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MATHEMATICAL MODELING
IN NUCLEAR TECHNOLOGIES

Model of Dilaton Gravity with Dynamical Boundary: Results and Prospects

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Abstract— We consider a model of two-dimensional dilaton gravity where the strong coupling region is cut off by the dynamical boundary making its causal structure similar to the spherically symmetric sector of the higher dimensional gravity. It is shown that the classical dynamics is fully determined by a single ordinary differential equation which possesses an infinite number of exact solutions. All solutions describe either the solutions describing the full reflection regime at subcritical energies or the black hole formation regime at larger energies. Black hole evaporation effect is taken into account by introduction of a new field mimicking the one-loop conformal anomaly. The semiclassical solutions become nonanalytic and ambiguous. It is proposed to perform analytic continuation of the subcritical solutions describing the full reflection through the complex domain to bypass singularities of real solutions describing collapse. It is supposed that this may lead to the correct saddle point solution saturating the path integral for gravitational scattering amplitude at enough energy for a black hole to form in the classical theory.

Keywords: quantum field theory, two-dimensional gravity, black holes, information paradox

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INTRODUCTION

Two-dimensional models serve as excellent tools to improve our understanding of different aspects of quantum field theory. In particular, the appropriate model can be a perfect arena for studying the quantum gravity. The clarification of the situation with the apparent unitarity violation in the scattering processes with intermediate black holes [1] can be an intriguing possibility. Rigorous treatment of this problem met great difficulties, and, therefore, the consensus about its solution has not been achieved. It is favorable to have in a possession a simple model describing black hole evaporation where the problem can be formulated properly.

We draw our attention to the Callan–Giddings–Harvey–Strominger (CGHS) model [2] with a Lagrangian

$$\mathcal{L}_{CGHS} = e^{-2\phi}(R + 4(\nabla\phi)^2 + 4\lambda^2) - \frac{1}{2}\sum_{j=1}^{N}(\nabla f_j)^2, \quad (1)$$

describing interaction of two-dimensional metrics $g_{\mu\nu}$, dilaton ϕ, and N massless scalar fields f_j. In addition to the fact that two-dimensional gravity is itself renormalizable [3], the CGHS model is of great advantage in the respect that its solutions are expressed in the closed integral form.

The matter fields f_j obey the conformally invariant wave equation

$$\Box f_j = 0, \quad (2)$$

whose solutions decouple from the metrics and represent non-interacting in- and out-sectors

$$f_j(v,u) = f_j^{\text{in}}(v) + f_j^{\text{out}}(u) \quad (3)$$

in the light-cone frame.

Unfortunately, this model has its own pitfalls. Penrose diagram of the Minkowski vacuum shown in Fig. 1 is similar to that appeared in the spherically-symmetric Einstein gravity but with a new kind of asymptotic infinities $S^{\pm}$. The region of space-time $\mathcal{M}$ in the neighborhood of $\mathcal{I}^{\pm}$ is asymptotically flat: the effective gravitational constant $g_s \sim e^{\phi} \to 0$ what follows from Lagrangian (1) and vacuum solution $\phi = -\lambda r$. In contrast to the reduced higher dimensional gravity, there is a strong coupling region near the conformal boundary $S^{\pm}$, where fluctuations of the metrics and the dilaton field become large and the mean-field approximation fails.

It is natural to impose boundary conditions on the line $\phi = \phi_0$ outside the strong coupling region. The

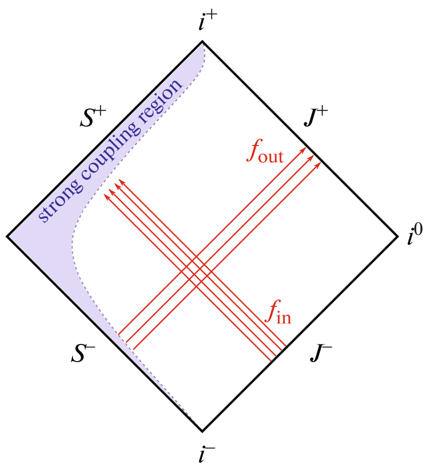

Fig. 1. Penrose diagram for Minkowski space in the CGHS model. Two wave packets f_{in} and f_{out} schematically depict noninteracting in- and out- wave sectors.

