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On the Breakdown of Charge Independence and Charge Symmetry of the Pion–Nucleon Coupling Constant

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Abstract—A simple, physically validated, model of charge-independence and charge-symmetry breaking is proposed for the pion–nucleon coupling constant. Within this model, the pion–nucleon coupling constants are assumed to be in direct proportion to the masses of nucleons and pions involved in the interaction. The charge dependence of the pion–nucleon coupling constants and low-energy parameters of nucleon–nucleon scattering in the $^1S_0$ spin-singlet state is studied on the basis of Yukawa's meson theory. By using the value of $f^2_{pp\pi^0} = 0.0749\,(7)$ established reliably for the pseudovector pion–nucleon coupling constant, which characterizes the strength of the nuclear proton–proton interaction, the values of $f^2_c = 0.0802\,(7)$ and $f^2_0 = 0.0750\,(7)$ were calculated for, respectively, the charged and neutral pion–nucleon coupling constants, along with the value of $f^2_{nn\pi^0} = 0.0751\,(7)$, which characterizes the strength of the nuclear neutron–neutron interaction. The values calculated for the low-energy parameters of neutron–proton and neutron–neutron scattering with the aid of the above constants agree well with experimental data.

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1. INTRODUCTION

The pion–nucleon coupling constants are fundamental physics features of strong nuclear interaction. There are two kinds of them—the pseudovector, f_π, and pseudoscalar, g_π, pion–nucleon coupling constants, which are related by the equation $g_\pi = (2M_N/m_{\pi^\pm})\, f_\pi$, where M_N and m_π are the masses of the nucleon and pion, respectively, involved in the interaction. The pion–nucleon coupling constants play an important role in studying nucleon–nucleon and pion–nucleon scattering processes. In view of this, much attention has been given to studying them and to refining their values [1–25]. The pion–nucleon coupling constants are especially important for low-energy nuclear physics owing to the fact that the pions are the lightest mesons, so that their exchange determines the most well known and, simultaneously, most long-range part of the nucleon–nucleon interaction—its so-called one-pion tail.

The aforementioned fact that the pion is the meson of greatest importance for obtaining deeper insight into the properties of nuclear forces explains why the understanding and a precise quantitative description of pion–nucleon interaction would provide a clue to constructing the theory of strong nuclear interaction as such [1–3]. In the 1970s and the early 1980s, there was some kind of consensus on the value of the pion–nucleon coupling constant g^2_π, which was generally thought to be charge-independent and approximately equal to 14.5 [2–8], but, later on, the situation changed substantially and became more ambiguous [9–24].

For example, a Nijmegen group of physicists published a series of articles in the 1990s [9–11], where, on the basis of an energy-dependent partial-wave analysis of data on nucleon–nucleon scattering, they obtained the values of $g^2_{\pi^0} = 13.47\,(11)$ and $g^2_{\pi^\pm} = 13.54\,(5)$ for, respectively, the neutral and charged pion–nucleon coupling constants. These values are approximately 7% smaller than their counterparts obtained earlier. The values presented by the Nijmegen group for the neutral and charged coupling constants are in close agreement within the errors, which confirms to some extent the charge independence of the pion–nucleon coupling constant. Similar smaller values of the charged pion–nucleon coupling constants, $g^2_{\pi^\pm} \sim 13.7$–13.8, were obtained in a number of other studies [14–17].

At the same time, the Uppsala group for neutron studies [18] obtained, for the charged pion–nucleon coupling constant, a much greater value of $g^2_{\pi^\pm} = 14.52\,(26)$, which exceeds substantially the generally accepted value obtained for the neutral pion–nucleon coupling constant, $g^2_{\pi^0} = 13.55\,(13)$ [the respective value of the pseudovector coupling constant is $f^2_{\pi^0} =$

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0.0749 (7)], in [9] on the basis of a partial-wave analysis of proton–proton scattering in the energy range of $T_{\rm lab} \leqslant 350$ MeV. Other studies of the Uppsala group [19, 20] confirmed a rather "large" value of the charged pion–nucleon coupling constant, $g^2_{\pi^\pm} \sim 14.5$, this being in agreement with the results of some earlier studies reported in [4–6, 8] and devoted to determining the charged pion–nucleon coupling constants.

It follows that, at the present time, the problem of the possible charge dependence of the pion–nucleon coupling constant is of paramount fundamental importance. In other words, this is problem of the distinction between the pion–nucleon coupling constants for neutral and charged pions. By and large, quite a detailed history of the change in the situation around the pion–nucleon coupling constants can be found in [3, 11–13, 21, 22]. The problem of the possible breakdown of charge symmetry of the pion–nucleon coupling constant is even subtler and has not yet received adequate study. This is the problem of the distinction between the pion–nucleon coupling constants corresponding to proton–proton and neutron–neutron interaction. The point is that, because of the absence of neutron targets, direct experiments aimed at studying neutron–neutron scattering have to date remained impossible, as is well known. Anywhere, however, the amount of the breakdown of charge symmetry of the pion–nucleon coupling constant should obviously be less than the amount of the breakdown of its charge independence.

