ISSN 1063-7788, Physics of Atomic Nuclei, 2021, Vol. 84, No. 4, pp. 401–407. © The Author(s), 2021. This article is an open access publication.
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NUCLEI
Theory

Study of the Influence of Different Methods of Taking into Account the Coulomb Interaction on Determining Asymptotic Normalization Coefficients within the Framework of Exactly Solvable Model

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Received December 25, 2020; revised December 25, 2020; accepted December 25, 2020

Abstract—It is shown that the Schrödinger equation for the sum of the potential of a square well and the Coulomb potential of a uniformly charged sphere admits an analytical solution for arbitrary values of the orbital angular momentum. An explicit form of this solution has been found. Using the obtained solution, the influence of the Coulomb interaction for both point and distributed nuclear charges on the values of asymptotic normalization coefficients for various nuclear systems is investigated. It is shown that taking into account the non-point distribution of the nuclear charge has little effect on the calculated values of the asymptotic normalization coefficients, provided that the binding energy of the system is assumed to be fixed.

DOI: 10.1134/S1063778821040098

1. INTRODUCTION

Asymptotic normalization coefficients (ANC) determine the asymptotics of nuclear wave functions in binary channels at distances between fragments exceeding the radius of nuclear interaction. In terms of ANCs, the cross sections of peripheral nuclear processes are parameterized, such as reactions with charged particles at low energies, when, due to the Coulomb barrier, the reaction occurs at large distances between fragments. The most important class of such processes is astrophysical nuclear reactions occurring in the cores of stars, including the Sun. The important role of ANCs in nuclear astrophysics was first noted in the works [1, 2], in which it was shown that ANCs determine the overall normalization cross sections of peripheral reactions of radiation capture (see also works [3, 4]).

When extracting the ANC values from the data on phase shifts of elastic scattering, it is very important to take into account the effects of the Coulomb interaction between colliding particles at energies near zero (see, for example, [5, 6]). These effects are associated with the long-range nature of the Coulomb interaction, which for point charges has the form $V_{\text{coul}} = Z_1 Z_2 e^2/r$ for all values of r. Here $Z_i e$ is the charge of particle i and r is the distance between the centers of mass of the colliding particles. However, when analyzing data on the cross sections of nuclear reactions within the framework of the distorted wave Born approximation, the non-point distribution of the electric charge in the nucleus is often taken into account; in this case, as a rule, the potential of a uniformly charged sphere is taken as the Coulomb interaction. Since the distorted wave Born approximation is often used to obtain information about the ANCs [3], the question arises about the influence of the non-point nuclear charge on the ANC values. For brevity, below we will call the Coulomb potential for a point charge and for a uniformly charged sphere a point and non-point Coulomb interaction, respectively.

In the works [5–7], along with the determination of the ANC values from the data on phase-shift analyses, the study of qualitative Coulomb effects in low-energy elastic scattering was carried out. In this case, the potential of a rectangular well was chosen as the nuclear interaction, which, unlike other types of potentials, allows one, in superposition with the point Coulomb interaction, to obtain an analytical solution of the Schrödinger equation for arbitrary values of the orbital angular momentum l. In this paper, it is shown that an analytical solution of the Schrödinger equation for arbitrary values of l can also be obtained for a combination of the potential of a rectangular well and the Coulomb potential of a uniformly charged sphere. Within the framework of this exactly solvable model, we investigate the influence of the Coulomb interaction on the values of the ANC for both point and distributed electric charge of the nucleus. Note that the influence of the point Coulomb interaction on the ANC values was discussed in the framework

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of the two-particle potential model with separable s-wave interaction in [8] using the example of ^{3}H and ^{3}He nuclei.

It should be emphasized that when considering low-energy processes that are of interest to us, the results are insensitive to the detailed structure of the nuclear potential at distances $r \ll 1/k$, where k is the relative momentum of the interacting particles [9]. Therefore, the conclusions obtained in this work, especially qualitative ones, should not depend on the specific form of the strong interaction, be it a rectangular well, the Woods–Saxon potential, or something else.