simplest variant is to set reflecting conditions for the matter fields f_j. The spacetime boundary $\partial\mathcal{M}$ becomes dynamical and the strong coupling region is removed. The similar approach was conducted in previous researches of the CGHS model (see, e.g., [4, 5] (the Russo–Susskind–Thorlacius (RST) model) and [6, 7]).

For a large number N of scalar fields, the black hole evaporation is a one-loop effect and it can be effectively incorporated by addition of the Liouville–Polyakov term,

$$-\frac{N}{48}\int d^2x\sqrt{-g}\int d^2x'\sqrt{-g'}R\Box^{-1}(x,x')R', \qquad (4)$$

where $\Box^{-1}(x,x')$ is the Green's function of the d'Alembert operator. $N \gg 1$ allows us to neglect contributions from the ghost fields, metrics, and dilaton. In the following, we use a trick from Ref. [8] replacing Eq. (4) with an additional scalar field nonminimally interacting with metric in such a way that its on-shell action is equivalent to Eq. (4). This allows us to work with the explicitly covariant and local expressions.

Unitarity of the quantum gravity received a substantial support from the AdS/CFT correspondence [9]. Nevertheless, it did not shed any light on the explicit mechanism behind the quantum coherence preservation. In addition, the information paradox received a new boost after formulation of the firewall problem [10] signaling severe violation of the equivalence principle. The numerical studies of Ashtekar et al. [11] showed that the CGHS model is still capable of surprising us. Thus, we still need simple two-dimensional gravity models where one can study the firewall problem without invoking any kind of holographic reasoning.

There are several new-fashioned semiclassical methods which allow reconsidering the problems described above. Among them is the technique of expansion around the instantons [12]. This method was applied at the beginning of the 1990s to describe tunneling transitions induced by multiparticle scattering. The gravitational collapse of a large number of particles into the black hole and their subsequent evaporation in larger number of soft Hawking quanta can be treated as such a process.

The second method deals with complex solutions of the classical field equations with regularized energy $E \mapsto E + i\varepsilon$, $\varepsilon > 0$. The suppression exponent of the gravitational scattering was calculated by exploiting this method in the thin shell models [13]. One cannot apply this method to the original CGHS model, because there is no classical reflection regime and the black holes form at arbitrarily low energies. Thus, the dynamical boundary is necessary (i) to cut off the strong coupling region, (ii) to ensure the existence of the full reflection regime, and (iii) to couple the in- and out-sectors of scalar fields f_j.

We aimed to construct a stringent Lagrangian formulation of the CGHS model with boundary, what was achieved recently in paper [14]. Despite breaking of exact solvability, dynamics of the model with boundary is fully governed by a single ordinary differential equation. We are able to solve Cauchy problem analytically for an infinite class of initial conditions representing incident wave packets of the scalar field f_{in}. We suppose that until now researchers were mostly studying delta-functional or piecewise-constant wave packets. Now, a fascinating possibility to study analytically smooth exact solutions becomes available. In addition, by investigating the critical behavior on the threshold of black hole formation, we came to the conclusion that this model is non-integrable.

Further, we take into account the quantum corrections by adding an auxiliary scalar field. Using the analytical results [14], it is still possible to derive smooth exact solutions. However, from physical reasoning it is clear that the obtained solutions describe the black hole evaporation consistently only to the past of the last ray of Hawking radiation which emanates from the endpoint where the event horizon meets the black hole singularity.

Near this last ray, an imminent quantum region arises, and, therefore, the full effective solution cannot be found without additional input concerning nonperturbative effects near singularity. It is unusual that large fluctuations near singularity exist despite the fact that the strong coupling region where $e^{\phi_0} \gg 1$ was removed. As a result, we lose predictability and analyticity of solutions in the mean-field theory, and the information loss appears to be inevitable.