In our earlier studies [23, 24], the pion–nucleon coupling constant and the breakdown of charge independence of nuclear forces were studied in the approximation of one-pion exchange by employing the standard classic Yukawa model [1–3]. According to Yukawa's meson theory, strong nuclear interaction between two nucleons at low energies is due primarily to the exchange of virtual pions, which determines the long-range part of the nucleon–nucleon (NN) interaction and, accordingly, governs NN scattering in the low-energy region. In that case, the pion–nucleon coupling constants, which are fundamental physics parameters of strong nuclear interaction, control the strength of nuclear interaction. The results obtained in those studies are indicative of a substantial charge dependence of the pion–nucleon coupling constants, which characterize the NN interaction. On the basis of a simultaneous analysis of low-energy pion–nucleon and nucleon–nucleon parameters, we showed that the breakdown of charge independence of nuclear forces stems primarily from the mass difference between the charged and neutral pions. In the present article, which, in fact, reports on a continuation of our earlier studies quoted above [23, 24], we describe charge-independence and charge-symmetry breaking for nuclear forces in the pion–nucleon coupling constant and low-energy parameters of nucleon–nucleon scattering on the basis of a simple phenomenological model that takes into account the mass difference between the charged and neutral pions, as well as the mass difference between the neutron and proton, and which is compatible with the standard classic Yukawa meson model.

2. DERIVATION AND DISCUSSION OF BASIC FORMULAS THAT ESTABLISH A RELATION BETWEEN DIFFERENT KINDS OF PION–NUCLEON COUPLING CONSTANTS

With allowance for electric-charge conservation in pion–nucleon and nucleon–nucleon systems, it is necessary in general to discriminate between four kinds of the elementary pseudovector pion–nucleon coupling constants [11, 22]:

$$f_{p\pi^0\to p}, \quad f_{n\pi^0\to n}, \quad f_{p\pi^-\to n}, \quad f_{n\pi^+\to p}. \qquad (1)$$

These kinds correspond to four possible types of elementary interaction vertices: $p\pi^0 \to p$, $n\pi^0 \to n$, $p\pi^- \to n$, and $n\pi^+ \to p$. Here, the $p\pi^- \to n$ vertex, for example, corresponds to the process in which proton annihilation and neutron creation occur, along with π^--meson annihilation or π^+-meson creation. The remaining pion–nucleon interaction vertices admit a similar interpretation. It is noteworthy that, instead of the detailed notation $f_{p\pi^0\to p}$, $f_{n\pi^0\to n}$, $f_{p\pi^-\to n}$, and $f_{n\pi^+\to p}$ for the elementary pion–nucleon coupling constants, use is made of the notation f_p, f_n, f_-, and f_+, respectively, in some articles [11, 22]. For the sake of convenience, we will henceforth use the notation $f_{p\pi^0}$, $f_{n\pi^0}$, $f_{p\pi^-}$, and $f_{n\pi^+}$, where the subscripts correspond to the initial channel of the $N\pi \to N'$ reaction.

It should be noted that the normalization of the elementary coupling constants in (1) is chosen in such a way that, in the limiting particular case of exact fulfillment of charge independence (CI), all these coupling constants are coincident with one another; that is,

$$f^{\rm CI}_{p\pi^0} = f^{\rm CI}_{n\pi^0} = f^{\rm CI}_{p\pi^-} = f^{\rm CI}_{n\pi^+}. \qquad (2)$$

At the same time, we emphasize that, in the present study, we assume that all coupling constants in (1) are different in general, since, in the processes that they describe, one deals with particles of different mass corresponding to the nucleon, M_N ($N = p, n$), and pion, m_π ($\pi = \pi^0, \pi^+, \pi^-$) masses.

The pion–nucleon coupling constants that characterize the strength of nuclear interaction between two nucleons are important combinations of the elementary coupling constants in (1). They are defined as follows [11, 22]:

$$f^2_{pp\pi^0} = f_{p\pi^0} f_{p\pi^0}, \tag{3}$$

$$f^2_{nn\pi^0} = f_{n\pi^0} f_{n\pi^0}, \tag{4}$$

$$f^2_0 = f_{p\pi^0} f_{n\pi^0}, \tag{5}$$

$$f^2_c = f_{p\pi^-} f_{n\pi^+}. \tag{6}$$

The coupling constants in (3) and (4) characterize the strength of the nuclear interaction in the $^1S_0$ spin-singlet state between two protons and between two neutrons, respectively, via the exchange of neutral pions. In the case of neutron–proton interaction, the exchange of both neutral and charged pions occurs. In the latter case, it is necessary to employ the neutron–proton coupling constant $f^2_{np\pi}$ defined as the average of the neutral, f^2_0, and charged, f^2_c, pion–nucleon coupling constants [23–26]; that is,

$$f^2_{np\pi} \equiv \frac{1}{3}\left(f^2_0 + 2f^2_c\right). \tag{7}$$

In [24], we showed that the following approximate relation holds for the pseudovector pion–nucleon coupling constant:

$$\frac{f_{\pi^\pm}}{f_{\pi^0}} = \frac{m_{\pi^\pm}}{m_{\pi^0}}. \tag{8}$$

This relation, which is highly precise, indicates that the charge splitting of the pion–nucleon coupling constant is nearly identical to the charge splitting of the pion mass. The analysis in [24] is performed in the approximation of exact fulfillment of charge symmetry (CS); that is,

$$f^2_{\pi^0} \equiv f^2_{pp\pi^0} = f^2_{nn\pi^0} = f^2_0, \quad f^2_{\pi^\pm} = f^2_c.$$

Relation (8) has a straightforward physical validation [24]. Indeed, we note that, since the pion–nucleon coupling constant f_π measures the strength of the pion-field action on a nucleon, the higher the pion mass m_π, the stronger this action. Thus, the meson field of charged pions having the mass $m_{\pi^\pm}$ in excess of the neutral-pion mass m_{π^0} and surrounding a nucleon has a stronger effect on it than the neutral-pion field. In other words, relation (8) can also be recast into the form

$$f_\pi = Cm_\pi. \tag{9}$$

This form reflects directly the above physical property of the pion–nucleon system. We would like to emphasize that relations (8) and (9), taken in one form or another, attracted the attention of the authors of some earlier studies [12, 13, 27, 28].