The paper is organized as follows. Section 2 presents the general formalism of the problem. Section 3 is devoted to the application of this formalism to analyze the influence of various forms of the Coulomb interaction on the ANC values for specific nuclear systems. The results are briefly discussed in Section 4.

Throughout the paper we use the system of units in which $\hbar = c = 1$.

2. GENERAL FORMALISM

2.1. The Form of Interaction and the Structure of the Solution of the Schrödinger Equation

We investigate the ANC for splitting the bound state of the nucleus a into two fragments b and c ($a \to b + c$). Specifically, a proton or an α-particle will be considered as a light fragment c. The Coulomb interaction between b and c is described by the potential of a uniformly charged sphere of radius R_c, and also, for comparison, by a point Coulomb potential, which formally corresponds to $R_c = 0$. An attractive rectangular well with depth V_0 and radius R is chosen as the nuclear interaction. The selection of the potential parameters is described in Section 3. In accordance with the conclusions of the works [10, 11], we will assume without loss of generality that $R_c \le R$.

The problem of finding the ANC is reduced to finding the coordinate asymptotics of the solution to the radial Schrödinger equation

$$\frac{d^2\chi_l}{dr^2} + \left[2\mu E - 2\mu V(r) - \frac{l(l+1)}{r^2}\right]\chi_l = 0 \quad (1)$$

for a potential of the form

$$V(r) = -V_0 + \alpha(3 - r^2/R_c^2)/(2R_c), \quad 0 \le r < R_c, \quad (2)$$

$$V(r) = -V_0 + \alpha/r, \quad R_c \le r < R, \quad (3)$$

$$V(r) = \alpha/r, \quad r \ge R, \quad (4)$$

where $\alpha = Z_b Z_c e^2$. We consider the bound state a of particles b and c with energy $E = -\varepsilon = -\varkappa^2/2\mu$, $\varkappa$ is the wave number of the bound state and μ is the reduced mass of b and c.

For $r \le R_c$ $\chi_l(r) = a_1\varphi_1(r)$; the explicit form of $\varphi_1(r)$ is discussed below in subsection 2.2. In the region $R_c \le r < R$ $\chi_l(r) = a_2\varphi_2(r) + a_3\varphi_3(r)$, where $\varphi_2(r)$ and $\varphi_3(r)$ are regular and irregular Coulomb solutions corresponding to the energy $\tilde{E} = V_0 - \varepsilon$. Finally, for $r \ge R$ $\chi_l(r) = a_4 W_{-\eta,l+1/2}(2\varkappa r)$, $W_{\beta,\gamma}(z)$ is the Whittaker function, $\eta = \alpha\mu/\varkappa$ is the Coulomb parameter. The asymptotics of $W_{-\eta,l+1/2}(2\varkappa r)$ for $r \to \infty$ has the form:

$$W_{-\eta,l+1/2}(2\varkappa r)|_{r\to\infty} = e^{-\varkappa r}/(2\varkappa r)^{\eta}. \quad (5)$$

The constants a_i ($i = 1$–4) and the binding energy ε are found from the conditions for matching (continuity) values and first derivatives of the function $\chi_l(r)$ at the points $r = R_c$ and $r = R$ and the normalization condition:

$$\int_0^\infty \chi_l^2(r)dr = 1. \quad (6)$$

We emphasize that, by definition, the constant a_4 coincides with the required ANC.

In the particular case $R_c = R$, the second region degenerates into a point, and the matching is carried out at one point $r = R$.

2.2. Solution of the Schrödinger Equation for the Superposition of the Potentials of a Rectangular Well and a Uniformly Charged Sphere

$\varphi_1(r)$ is a solution of the Schrödinger equation with the potential (2), which can be written as

$$\frac{d^2\varphi_1}{dr^2} + \left[2\mu\left(E + V_0 - \frac{3\alpha}{2R_c}\right) + \frac{\mu\alpha}{R_c^3}r^2 - \frac{l(l+1)}{r^2}\right]\varphi_1 = 0. \quad (7)$$

By the form of the radial dependence of the interaction, this equation is similar to the equation for the potential of a three-dimensional harmonic oscillator, which has the form

$$\frac{d^2\chi_l}{dr^2} + \left[2\mu E - \mu^2\omega^2 r^2 - \frac{l(l+1)}{r^2}\right]\chi_1 = 0. \quad (8)$$