We have mentioned the main theses of this paper and now proceed to a more detailed discussion of the model with boundary.

SEMICLASSICAL MODEL WITH PARAMETER Q

Let us consider the toy model of dilaton gravity describing evaporating black holes with action

$$S = \int_{\mathcal{M}} d^2x\sqrt{-g}\left[\mathcal{L}_{CGHS} - Q^2\phi R + Q\chi R - \frac{(\nabla\chi)^2}{2}\right] \times \theta(\phi_0 - \phi) + 2\oint_{\partial\mathcal{M}} d\tau[(e^{-2\phi} - Q^2\phi + Q\chi)K + 2\lambda e^{-2\phi} + \lambda Q^2 + \Lambda_\chi\partial_\tau\chi + \Lambda_\phi(\phi_0 - \phi)], \quad (5)$$

where χ is the auxiliary scalar field with the bulk equation

$$\Box\chi + QR = 0. \quad (6)$$

We assume that $N-1$ massless scalar fields f_j are in vacuum. It allows us to keep in the action (5) explicitly only one field $f_1 \equiv f$. The Liouville–Polyakov term (4) is encoded in the new scalar field χ. The local RST term ϕR is required by solvability of bulk field equations.

The boundary action is the Gibbons–Hawking term. It contains the extrinsic curvature $K = g^{\mu\nu}\nabla_\mu n_\nu$, where $n^\mu \propto \partial^\mu\phi$ is the outer normal to the boundary. The proper time τ is the invariant integration measure along $\partial\mathcal{M} = \{\phi = \phi_0\}$. It was also necessary to add the mass $\propto\lambda$ at the boundary to match the vacuum solution $\phi = -\lambda r$ with the corresponding boundary condition

$$\nabla_n\phi|_{\partial\mathcal{M}} = \lambda, \quad (7)$$

where $\nabla_n = n^\mu\nabla_\mu$. Next, we introduced the Lagrange multiplier for the field χ to impose a specific boundary condition

$$\partial_\tau\chi|_{\partial\mathcal{M}} = 0. \quad (8)$$

The last condition on the field f is of Neumann type:

$$\nabla_n f|_{\partial\mathcal{M}} = 0. \quad (9)$$

The action of the field χ is equivalent to (4) if one fixes $Q^2 = N/24$. Model with arbitrary parameter Q could be interesting on its own allowing us to study different regimes of evaporation. We need to stress that Eq. (8) is incorrect at quantum level, because it contradicts Wess–Zumino consistency condition.

For the sake of completeness, we write the bulk equations obtained by varying with respect to dilaton

$$2e^{-2\phi}(R + 4(\Box\phi - (\nabla\phi)^2 + \lambda^2)) + Q^2R = 0, \quad (10)$$

and the metric

$$4e^{-2\phi}(\nabla_\mu\nabla_\nu\phi + g_{\mu\nu}((\nabla\phi)^2 - \Box\phi - \lambda^2)) - 2Q^2\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)\phi = T^f_{\mu\nu} + T^\chi_{\mu\nu}, \quad (11)$$

where

$$T^f_{\mu\nu} = \nabla_\mu f\nabla_\nu f - \frac{g_{\mu\nu}}{2}(\nabla f)^2, \quad (12)$$

$$T^\chi_{\mu\nu} = \nabla_\mu\chi\nabla_\nu\chi - \frac{g_{\mu\nu}}{2}(\nabla\chi)^2 - 2Q(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\chi \quad (13)$$

are the energy-momentum tensors of the fields f and χ.