Taking now into account the presence of four forms of the pion–nucleon coupling constants in the general case, as well as the finiteness of the nucleon masses and the difference in them, we then obtain the following relations for the elementary pion–nucleon coupling constants $f_{N\pi} \equiv f_{N\pi\to N'}$, which characterize the strength of the nuclear nucleon–pion interaction:

$$f_{p\pi^0} = CM_p m_{\pi^0}, \tag{10}$$

$$f_{n\pi^0} = CM_n m_{\pi^0}, \tag{11}$$

$$f_{p\pi^-} = CM_p m_{\pi^-}, \tag{12}$$

$$f_{n\pi^+} = CM_n m_{\pi^+}. \tag{13}$$

They provide a natural generalization of relation (9).

We note that, to some extent, relations (10)–(13) for pion–nucleon systems appear to be an analog of Newton's law of universal gravitation and Coulomb's law, since, in these relations, some feature of the interaction strength in the system being considered—namely, the pion–nucleon coupling constant—is in direct proportion to the product of what plays the role of charges in the system. In this connection, it is worth recalling that Yukawa's meson theory [1–3] was originally constructed by analogy with electromagnetic-interaction theory for the strong-interaction case where the interaction is mediated by massive mesons rather than by massless photons. In general, relations (10)–(13) admit the same physics interpretation and have the same validation as relations (8) and (9), which are their particular case.

It is of importance that, from the proposed relations in (10)–(13), it obviously follows that, in the particular limiting case where the nucleon masses are equal to each other ($M_n = M_p = M_N$) and where, simultaneously, all of the pion masses are equal to one another ($m_{\pi^\pm} = m_{\pi^0} = m_\pi$), all elementary pion–nucleon coupling constants in (1) are equal to one another, and so are therefore all of the constants in (3)–(7); that is, the charge independence of nuclear forces holds exactly and fully in this particular case for the pion–nucleon coupling constant. Further, charge-independence and charge-symmetry breaking for nuclear forces in the simple model being considered turns out to be due fully and directly to, as follows from relations (10)–(13), the mass difference between the particles involved in the interaction (nucleons) and the mass difference between the mediators of the interaction (pions).

The ensuing calculations and conclusions drawn from them show that the proposed hypothesis leads to a series of reasonable results and implications, which, in a number of cases, agree well with experimental data. In passing, we emphasize that the hypothesis that charge-independence breaking for nuclear forces in nucleon–nucleon systems stems primarily from the mass difference between the charged and neutral pions has a rather long and rich history and a sound validation [12, 13, 28–36]. However, it has yet to be proven conclusively.

Ultimately, charge-independence breaking for nuclear forces, in general, and for the pion–nucleon coupling constant, in particular, is due, from the microscopic point of view of quantum chromodynamics (QCD), to the mass and charge difference between u- and d-quarks, which are elementary constituents of hadrons, as well as to microscopic electromagnetic effects in quark–gluon interaction that are induced by these distinctions. Thus, the modern generally accepted microscopic theory of strong interaction in the form of QCD predicts that charge-independence breaking inevitably occurs for the pion–nucleon coupling constant, and there only remains the question of the amount of this breaking for various forms of this constant.

As was indicated earlier, the proton–proton coupling constant $f^2_{pp\pi^0}$ has been determined to date most reliably and most precisely from experiments [9]. There is no substantial discrepancies between its known values. Within the proposed model, the neutron–neutron coupling constant $f^2_{nn\pi^0}$, the neutral coupling constant f^2_0, and the charged coupling constant f^2_c are expressed, according to Eqs. (3)–(6) with allowance for Eqs. (10)–(13), in terms of the coupling constant $f^2_{pp\pi^0}$ as

$$f^2_{nn\pi^0} = \frac{M_n^2}{M_p^2} f^2_{pp\pi^0}, \quad (14)$$

$$f^2_0 = \frac{M_n}{M_p} f^2_{pp\pi^0}, \quad (15)$$

$$f^2_c = \frac{M_n}{M_p}\frac{m^2_{\pi^\pm}}{m^2_{\pi^0}} f^2_{pp\pi^0}. \quad (16)$$

From Eqs. (7), (15), and (16), it follows that the neutron–proton coupling constant has the form

$$f^2_{np\pi} = \frac{1}{3}\frac{M_n}{M_p}\left(1 + 2\frac{m^2_{\pi^\pm}}{m^2_{\pi^0}}\right) f^2_{pp\pi^0}. \quad (17)$$