However, the fundamental difference between equations (7) and (8) consists in the opposite signs of the terms proportional to r^2. We were unable to find in the literature an explicit form for the solution of the

equation (7). One can, however, proceed as follows. The solution (8), which behaves like r^{l+1} for $r \to 0$, is known [12]:

$$\chi_l(r) = C_1\, {}_1F_1\left(\frac{1}{2}(l+\frac{3}{2}-\nu), l+\frac{3}{2}; \lambda r^2\right) \times e^{-\frac{\lambda}{2}r^2} r^{l+1}, \quad \lambda = \mu\omega, \quad \nu = \frac{E}{\omega}, \tag{9}$$

where ${}_1F_1(a,b;z)$ is a degenerate hypergeometric function. The equation (7) is obtained from (8) by a formal replacement:

$$E \to E + V_0 - \frac{3\alpha}{2R_c}, \quad \omega^2 \to -\frac{\alpha}{\mu R_c^3}, \quad \omega \to i\sqrt{\frac{\alpha}{\mu R_c}}\frac{1}{R_c}. \tag{10}$$

Therefore, in view of the analytical dependence of the solution (9) on its parameters, we assume that the solution to the equation (7) can be obtained from the expression (9) by replacing

$$\nu \to \tilde{\nu} = -i\sqrt{\frac{\mu R_c}{\alpha}} R_c \left(E + V_0 - \frac{3\alpha}{2R_c}\right), \quad \lambda \to \tilde{\lambda} = i\sqrt{\frac{\mu\alpha}{R_c}}\frac{1}{R_c}. \tag{11}$$

The solution $\xi(r)$ thus obtained is regular at zero but complex, and we need a real solution for a bound state. All coefficients in (7) are real. Therefore, if for Im $r = 0$ the function $\xi(r)$ is a solution, then $\xi^*(r)$ is also the solution and Re $\xi(r)$ and Im $\xi(r)$ are also solutions. Therefore, one can take

$$\varphi_1(r) = \mathrm{Re}\left\{ {}_1F_1\left(\frac{1}{2}(l+\frac{3}{2}-\tilde{\nu}), l+\frac{3}{2}; \tilde{\lambda} r^2\right) \times e^{-\frac{\tilde{\lambda}}{2}r^2} r^{l+1} \right\} \tag{12}$$

as the function $\varphi_1(r)$ we need.

2.3. ANC Calculations for Specific Nuclear Systems

In this section, within the framework of the above scheme, comparative ANC calculations for specific nuclear systems are carried out. Nuclear interaction is described by the potential of a rectangular well. Both the potential of a point charge and the potential of a uniformly charged sphere are used as the Coulomb interaction. Calculations are also performed neglecting the Coulomb interaction. Systems with both small and large values of the binding energy ε and the Coulomb parameter η are considered. Since the main goal of the work is to study qualitative patterns, and not to determine the ANC values with high accuracy, to simplify the calculations, the values of the radii of the charged sphere and rectangular well are assumed to be the same: $R_c = R$.

In addition to the ANC defined above, which has the dimension fm$^{-1/2}$ and which we will denote by the letter C, in the literature, first of all, in the microscopic calculations of ANCs, the dimensionless ANC $\tilde{C}$ is also used which is related with ANC C as follows: $C = \sqrt{2\varkappa}\tilde{C}$. We also note that for large charges Z and/or small values of the binding energy ε and, accordingly, large values of the Coulomb parameter η, the ANC C values can be very large, which is explained by the presence of the barrier Coulomb factor $\Gamma(l+1+\eta)/l!$. In this regard, in the work [13], for the convenience of calculations, the renormalized ANC C_r was introduced, which does not contain this factor: $C_r = l!/\Gamma(l+1+\eta)C$. Using C_r instead of C also makes it easier to compare ANC values for mirror nuclei [13]. Note that the value of C_r is directly related to the residue of the renormalized Coulomb-nuclear partial-wave elastic scattering amplitude [14]. For all systems considered below, we calculated and presented in the form of tables the values of ANCs C, $\tilde{C}$, C_r, as well as $\tilde{C}_r = C_r/\sqrt{2\varkappa}$.