GENERAL SOLUTION

Let us proceed to the conformal gauge (the light-cone frame), where $ds^2 = -e^{2\rho}dvdu$. Since $R = 8e^{-2\rho}\partial_v\partial_u\rho$, one obtains

$$\chi(v,u) = \chi_{\rm in}(v) + \chi_{\rm out}(u) + 2Q\rho(v,u). \quad (14)$$

Let the boundary in the light-cone coordinates move according to the law $u = U(v)$. Using the boundary conditions (8) and (9), one can relate in- and out-components of the solutions (3) and (14). For the field f, one has

$$\nabla_n \propto \partial^\mu\phi\nabla_\mu, \quad \Rightarrow \quad \partial_v\phi\partial_u f + \partial_u\phi\partial_v f = 0,$$

what yields the reflection law,

$$f_{\rm in}(v) - f_{\rm out}(U(v)) = f_0. \quad (15)$$

For χ, we obtain the analogous relation

$$\chi_{\rm in}(v) + \chi_{\rm out}(U(v)) = \chi_0 - 2Q\phi_0. \quad (16)$$

The integration constants f_0 and χ_0 are fixed correspondingly for separate timelike segments of the boundary in order to sustain continuity of the fields f and χ.

The residual gauge degrees of freedom may be fixed completely by fixing $\rho = \phi$. Solving Eqs. (10) and (11) one obtains the general bulk solution,

$$\Omega \triangleq e^{-2\phi} + Q^2\phi = -\lambda^2vu + g(v) + h(u), \quad (17)$$

where

$$g(v) = \frac{1}{2}\int_{\frac{1}{\lambda}}^{v} dv'\int_{v'}^{+\infty} dv''[(\partial_v f_{\rm in}(v''))^2 + (\partial_v\chi_{\rm in}(v''))^2 + 2Q\partial_v^2\chi_{\rm in}(v'')], \quad (18)$$

$$h(u) = -\frac{1}{2}\int_{-\frac{1}{\lambda}}^{u} du' \int_{-\infty}^{u'} du''[(\partial_u f_{\text{out}}(u''))^2 + (\partial_u \chi_{\text{out}}(u''))^2 + 2Q\partial_u^2 \chi_{\text{out}}(u'')]. \tag{19}$$

The linear dilaton vacuum $\phi = -\lambda r$ satisfies the field equations. Using the vacuum solution $\phi = -\frac{1}{2}\ln(-\lambda^2 vu)$ in the gauge $\rho = \phi$, one finds

$$\chi_{\text{in}}(v) = Q\ln(\lambda v), \quad \chi_{\text{out}}(u) = Q\ln(-\lambda u) \tag{20}$$

because of the nontrivial transformation law for ρ in (14). In this case all scalar fields are in vacuum, $f = 0 = \chi$.

BOUNDARY EQUATION

The derivation of the differential equation for $U(v)$ is the final step. Using the dilaton boundary condition (7), one obtains the Riccati equation

$$\partial_v U(v) = \frac{e^{2\phi_Q}}{\lambda^2}(\partial_v g + Q\partial_v \chi_{\text{in}} - \lambda^2 U(v))^2, \tag{21}$$

where $e^{-2\phi_Q} \triangleq e^{2\phi_0}(e^{-2\phi_0} + Q^2/2)^2$. Substituting into Eq. (21)

$$\lambda^2 U = \partial_v g + Q\partial_v \chi_{\text{in}} - e^{-2\phi_Q}\partial_v \psi/\psi, \tag{22}$$

one simplifies it to equation on the new function ψ,

$$\partial_v^2 \psi = -\frac{e^{2\phi_Q}}{2}\left(\frac{Q^2}{v^2} + (\partial_v f_{\text{in}})^2\right)\psi, \tag{23}$$

where we have used the vacuum value $\chi_{\text{in}}(v)$ from (20).