The pseudoscalar pion–nucleon coupling constant $g_{N\pi}$ and the pseudovector coupling constant $f_{N\pi}$ are related by the well-known equivalence equation [1, 3]

$$g_{N\pi} = \frac{2M_N}{m_{\pi^\pm}} f_{N\pi}. \quad (18)$$

We note in passing that many authors use the notation $g^2/4\pi$ for the pseudoscalar pion–nucleon coupling constant instead of our notation g^2. Our notation, which is also used quite widely, is obtained from the aforementioned one by means of the simple scaling transformation $g \to g\sqrt{4\pi}$ [22], which is quite obvious. Taking into account Eq. (18) and also employing relations (3)–(6) and (14)–(17), we obtain the following expressions for the pseudoscalar pion–nucleon coupling constants

$$g^2_{pp\pi^0} = \left(\frac{2M_p}{m_{\pi^\pm}}\right)^2 f^2_{pp\pi^0}, \quad (19)$$

$$g^2_{nn\pi^0} = \frac{M_n^4}{M_p^4} g^2_{pp\pi^0}, \quad (20)$$

$$g^2_0 = \frac{M_n^2}{M_p^2} g^2_{pp\pi^0}, \quad (21)$$

$$g^2_c = \frac{M_n^2}{M_p^2}\frac{m^2_{\pi^\pm}}{m^2_{\pi^0}} g^2_{pp\pi^0}, \quad (22)$$

$$g^2_{np\pi} = \frac{1}{3}\frac{M_n^2}{M_p^2}\left(1 + 2\frac{m^2_{\pi^\pm}}{m^2_{\pi^0}}\right) g^2_{pp\pi^0}. \quad (23)$$

3. NUMERICAL RESULTS AND DISCUSSION ON CHARGE-INDEPENDENCE BREAKING FOR THE PION–NUCLEON COUPLING CONSTANT

Taking into account Eqs. (14)–(17) and employing the reliably established experimental value of the proton–proton coupling constant [9],

$$f^2_{pp\pi^0} = 0.0749\,(7), \quad (24)$$

and the experimental values of the nucleon and pion masses [37],

$$M_p = 938.272046\ \mathrm{MeV}/c^2, \quad (25)$$
$$M_n = 939.565379\ \mathrm{MeV}/c^2,$$

$$m_{\pi^0} = 134.9766\ \mathrm{MeV}/c^2, \quad (26)$$
$$m_{\pi^\pm} = 139.57018\ \mathrm{MeV}/c^2,$$

we obtain the following values for the pion–nucleon coupling constants $f^2_{nn\pi^0}$, f^2_0, f^2_c, and $f^2_{np\pi}$:

$$f^2_{nn\pi^0} = 0.0751\,(7), \quad (27)$$

$$f^2_0 = 0.0750\,(7), \quad (28)$$

$$f^2_c = 0.0802\,(7), \quad (29)$$

$$f^2_{np\pi} = 0.0785(7). \quad (30)$$

In this case, the use of Eqs. (19)–(23) leads to the following results for the pseudoscalar pion–nucleon coupling constants:

$$g^2_{pp\pi^0} = 13.54\,(13)\,, \quad (31)$$

$$g^2_{nn\pi^0} = 13.61\,(13)\,, \quad (32)$$

$$g^2_0 = 13.58\,(13)\,, \quad (33)$$

$$g^2_c = 14.52\,(13)\,, \quad (34)$$

$$g^2_{np\pi} = 14.20(13). \quad (35)$$

The value in (34) that we found for the charged pseudoscalar pion–nucleon coupling constant g^2_c in the way outlined above on the basis of the proposed model is in perfect agreement with the experimental value

$$g^2_c = 14.52\,(26) \quad (36)$$

obtained by the Uppsala group for neutron studies [18]. The value in (34) is also in very good agreement with the value of $g^2_{\pi^\pm} = 14.55\,(13)$ that we obtained in [23, 24] on the basis of Yukawa's pion–nucleon model by employing the low-energy parameters of *pp* and *np* scattering in the approximation of exact fulfillment of charge symmetry ($g^2_{\pi^0} \equiv g^2_{pp\pi^0} = g^2_{nn\pi^0} = g^2_0$ and $g^2_{\pi^\pm} = g^2_c$). The value in (34) obtained for the coupling constant g^2_c within the model being considered also agrees with a number of other values deduced for the charged pion–nucleon coupling constant from an analysis of experimental data on nucleon–nucleon and pion–nucleon interaction [4–6, 8, 19, 20].

At the same time, some other experimental determinations [10, 11, 14–17, 38–40] led to substantially smaller values for the charged coupling constant g^2_c, which are close to the value of the neutral pion–nucleon coupling constant in (33), $g^2_0 = 13.58\,(13)$; this may suggest the possible charge independence of the pion–nucleon coupling constant. Thus, the problem of charge dependence or charge independence of the pion–nucleon coupling constants f^2 and g^2 is open at the present time and calls for further experimental and theoretical investigations [9–24, 38–43]. Nevertheless, the results of the present study and the results of our earlier studies reported in [23, 24] and based on Yukawa's meson theory are indicative of charge-independence breaking for the pion–nucleon coupling constant, fully in agreement with the concepts adopted in QCD, which underlies generally accepted microscopic strong-interaction theory. The recently published review article of Matsinos [22] provides a more detailed description of the situation around the pion–nucleon coupling constants and their determination.