ANC calculations were carried out for five different versions.

1. (Version 1). The values of the parameters V_0 and R were chosen so as to reproduce the experimental value of the binding energy ε and the value of the ANC C known from the literature for the given channel $a \to b + c$ when using the point Coulomb potential.

2. (Version 2). For the same values of V_0 and R as in version 1, the Coulomb interaction was taken in the form of the potential of a uniformly charged sphere. The calculated value of ε was naturally changed in this case.

3. (Version 3). With the same value of R as in version 1 and using the potential of a uniformly charged sphere, the value of V_0 was chosen so as to reproduce the experimental value of ε. The purpose of this action is to exclude the influence of changes in the binding energy on the ANC values.

4. (Version 4). At the same values of V_0 and R as in version 1, the Coulomb interaction was turned off. The calculated value of ε has been changed.

5. (Version 5). For the same value of R as in version 1 and the Coulomb interaction being switched off, the value of V_0 was chosen so as to reproduce the experimental value of ε.

In all the calculations performed, it was assumed that the daughter nucleus b is in the ground state.

Table 1. ANC for $^{17}\mathrm{F}(\mathrm{ground}) \to {}^{16}\mathrm{O} + p$

Version	V_0, MeV	η	ε, MeV	C, fm$^{-1/2}$	$\tilde{C}$	C_r, fm$^{-1/2}$	$\tilde{C}_r$
1	30.46	1.583	0.6005	1.042	1.8142	0.1596	0.2779
2	30.46	1.271	0.9310	1.0808	1.6860	0.2537	0.3958
3	30.07	1.583	0.6005	1.0337	1.7995	0.1583	0.2756
4	30.46	0	3.9351	0.9010	0.9803	0.9010	0.9803
5	25.76	0	0.6005	0.0955	0.1663	0.0955	0.1663

Table 2. ANC for $^{17}\mathrm{F}^*(0.495\ \mathrm{MeV}) \to {}^{16}\mathrm{O} + p$

Version	V_0, MeV	η	ε, MeV	C, fm$^{-1/2}$	$\tilde{C}$	C_r, fm$^{-1/2}$	$\tilde{C}_r$
1	8.2691	3.782	0.1052	77.33	208.08	4.452	11.980
2	8.2691	1.300	0.8898	6.23	9.82	5.337	8.420
3	7.1838	3.782	0.1052	73.41	197.55	4.227	11.373
4	8.2691	0	3.6052	2.92	3.2445	2.920	3.2445
5	2.8663	0	0.1052	0.4438	1.1943	0.4438	1.1943

2.4. ANC for $^{17}F \to {}^{16}O + p$

The ^{17}F nucleus has two bound states: the ground state $^{17}\mathrm{F}(5/2^+;\ \mathrm{ground})$ $(l = 2)$ and the excited state $^{17}\mathrm{F}^*(1/2^+;\ 0.495\ \mathrm{MeV})$ $(l = 0)$. For the ground state, $\varepsilon = 0.6005$ MeV. The value $C = 1.042$ fm$^{-1/2}$ for this state was taken from the work [15] and was used within the version 1 to fit the values of V_0 and R. These values of ε and C lead to the values $V_0 = 30.46$ MeV and $R = 4.238$ fm.

The calculation results are presented in Table 1.

For the excited state $^{17}\mathrm{F}^*(0.495\ \mathrm{MeV})$ $\varepsilon = 0.1052$ MeV. As ANC C for calculations in version 1, we take $C = 77.33$ fm$^{-1/2}$, which coincides with one of the values obtained in [16] from the analysis of the reaction $^{16}\mathrm{O}(^3\mathrm{He}, d)^{17}\mathrm{F}$ in the framework of the distorted wave Born approximation. These values of ε and C for version 1 correspond to the values $V_0 = 8.2691$ MeV and $R = 4.984$ fm. The calculation results are presented in Table 2.