The function ψ has a simple relation with the proper time of the boundary. Since

$$d\tau^2 = e^{2\phi_0}\partial_v U(v) dv^2, \quad \Rightarrow \quad \psi = \psi_0 e^{\lambda\tau}. \tag{24}$$

For the linear dilaton vacuum, the boundary equation (21) has a solution $U(v) = -e^{-2\phi_0}/(\lambda^2 v)$. It appears not to be a unique vacuum solution of Eq. (21). Consider the corresponding general solution of Eq. (23):

$$\psi = c_+(\lambda v)^{k_+} + c_-(\lambda v)^{k_-}, \qquad k_\pm = \frac{1 \pm \sqrt{1 - 2e^{2\phi_Q}Q^2}}{2}. \tag{25}$$

Indeed, the solution with $c_+ \neq 0 = c_-$ yields the boundary $U(v) = -e^{-2\phi_0}/(\lambda^2 v)$. However, there is also a second branch with $c_+ = 0 \neq c_-$, which corresponds to the boundary also in the motionless state, $U(v) = -e^{2\phi_0}Q^4/(4\lambda^2 v)$.

By direct substitution one can verify that the first boundary trajectory is matched onto the flat solution with the metric $g = \eta$ and zero curvature $R = 0$,

$$\Omega = -\lambda^2 vu - \frac{Q^2}{2}\ln(-\lambda^2 vu).$$

The second boundary trajectory yields the solution

$$\Omega = \frac{M^*}{2\lambda} - \lambda^2 vu - \frac{Q^2}{2}\ln(-\lambda^2 vu), \tag{26}$$

with non-zero mass parameter $M^* = 2\lambda e^{-2\phi_0} + 4\lambda Q^2\phi_0 - \frac{1}{2}\lambda e^{2\phi_0}Q^4 + 2\lambda Q^2 \ln\frac{Q^2}{2}$.

EVAPORATING BLACK HOLES

In the case of low energy of the field f_{in}, the solution describes the full reflection off the boundary without appearance of the event horizon. If Q is sufficiently small in comparison with $e^{-\phi_0}$, the corresponding solutions far before the end of evaporation only slightly differ from those studied in [14]. Note that the accelerated motion of the boundary also generates a nonzero flux of the field χ in $\mathscr{I}^+$ representing quantum creation of particles. The asymptotic boundary behavior $U \sim -e^{-2\phi_0}/(\lambda^2 v)$ at $v \to 0, +\infty$ guarantees the total energy conservation with no regard to the presence of a black hole.

With growing energy, the boundary $\phi = \phi_0$ becomes spacelike, what can be interpreted as a singularity absorbing the matter fields. Then, the black hole begins to evaporate producing field χ with flux $\sim \lambda^2 Q^2$. During evaporation process, the singularity $\phi = \phi_0$ moves toward a turning point (v_e, u_e) and finally meets the horizon and again becomes timelike. One can show that this curve does not satisfy Eq. (21) and, therefore, cannot be a regular part of the boundary, violating the reflecting conditions, Eqs. (15) and (16). Thus, the full analytical solution is incompatible. One also notices that the matter fields at $\mathscr{I}^+$ violate energy conservation and are in a causal contact with the black hole interior, what is unphysical. The only way to avoid this is to attach the new solution after the last ray $u = u_e$ satisfying the equation of motion (21) with the vacuum asymptotics at $v \to +\infty$.

The boundary condition could be such that a new branch of the timelike boundary $U'(v)$ emerges at the endpoint (v_e, u_e) automatically satisfying Eq. (21). The corrected solution is unique. Using the reflecting conditions, one obtains the functions χ'_{out} and f'_{out}, which have to be matched with χ_{out} and f_{out} on the last ray with use of free integration constants f_0 and χ_0 in (15) and (16). Otherwise, the fields χ_{out} and f_{out} suffer from discontinuity at $u = u_e$, leading to a malicious singularity in the energy density $\propto \delta^2(u - u_e)$.

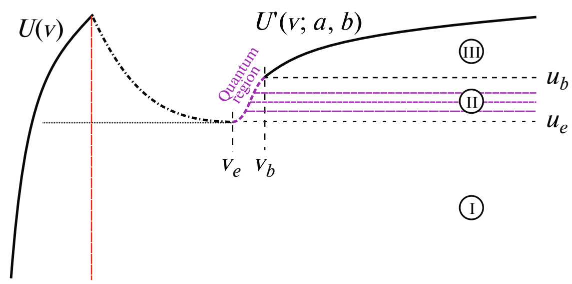

Fig. 2. Spacetime of evaporating black hole. Two regular parts of the boundary correspond to the solution in regions I and III. II is the quantum region.

