gamma_{\gamma,\beta\mu} + \Gamma_{\beta,\gamma\mu}) dx^\mu, \tag{C.5}
 \end{aligned}$$

where  $\Gamma_{\gamma,\beta\mu}$  are the Cristoffel symbols of the first kind, while the last integral in the right-hand side is the usual (linear) integral taken in the  $d$  dimensional coordinate space. This formula can be generalized to the two-, three- and multi-dimensional metric integrals. For instance, in the two-dimensional case we have to replace  $dg_{\beta\gamma}dg_{\lambda\sigma} = (\Gamma_{\gamma,\beta\mu} + \Gamma_{\beta,\gamma\mu})(\Gamma_{\sigma,\lambda\nu} + \Gamma_{\lambda,\sigma\nu})dx^\mu dx^\nu$ .

However, it is easy to note that the both sides of Eq. (C.5) are transformed differently during general transformations of the  $x^\alpha$  coordinates. Indeed, in the left-hand side of this equation we have an absolute (or first-class) variable  $g_{\beta\gamma}$ , while in the right-hand side of the same equation we have a set of usual (or second-class) variables  $x^\mu$ , where  $\mu = 0, 1, \dots, d - 1$ . In general, by varying these  $x^\mu$  coordinates arbitrarily one quickly arrives to fundamental contradictions with the use of Eq. (C.5). To avoid such situations and define all multi-chain integrals over different components of the metric tensor correctly and uniformly we need to introduce some special sets of  $x^\mu$  variables (or coordinates) which are called the Killing's sets, or curves. The Killing's curves consist only those  $x^\mu$  variables which do not change metric, i.e. all components of metric tensor, under infinitesimal transformations. For general transformations of variables in metric GR which are written in the form  $x^\mu \rightarrow x^\mu + \xi^\mu$ , where  $\xi^\mu$  are the small values, the Killing's criterion is written in the form  $\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu = 0$ , where  $\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu$  is the sum of contravariant derivatives of the  $\xi^\nu$  and  $\xi^\mu$  values (for more details, see [9]). In turn, the Killing's criterion is equivalent to the diffeomorphism gauge symmetry (or diffeomorphism conservation) in the metric GR [4, 36]. Thus, the diffeomorphism plays a central role for the both Hamiltonian formulation(s) of the metric GR and for correct and uniform definition of the chain integrals over components of the metric tensor. The central role of diffeomorphism in the metric gravity follows from the fact that the variables in metric gravity are the components of metric tensor  $g_{\alpha\beta}$  (or  $g^{\alpha\beta}$ ), but not the physical coordinates  $x^\mu$ . The diffeomorphism guarantees that the real connection between the components of metric tensor  $g_{\alpha\beta}$  (or  $g^{\alpha\beta}$ ) and coordinates  $x^\mu$  with each other is unambiguous and analytical (or smooth) at any time.

## REFERENCES

1. P. A. M. Dirac, Proc. Roy. Soc. **246**, 333 (1958).
2. F. A. E. Pirani, A. Schild, and R. Skinner, Phys. Rev. **87**, 452 (1952).
3. P. A. M. Dirac, Can. J. Math. **2**, 129 (1950).
4. N. Kiriushcheva, S. V. Kuzmin, C. Racnkor, and S. R. Valluri, Phys. Lett. A **372**, 5101 (2008).
5. A. M. Frolov, N. Kiriushcheva, and S. V. Kuzmin, Gravit. Cosmol. **17**, 314 (2011).
6. A. M. Frolov, Can. J. Phys. **98**, 405 (2020).
7. N. E. Kochin, *Vector Calculus and the Principles of Tensor Calculus*, 9th ed. (USSR Acad. of Sciences Publishing, Moscow, 1965), Chap. IV.
8. A. M. Frolov, N. Kiriushcheva and S. V. Kuzmin, Grav. Cosmol. **16**, 181 (2010).
9. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Pergamon Press, 1975), Chap. 11.
10. M. Carmeli, *Classical Fields: General Relativity and Gauge Theory* (World Scientific Publ. Co., Singapore, 2002).
11. P. A. M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Sciences, Yeshiva University, New York 1964).
12. F. R. Gantmakher, *Lectures on Analytical Mechanics* (Nauka, Moscow, 1966), Chap. III and IV.
13. P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th ed. (Oxford at Clarendon Press, 1958), Chap. IV, p. 21.
14. V. A. Fock, *Foundations of Quantum Mechanics*, 2nd ed. (Nauka, Moscow, 1976), Chpt. III.
15. I. M. Gelfand and S. V. Fomin, *Calculus of Variations* (Dover Publ., Inc., Mineola, New York, 1990), Chapt. 4.
16. D. M. Gitman and I. V. Tyutin, *Canonical Quantization of Fields with Constraints* (Springer-Verlag, Berlin-Heidelberg, 1990).
17. L. Castellani, Ann. Phys. (N. Y.) **143**, 357 (1982).
18. C. W. Misner and J. A. Wheeler, Ann. Phys. (NY) **2**, 525 (1957).
19. R. Arnowitt, S. Deser, and C. Misner, Phys. Rev. **117**, 1595 (1960).
20. S. Samanta, Int. J. Theor. Phys. **48**, 1436 (2009).
21. H. Flanders, *Differential Forms with Applications to the Physical Sciences* (Dover Publ., New York, 1989).
22. H. Poincaré, *New Methods of Celestial Mechanics* (Springer-Verlag, Berlin-Heidelberg, 1977).
23. G. D. Birkhoff, *Dynamical Systems* (Reprinted by AMS, Colloquium Publications, Vol. 9, Providence, Rhode Island, 2008).
24. V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Nauka, Moscow, 1974), Part III [in Russian].
25. F. Tricomi, *Lectures on Equations in Partial Derivatives* (International Literature Publishing, Moscow, 1957).
26. M. V. Fedoryuk, *Ordinary Differential Equations* (Nauka, Moscow, 1980), Chap. 5.
27. R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. II (Interscience Inc., New York, 1962), Chap. 2, Sec. 8.

28. N. Kiriushcheva and S. V. Kuzmin, *Cent. Eur. J. Phys.* **9**, 576 (2011).
29. R. Arnowitt, S. Deser, and C. W. Misner, *Gravitation: An Introduction to Current Research*, Ed. by L. Whitten (John Wiley, New York, 1964).
30. T. Regge and C. Teitelboim, *Ann. Phys. (NY)* **88**, 286 (1974).
31. J. Lee and R. M. Wald, *J. Math. Phys.* **31**, 725 (1990).
32. R. M. Wald and A. Zoupas, *Phys. Rev. D* **61**, 084027 (2000).
33. C. J. Isham and K. V. Kuchar, *Ann. Phys. (NY)* **164**, 316 (1985).
34. S. W. Hawking, *General Relativity. An Einstein Centenary Survey*, Eds. S. W. Hawking and W. Israel (Cambridge Univ. Press., Cambridge, 1979), p. 476.
35. C. Lanczos, *The Variational Principles of Mechanics*, 4th Ed. (Dover Publ., New York, 1986), Chap. VI, VII and Appendix I.
36. L. P. Grishchuk, A. N. Petrov, and A. D. Popova, *Commun. Math. Phys.* **94**, 379 (1984).