Note that, within the framework of the shell model, the discussed excited state ^{17}F can be written as $2s_{1/2}$, that is, the potential of the shell model contains the $1s$-state of lower energy. In such cases, when describing real nuclear states, potentials with forbidden states are often used. It was shown in [17] that, within the framework of the potential model, the ANC values calculated at a fixed binding energy can significantly depend on the number of forbidden states taken into account. In this regard, to study the influence of forbidden states on the Coulomb effects in determining the ANCs, we performed calculations that differ from the calculations presented in Table 2 by the fact that in determining the potential parameters V_0 and R in version 1, it was assumed that in the potential used, in addition to the s-state with the experimental binding energy, there is one more lower-located (forbidden) s-state. This assumption leads to the adjusted in version 1 values $V_0 = 32.06$ MeV and $R = 4.658$ fm. The binding energy of the forbidden state is 20.17 MeV.

The results of ANC calculations for the excited state ^{17}F in the presence of a forbidden state are given in Table 3. Comparison of Tables 2 and 3 shows that their results are qualitatively the same.

2.5. ANC for $^{16}O \to {}^{12}C + \alpha$

For the transition $^{16}\mathrm{O} \to {}^{12}\mathrm{C} + \alpha$ from the ^{16}O ground state , there is no ANC value in the literature. Therefore, as the initial nucleus, we considered two excited states of ^{16}O: 0^+ (6.049 MeV) $(l = 0)$ and 1^- (7.117 MeV) $(l = 1)$. The experimental values of the binding energy in the $^{12}\mathrm{C} + \alpha$ channel for these states are 1.113 and 0.045 MeV, respectively.

In the case of an excited 0^+-state, to determine V_0 and R in the framework of the first version of calculations, in addition to the binding energy $\varepsilon = 1.113$ MeV, the value $C = 1560$ fm$^{-1/2}$ [18] was used. The obtained $V_0 = 18.07$ MeV and $R = 4.903$ fm. For these values of the parameters, in addition to the state under consideration, the system also has a deeper 0^+-state associated with the ground state ^{16}O. The calculation results are presented in Table 4.

For the 1^--state, $C = 2.10 \times 10^{14}$ fm$^{-1/2}$ [18] was taken as the initial value. This ANC value combined

Table 3. ANC for $^{17}\mathrm{F}^*(0.495\ \mathrm{MeV}) \to {}^{16}\mathrm{O} + p$ if a forbidden state is available

Version	V_0, MeV	η	ε, MeV	C, fm$^{-1/2}$	$\tilde{C}$	C_r, fm$^{-1/2}$	$\tilde{C}_r$
1	32.06	3.782	0.1052	77.33	208.08	4.452	11.980
2	32.06	0.7956	2.376	7.6700	9.466	8.245	10.177
3	28.62	3.782	0.1052	77.357	208.16	4.454	11.984
4	32.06	0	5.294	4.890	4.930	4.890	4.940
5	23.33	0	0.1052	0.4449	1.197	0.4449	1.197

Table 4. ANC for $^{16}\mathrm{O}^*(1.113\ \mathrm{MeV}) \to {}^{12}\mathrm{C} + \alpha$

Version	V_0, MeV	η	ε, MeV	C, fm$^{-1/2}$	$\tilde{C}$	C_r, fm$^{-1/2}$	$\tilde{C}_r$
1	18.07	3.103	1.113	1560	1745	228	255
2	18.07	1.450	4.767	817	635	615	478
3	13.97	3.103	1.113	1472.17	1646	215.3	240.75
4	18.07	0	9.272	117.6	77.44	117.6	77.44
5	8.591	0	1.113	3.442	3.850	3.442	3.850

Table 5. ANC for $^{16}\mathrm{O}^*(0.045\ \mathrm{MeV}) \to {}^{12}\mathrm{C} + \alpha$

Version	V_0, MeV	η	ε, MeV	C, fm$^{-1/2}$	$\tilde{C}$	C_r, fm$^{-1/2}$	$\tilde{C}_r$
1	12.61	15.43	0.045	2.10×10^{14}	5.24×10^{14}	2.973	7.415
2	12.61	3.359	0.950	639.2	743.7	15.31	17.81
3	11.54	15.43	0.045	1.95×10^{14}	4.87×10^{14}	2.763	6.892
4	12.61	0	6.261	14.28	10.37	14.28	10.37
5	4.652	0	0.045	0.1518	0.3788	0.1518	0.3788

with $\varepsilon = 0.045$ MeV leads to $V_0 = 12.61$ MeV, $R = 3.892$ fm. The calculation results are presented in Table 5.