QUANTUM REGION

Unfortunately, the classical solution described in the section above will suffer from the quantum pathology known as the thunderbolt problem [15]. We consider the behavior of the correlation functions $G(x,x') =: \langle \hat{f}(x)\hat{f}(x')\rangle :_x$. Vacuum correlation of the scalar field $f_{\rm in}$

$$G_{\rm in}(\bar{v},\bar{v}') \propto \ln|\bar{v}-\bar{v}'|$$

has a logarithmic singularity at $\bar{v}$ and $\bar{v}'$. Using the reflecting condition, one finds the correlation function for the field $f_{\rm out}$

$$G_{\rm out}(\bar{u},\bar{u}') = G_{\rm in}(\bar{v}(\bar{u}),\bar{v}(\bar{u}')).$$

Now, we consider two points lying on opposite sides of the last ray: the first one is at $\bar{u} = \bar{u}_e - \varepsilon$ and $\bar{v} \approx \bar{v}_{hor}$ on the boundary before it went under the horizon, and the second one is at $\bar{u}' = \bar{u}_e + \varepsilon$ and $\bar{v}' \approx \bar{v}_e$ on the boundary after its re-emerging at the endpoint (Fig. 2). One can see that

$$G_{\rm out}(\bar{u}_e-\varepsilon,\bar{u}_e+\varepsilon) \propto \ln(\lambda t_{\rm evap}), \qquad t_{\rm evap} \sim \bar{v}_e - \bar{v}_{hor},$$

differs significantly from the vacuum one for any ε. This means that there must exist particles with arbitrarily large momenta $k \sim \varepsilon^{-1}$ in the last ray vicinity of size ε. The thunderbolt is a burst of particles with infinite energy.

One may provide a following interpretation of such a disaster. Delta-functional energy density on the last ray indicates that semiclassical approximation is not applicable there. The thunderbolt appears in this case as an artifact signaling large quantum fluctuations of the fields near the endpoint. One assumes that backreaction taken into account dissolves the thunderbolt in a region of size $\varepsilon \sim \dfrac{Q}{M_{\rm cr}}$, which can be estimated by using energy conservation. Hence, the solution in this quantum region cannot be determined uniquely and the mean-field approximation is not meaningful even in the weak coupling limit.

It is well known that the evaporating black holes violate global charge conservation laws. Massless scalar field possesses a shift charge

$$Q_f = \int d\bar{u}\,\partial_{\bar{u}} f_{\rm out}.$$

One may wonder if this charge absorbed by a black hole could be restored because of uncertainty of the solution in the quantum region. Nevertheless, this is impossible. It is easy to derive a Cauchy-type inequality $Q_f \le \varepsilon E$, where Q_f is the shift charge hidden inside the quantum region with energy E and size ε. This estimate reads $Q_f \lesssim Q$, so that is impossible to restore an arbitrary amount of global charge absorbed by the black hole.

CONCLUSIONS

We showed that the standard mean-field theory suffers a defeat in the weakly-coupled model, Eq. (5). For any solution with sufficiently large energies, there appears a quantum region of finite width where the usual semiclassical approximation does not work.

Is there any possible approach to the CGHS problem to solve it without complete rejection of semiclassics? An interesting possibility arose from the method of complex classical trajectories applied to the CGHS model with boundary [14]. The idea is to start from reflecting solution with subcritical energy $E < E_{\rm cr}$ and then perform analytic continuation avoiding singularities of collapsing solution at energies with $\mathrm{Re}\,E > E_{\rm cr}$.

One may use the obtained solution to estimate the suppression exponent of the multiparticle scattering of the coherent states corresponding to the wave packets $f_{\rm in,out}$ at energies sufficient for black hole formation in a purely classical setup. If this method becomes successful, it would be possible to test unitarity of the gravitational S-matrix in this model.

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