The value obtained within the proposed model for the ratio of the neutral pseudoscalar pion–nucleon coupling constants $g_{n\pi^0} \equiv g_{nn\pi^0}$ and $g_{p\pi^0} \equiv g_{pp\pi^0}$ corresponding to the neutron and proton,

$$\frac{g_{n\pi^0}}{g_{p\pi^0}} = \frac{M^2_n}{M^2_p} = 1.0028, \quad (37)$$

agrees well with the value

$$\frac{g_{n\pi^0}}{g_{p\pi^0}} = 1.0038, \quad (38)$$

obtained in [44] on the basis of the chiral Cloudy Bag Model (CBM) and with the value

$$\frac{g_{n\pi^0}}{g_{p\pi^0}} = 1.0023, \quad (39)$$

found in [45] by the method of Feynman graphs.

On the whole, the problem of the value of the pion–nucleon coupling constant $g_{nn\pi^0}$ corresponding to the neutron–neutron interaction has not yet received adequate study, and the number of investigations devoted to it is rather small. In any case, the distinction between the neutron, $g_{nn\pi^0}$, and proton, $g_{pp\pi^0}$, coupling constants is small, as can be seen from the values in (37)–(39), but the spread of the specific values obtained for $g_{nn\pi^0}/g_{pp\pi^0}$ within various models is quite large. The value obtained for this quantity on the basis of our model and presented in (37) agrees with the results based on some of the aforementioned models but disagrees with the results produced by some other models. A more detailed investigation of the neutron coupling constant $g_{nn\pi^0}$ and a comparison with the results of the calculations already performed on the basis of various models will be the subject of our subsequent studies.

Let us now consider numerical characteristics of charge-independence and charge-symmetry breaking for the pion–nucleon coupling constants f^2 and g^2 within the proposed model. From Eqs. (14)–(23), it follows that

$$\frac{f^2_c}{f^2_0} = \frac{g^2_c}{g^2_0} = \frac{m^2_{\pi^\pm}}{m^2_{\pi^0}} = 1.0692, \quad (40)$$

$$\frac{f^2_{np\pi}}{f^2_0} = \frac{g^2_{np\pi}}{g^2_0} = \frac{1}{3}\left(1 + 2\frac{m^2_{\pi^\pm}}{m^2_{\pi^0}}\right) = 1.0461, \quad (41)$$

$$\frac{f^2_{nn\pi^0}}{f^2_{pp\pi^0}} = \frac{M^2_n}{M^2_p} = 1.0028, \quad (42)$$

$$\frac{g^2_{nn\pi^0}}{g^2_{pp\pi^0}} = \frac{M_n^4}{M_p^4} = 1.0055. \quad (43)$$

These important relations characterize the amount of charge-independence and charge-symmetry breaking for the pion–nucleon coupling constants. The degree of charge-independence breaking for the pion–nucleon coupling constants is determined by the mass difference between the charged and neutral pions ($m_{\pi^\pm} > m_{\pi^0}$), while the degree of charge-symmetry breaking is completely determined by the mass difference between the neutron and proton ($M_n > M_p$). In the particular case of exact charge symmetry, relation (40) written in the form $f_c/f_0 = m_{\pi^\pm}/m_{\pi^0}$ reduces to relation (8).

From Eqs. (40)–(43), one can see that the pion–nucleon coupling constants satisfy the following important inequalities:

$$f^2_{pp\pi^0} < f_0^2 < f^2_{nn\pi^0} < f^2_{np\pi} < f_c^2, \quad (44)$$

$$g^2_{pp\pi^0} < g_0^2 < g^2_{nn\pi^0} < g^2_{np\pi} < g_c^2. \quad (45)$$

The numerical values obtained above for the pion–nucleon coupling constants and presented in (24) and (27)–(35) illustrate and confirm the inequalities in (44) and (45). Under the condition of exact charge symmetry, we obtained the following inequalities in [24]:

$$f_0^2 < f_c^2, \quad g_0^2 < g_c^2. \quad (46)$$

They are a particular case of the inequalities in (44) and (45).

The inequalities in (44) and (45) have a clear physical validation and admit a straightforward interpretation. Namely, the fact that the pion–nucleon coupling constant $f^2_{NN'\pi}$ characterizes the strength of the nuclear interaction between respective nucleons in the $^1S_0$ spin-singlet state entails the conclusion that, with allowance for Eqs. (24), (27), (28) and (30), the inequalities in (44) and (45) are indicative of a substantially greater strength of the np interaction in relation to the pp- and nn-interaction strengths, as well as in relation to the averaged strength of pp and nn interactions. The important fact that the strength of the np interaction in the $^1S_0$ spin-singlet state is substantially greater than the strength of the pp interaction is beyond any doubt now [24]. The fact that the singlet np scattering length is larger in absolute value than the purely nuclear pp scattering length, $|a_{np}| > |a_{pp}|$, is an argument in support of the statement that the np interaction is stronger at low energies than the pp interaction. The hypothesis that the neutron–neutron interaction is stronger than the proton–proton interaction, fully in accord with the inequalities in (44) and (45), seems quite justified at the present time, even though there is no consensus on the value of the nn scattering length [13, 28, 33–36, 46–48]. In addition, we recall that, by definition, the coupling constant f_0^2 is the average of the coupling constants $f^2_{pp\pi^0}$ and $f^2_{nn\pi^0}$, while the coupling constant $f^2_{np\pi}$ is the average of the coupling constants f_0^2 and f_c^2, and this also leads to the respective inequalities in the chains of inequalities in (44) and (45).