2.6. ANC for $^{16}O \to {}^{15}N + p$

In the examples considered above, weakly coupled systems ($\varepsilon \le 1.2$ MeV) were studied. For comparison, we now carry out a similar study for the channel $^{16}\mathrm{O(ground)} \to {}^{15}\mathrm{N} + p$ with high binding energy ($\varepsilon = 12.13$ MeV, $l = 1$). To determine the parameters V_0 and R, the ANC value $C = 13.86$ fm$^{-1/2}$ [19] was used, which leads to the values $V_0 = 33.47$ MeV and $R = 4.119$ fm. The calculated ANC values are given in Table 6.

2.7. ANC for $^{209}Bi \to {}^{208}Pb + p$

As the last example, consider the ANC for a channel with a large Z value: $^{209}\mathrm{Bi(ground)} \to {}^{208}\mathrm{Pb} + p$. For this process, $\varepsilon = 3.799$ MeV, $l = 5$, but the ANC value is unknown. Therefore, we act differently than in previous cases. Namely, we choose as R the value 7.418 fm obtained by the formula $R = r_0 A^{1/3}$ for $r_0 = 1.25$ fm. After that, the value of V_0 at a given value of R is fitted to the binding energy, which leads to $V_0 = 51.29$ MeV. For these values of V_0 and R, the calculated ANC value C is equal to 0.1158×10^8 fm$^{-1/2}$. Of course, the authors do not pretend that this value of C is in good agreement with the real ANC for the channel under consideration; it is used only for a qualitative assessment of the Coulomb effects.

The calculation results are presented in Table 7.

3. ANALYSIS OF THE CALCULATION RESULTS AND CONCLUSIONS

In this paper, it is shown that for a combination of the potential of a rectangular well and the Coulomb potential of a uniformly charged sphere, which is often used in calculations within the distorted wave Born approximation, the solution of the Schrödinger equation can be obtained analytically for arbitrary values of the orbital angular momentum l. An explicit form of

Table 6. ANC for $^{16}\mathrm{O}(\mathrm{ground}) \to {}^{15}\mathrm{N} + p$

Version	V_0, MeV	η	ε, MeV	C, fm$^{-1/2}$	$\tilde{C}$	C_r, fm$^{-1/2}$	$\tilde{C}_r$
1	33.47	0.3076	12.13	13.86	11.39	11.83	9.721
2	33.47	0.2997	12.77	14.56	11.81	12.48	10.13
3	32.74	0.3076	12.13	13.58	11.16	11.58	9.522
4	33.47	0	15.71	11.21	8.635	11.21	8.635
5	29.38	0	12.13	7.408	6.089	7.408	6.089

Table 7. ANC for $^{209}\mathrm{Bi} \to {}^{208}\mathrm{Pb} + p$

Version	V_0, MeV	η	ε, MeV	C, fm$^{-1/2}$	$\tilde{C}$	C_r, fm$^{-1/2}$	$\tilde{C}_r$
1	51.29	6.634	3.799	1.158×10^7	1.253×10^7	7.282	7.881
2	51.29	5.739	5.075	0.2804×10^7	0.2823×10^7	15.90	16.00
3	49.92	6.634	3.799	1.102×10^7	1.192×10^7	6.928	7.498
4	51.29	0	23.49	166.3	114.11	166.3	114.1
5	29.86	0	3.799	0.2537	0.2746	0.2537	0.2746

this solution has been found. Within the framework of this exactly solvable model, we studied the influence of the Coulomb interaction on the values of the ANC for both the point and the distributed electric charge of the nucleus. Specific nuclear systems with different values of the binding energy, Coulomb parameter, and orbital angular momentum were considered.

In what follows, we will mark the value corresponding to the ith version of the calculation with the index i.