Thus, it follows from Eqs. (24)–(35) and (40)–(43) that, in the model being considered, charge-independence breaking (CIB) is mandatory for the pion–nucleon coupling constant. Along with Eqs. (40)–(43), the difference of the charged and neutral coupling constants is frequently viewed as the degree of CIB for pion–nucleon coupling constants; that is,

$$\Delta f^2_{\rm CIB} \equiv f_c^2 - f_0^2 \quad (47)$$
$$= \frac{M_n}{M_p}\left(\frac{m^2_{\pi^\pm}}{m^2_{\pi^0}} - 1\right) f^2_{pp\pi^0} = 0.0052,$$

$$\Delta g^2_{\rm CIB} \equiv g_c^2 - g_0^2 \quad (48)$$
$$= \frac{M_n^2}{M_p^2}\left(\frac{m^2_{\pi^\pm}}{m^2_{\pi^0}} - 1\right) g^2_{pp\pi^0} = 0.94.$$

The relative degree of CIB for the pion–nucleon coupling constants is given by

$$\delta f^2_{\rm CIB} \equiv \frac{\Delta f^2_{\rm CIB}}{f_0^2} = \frac{m^2_{\pi^\pm}}{m^2_{\pi^0}} - 1 = 0.069, \quad (49)$$

$$\delta g^2_{\rm CIB} \equiv \frac{\Delta g^2_{\rm CIB}}{g_0^2} = \delta f^2_{\rm CIB} = 0.069. \quad (50)$$

In the model being considered, it is quite significant, amounting to about 7% in relative units. It should be emphasized that, in the model being considered, the relative degree of CIB for the pion–nucleon coupling constants is fully determined by the mass difference between the neutral and charged pions.

The difference of the neutron–neutron and proton–proton coupling constants measures the degree of charge-symmetry breaking (CSB) for the pion–nucleon coupling constants; that is,

$$\Delta f^2_{\rm CSB} \equiv f^2_{nn\pi^0} - f^2_{pp\pi^0} \quad (51)$$
$$= \left(\frac{M_n^2}{M_p^2} - 1\right) f^2_{pp\pi^0} = 0.00021,$$

$$\Delta g^2_{\rm CSB} \equiv g^2_{nn\pi^0} - g^2_{pp\pi^0} \quad (52)$$
$$= \left(\frac{M_n^4}{M_p^4} - 1\right) g^2_{pp\pi^0} = 0.075.$$

The relative degree of CSB for the pseudovector and pseudoscalar pion–nucleon coupling constants is given by

$$\delta f^2_{\text{CSB}} \equiv \frac{\Delta f^2_{\text{CSB}}}{f^2_{pp\pi^0}} = \frac{M_n^2}{M_p^2} - 1 = 0.0028, \tag{53}$$

$$\delta g^2_{\text{CSB}} \equiv \frac{\Delta g^2_{\text{CSB}}}{g^2_{pp\pi^0}} = \frac{M_n^4}{M_p^4} - 1 = 0.0055. \tag{54}$$

Thus, the relative amount of CSB for the pion–nucleon coupling constants is fully determined by the mass difference between the neutron and proton. As might have been expected, it is rather small, amounting to about 0.5%.

The ratios of the relative amount of CIB for pseudovector and pseudoscalar pion–nucleon coupling constants to the relative amount of CSB for them are given by

$$\frac{\delta f^2_{\text{CIB}}}{\delta f^2_{\text{CSB}}} = \frac{m^2_{\pi^\pm}/m^2_{\pi^0} - 1}{M_n^2/M_p^2 - 1} = 25.09, \tag{55}$$

$$\frac{\delta g^2_{\text{CIB}}}{\delta g^2_{\text{CSB}}} = \frac{m^2_{\pi^\pm}/m^2_{\pi^0} - 1}{M_n^4/M_p^4 - 1} = 12.53. \tag{56}$$

The ratios of their absolute values have the form

$$\frac{\Delta f^2_{\text{CIB}}}{\Delta f^2_{\text{CSB}}} = \frac{M_n}{M_p}\frac{\delta f^2_{\text{CIB}}}{\delta f^2_{\text{CSB}}} = 25.13, \tag{57}$$

$$\frac{\Delta g^2_{\text{CIB}}}{\Delta g^2_{\text{CSB}}} = \frac{M_n^2}{M_p^2}\frac{\delta g^2_{\text{CIB}}}{\delta g^2_{\text{CSB}}} = 12.56. \tag{58}$$

Thus, we see that, according to (55)–(58), the degree of CSB for the pseudovector pion–nucleon coupling constant is approximately 25 times less than the degree of CIB for this quantity, while the degree of CSB for the pseudoscalar pion–nucleon coupling constant is less than the degree of CIB for this coupling constant by a factor of about 12.

4. ON THE CHARGE DEPENDENCE OF THE NUCLEON–NUCLEON SCATTERING LENGTH

The low-energy parameters of effective-range theory for nucleon–nucleon scattering, which include the scattering length and the effective range, are fundamental physics features of the nucleon–nucleon interaction and nuclear forces in general [1, 2, 49–56]. On the basis of the influence on the change in these parameters and on the basis of their values, one estimates, among other things, the degree of CIB and CSB for nuclear forces [13, 28, 33–36], as well as the properties of various nucleon–nucleon potentials and other physics parameters and properties of the nucleon–nucleon system [1, 2, 53–55]. In doing this, it is precisely the nucleon–nucleon scattering length that usually turns out to be the most sensitive (and, simultaneously, the most characteristic) parameter with respect to moderately small variations in the nucleon–nucleon potential or in some other physics features of the system being considered.

In order to calculate or estimate the low-energy parameters of nucleon–nucleon scattering within the proposed model, we now consider a description of the nucleon–nucleon interaction in terms of the nucleon–nucleon potential following from meson field theory—namely, the Yukawa potential, which contains the pion–nucleon coupling constant as an input parameter. For the NN interaction in the $^1S_0$ spin-singlet state, the Yukawa potential has a simple form [1–3]; that is,

$$V_Y(r) = -V_0\frac{e^{-\mu r}}{\mu r}. \tag{59}$$

In Eq. (59), r is the distance between the two nucleons involved, while μ is related to the pion mass m_π by the equation

$$\mu = \frac{m_\pi c}{\hbar}, \tag{60}$$

where c is the speed of light and $\hbar$ is the reduced Planck constant. The nuclear-force range $R \equiv 1/\mu \sim 1.4$ fm is in inverse proportion to the pion mass and is small. The potential depth V_0 in (59) is related to the dimensionless pseudovector pion–nucleon coupling constant f_π by the following simple equation [1–3, 23, 24, 33]:

$$V_0 = m_\pi c^2 f_\pi^2. \tag{61}$$

Thus, the pion mass m_π and the pion–nucleon coupling constant f_π are basic features of the pion–nucleon interaction, which play an important role in studying NN and πN interactions [1–3, 13, 28].

Two interacting protons or two interacting neutrons exchange neutral pions, in which case the parameters of the Yukawa potential in (59), μ_{pp} and V_0^{pp} in the former and μ_{nn} and V_0^{nn} in the latter case, are determined, according to Eqs. (60) and (61), by the neutral-pion mass m_{π^0} and the coupling constants $f^2_{pp\pi^0}$ and $f^2_{nn\pi^0}$. In the case of interaction between a neutron and a proton, both neutral and charged pions are exchanged, in which case the parameters μ_{np} and V_0^{np} of the potential in (59) should be determined by employing [23–26] the charge-averaged pion mass

$$\bar{m}_\pi \equiv \frac{1}{3}\left(m_{\pi^0} + 2m_{\pi^\pm}\right) \tag{62}$$

and the averaged neutron–proton coupling constant $f^2_{np\pi}$ in (7).

For input model parameters, we will use the rather well-known low-energy parameters of proton–proton scattering in just the same way as we earlier employed and specified the proton–proton pion–nucleon coupling constant $f^2_{pp\pi^0}$. Further, we will determine the parameters μ_{pp} and V_0^{pp} of the proton–proton Yukawa potential on the basis of the experimental values of the proton–proton scattering length a_{pp} and the proton–proton effective range r_{pp}. In doing this, it is necessary to remove corrections induced by the electromagnetic interaction from the real experimental values of the nuclear–Coulomb low-energy parameters of proton–proton scattering. After the removal of these corrections, the purely nuclear proton–proton scattering length, a_{pp}, and effective range, r_{pp}, take the following values [28]:

$$a_{pp}^{\rm expt} = -17.3\,(4)\ \text{fm}, \tag{63}$$

$$r_{pp}^{\rm expt} = 2.85\,(4)\ \text{fm}. \tag{64}$$

Employing the variable-phase approach [57] and the values of the proton–proton scattering parameters in (63) and (64) for the case of proton–proton interaction, we obtain the following values for the parameters of the Yukawa potential in (59):

$$\mu_{pp} = 0.8392\ \text{fm}^{-1}, \tag{65}$$

$$V_0^{pp} = 44.8259\ \text{MeV}. \tag{66}$$

It should be noted that all of the ensuing calculations of the low-energy parameters of nucleon–nucleon scattering with the aid of the Yukawa potential are performed here on the basis of the variable-phase approach developed in [57].

In [23, 24], we showed that the neutron–proton parameters μ_{np} and V_0^{np} of the potential in (59) were related to the analogous parameters μ_{pp} and V_0^{pp} of the proton–proton interaction by the equations

$$\mu_{np} = \frac{\bar{m}_\pi}{m_{\pi^0}}\mu_{pp}, \tag{67}$$

$$V_0^{np} = \frac{\bar{m}_\pi}{m_{\pi^0}}\frac{f^2_{np\pi}}{f^2_{pp\pi^0}}V_0^{pp}. \tag{68}$$