Let us begin the analysis of the calculation results with the binding energies. From the data given in the tables, it follows that for all considered examples in the case when the binding energy is not fixed, we have $\varepsilon_1 < \varepsilon_2 < \varepsilon_4$ (and, accordingly, $\eta_1 > \eta_2 > \eta_4$). This result is trivial, since taking into account the repulsive Coulomb interaction effectively weakens the attractive nuclear potential, and the transition from a point charge to a distributed one while maintaining the total charge is equivalent to weakening the Coulomb interaction, since the Coulomb potential inside a charged sphere is weaker than the potential of a point charge.

The situation with the ANCs is more complicated. Their values are largely determined by the asymptotics of the radial wave function, which, in accordance with (5), has the form

$$\chi_l(r)|_{r\to\infty} = Ce^{-\varkappa r}/(2\varkappa r)^{\eta}. \qquad (13)$$

In the absence of the Coulomb interaction ($\eta = 0$), with increasing the binding energy and, consequently, increasing $\varkappa$, the rate of decrease in the asymptotics (13) increases with increasing r, which, due to the conservation of the general normalization of the wave function should, generally speaking, lead to increasing the ANC C. This conclusion is certainly valid in the frequently used effective range approximation, in which $C = \sqrt{2\varkappa/(1-\varkappa r_e)}$, where r_e is the effective range (see, for example [20]). The ANC C increases with increasing $\varkappa$ also for the system described by the well-known Hulthén potential, for which the Schrödinger equation admits an analytical solution for $l = 0$ [21].

When the Coulomb interaction is turned on, the dependence of the ANCs on the binding energy becomes more complicated, since with a change in ε the factors $e^{-\varkappa r}$ and $(2\varkappa r)^{-\eta}$ on the right-hand side of (13) can change in opposite directions. So, for example, for the sum of the zero-radius potential and the point Coulomb potential, using the formulas of [8], one can obtain an explicit analytic expression for the ANC C in the form $C = \sqrt{2\varkappa}\tilde{C}(\eta)$, where $\tilde{C}(\eta)$ is a rapidly growing function of η, $\tilde{C}(0) = 1$. If the values of charges and masses are fixed, then with increasing ε the values of η and $\tilde{C}(\eta)$ decrease, but the ANC C may increase due to the multiplier $\sqrt{2\varkappa}$. In the examples we have considered, a similar situation is observed for the systems with small values of η presented in Tables 1 and 6, for which the transition from point to distributed charge slightly increases C: $C_2 > C_1$. For all other cases $C_2 < C_1$, and for loosely coupled systems with a large value of η, C_1 and C_2

may differ by an order of magnitude (see Tables 2 and 3) or even by many orders of magnitude (Table 5). For these systems, there is an even more general relation $C_1 > C_2 > C_4 > C_5$ and the similar relation for $\tilde{C}$ (see Tables 2–5, and 7). Note that for systems corresponding to Tables 1 and 6, even complete switching off the Coulomb interaction (version 4) does not change C significantly, which is associated with partial mutual compensation of the effects caused by decreasing η and increasing ε. In this case, the dimensionless ANC $\tilde{C}$, which differs from C by the factor $\sqrt{2\varkappa}$, changes more noticeably. From Tables 1 and 6, it is also seen that this compensation does not take place if, when the Coulomb interaction is switched off, the binding energy is preserved by changing the nuclear potential (version 5).

The most interesting and not obvious in advance result is the fact that for version 3, even in the case of a system with the maximum value of η (Table 5) C_3 differs from C_1 by only 7%; for other systems, this difference does not exceed 5%. Note that the same relative differences between versions 3 and 1 also take place for all other types of ANCs considered: $\tilde{C}$, C_r, and $\tilde{C}_r$. This implies an important conclusion that taking into account the non-point character of the nuclear charge distribution has little effect on the calculated values of ANCs, provided that the binding energy is considered fixed.

We emphasize that the use of renormalized ANCs C_r and $\tilde{C}_r$ is justified in the case of large values of η (see Tables 5 and 7).

ACKNOWLEDGEMENTS

The authors are grateful to S.Yu. Igashov, O.A. Rubtsova, and V.A. Khodyrev for helpful discussions.

FUNDING

This work was supported by the Russian Foundation for Basic Research grant no. 19-02-00014.

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