Similarly, the neutron–neutron parameters μ_{nn} and V_0^{nn} of the potential in (59) are related to the parameters μ_{pp} and V_0^{pp} of the proton–proton interaction by the equations

$$\mu_{nn} = \mu_{pp}, \tag{69}$$

$$V_0^{nn} = \frac{f^2_{nn\pi^0}}{f^2_{pp\pi^0}}V_0^{pp}. \tag{70}$$

Within the model being considered, expressions (68) and (70) for the potential depths V_0^{np} and V_0^{nn} can be recast with allowance for Eqs. (14) and (17) into the form

$$V_0^{np} = \frac{1}{3}\frac{M_n}{M_p}\frac{\bar{m}_\pi}{m_{\pi^0}}\left(1 + 2\frac{m^2_{\pi^\pm}}{m^2_{\pi^0}}\right)V_0^{pp}, \tag{71}$$

$$V_0^{nn} = \frac{M_n^2}{M_p^2}V_0^{pp}. \tag{72}$$

Relying on Eqs. (67)–(72) and employing the values of the parameters of the Yukawa potential for proton–proton interaction in (65) and (66), as well as the values of the nucleon and pion masses in (25) and (26), we calculate the parameters μ and V_0 of the Yukawa potentials for neutron–proton and neutron–neutron interactions. The results are the following:

$$\mu_{np} = 0.8583\ \text{fm}^{-1}, \tag{73}$$

$$V_0^{np} = 48.0246\ \text{MeV}, \tag{74}$$

$$\mu_{nn} = 0.8392\ \text{fm}^{-1}, \tag{75}$$

$$V_0^{nn} = 44.9496\ \text{MeV}. \tag{76}$$

The singlet neutron–proton scattering length, a_{np}, and effective range, r_{np}, calculated in this way on the basis of the proposed model with the values in (73) and (74) obtained for the parameters of the potential in (59) are the following:

$$a_{np} = -23.4\,(4)\ \text{fm}, \tag{77}$$

$$r_{np} = 2.70\,(5)\ \text{fm}. \tag{78}$$

They are in good agreement with their experimental counterparts [50, 51, 55, 56]

$$a_{np}^{\rm expt} = -23.715\,(8)\ \text{fm}, \tag{79}$$

$$r_{np}^{\rm expt} = 2.71\,(7)\ \text{fm}. \tag{80}$$

For the low-energy neutron–neutron scattering parameters a_{nn} and r_{nn}, we similarly obtain the following values by employing the values in (75) and (76) for the parameters of the Yukawa potential:

$$a_{nn} = -18.2\,(4)\ \text{fm}, \tag{81}$$

$$r_{nn} = 2.84\,(5)\ \text{fm}. \tag{82}$$

As a result, the values in (81) and (82) that we calculated for a_{nn} and r_{nn} on the basis of the proposed model agree well, with allowance for the errors, with the respective experimental values

$$a_{nn}^{\rm expt} = -18.6\,(5)\ \text{fm}, \tag{83}$$

$$r_{nn}^{\text{expt}} = 2.83\,(11)\ \text{fm}, \tag{84}$$

which the authors of [58] found in the reaction $\pi^- + d \to \gamma + n + n$. The values in (81) and (82) found for the low-energy neutron–neutron scattering parameters are also in very good agreement with the values of $a_{nn} = -18.38\,(55)$ fm and $r_{nn} = 2.84\,(4)$ fm that we obtained in [47, 48] on the basis of an analysis of the binding-energy difference between the ^{3}H and ^{3}He mirror nuclei. Thus, the value in (81) obtained for the neutron–neutron scattering length on the basis of the model being considered is in good agreement with the averaged experimental value of this quantity, $a_{nn}^{\text{expt}} \simeq -18.5$ fm, but deviates, at the same time, from its different averaged experimental value of $a_{nn}^{\text{expt}} \simeq -16.5$ fm. As is well known, the values obtained in the past decades for the neutron–neutron scattering length a_{nn} lie around the aforementioned two experimental values of this quantity [13, 28, 46–48, 58–61], which are markedly different.

Because of the presence of a virtual level at an energy close to zero in the system of two nucleons in the $^1S_0$ state, the nucleon–nucleon scattering length in this state is the most sensitive parameter with respect to moderately small variations in the nucleon–nucleon potential. For this reason, CIB for nuclear forces in the nucleon–nucleon system is often quantitatively measured [28, 36] in terms of the difference of the average of the proton–proton and neutron–neutron scattering lengths and the neutron–proton scattering length; that is,

$$\Delta a_{\text{CIB}} \equiv \frac{1}{2}(a_{pp} + a_{nn}) - a_{np}. \tag{85}$$

According to (63), (79), and (83), the experimental value of this difference is

$$\Delta a_{\text{CIB}}^{\text{expt}} = 5.8\,(3)\ \text{fm}, \tag{86}$$

which is about 30% in relative units. The value in (86) is far beyond the experimental errors and is indicative of the breakdown of the hypothesis of charge independence of nuclear forces [28, 33–36]. As was indicated earlier, the charge dependence of nuclear forces is usually associated with the mass difference between the charged and neutral pions [12, 13, 28–36], but only about one-half of the experimental difference $\Delta a_{\text{CIB}}^{\text{expt}}$ was then explained by the mass difference between the $\pi^{\pm}$ and π^0 mesons [12, 32–35].

The experimental value of the proton–proton scattering length a_{pp} in (63) and the values that we calculated for the neutron–proton, a_{np} [see (77)], and neutron–neutron, a_{nn} [see (81)], scattering lengths on the basis of the proposed model lead to the following value of Δa_{CIB} within this model:

$$\Delta a_{\text{CIB}}^{\text{theor}} = 5.7\,(4)\ \text{fm}. \tag{87}$$

The theoretical value obtained in this way for Δa_{CIB} and presented in (87) agrees very well with the experimental value in (86). Thus, we have seen that, within the model being considered, CIB for nuclear forces is due almost completely to the mass difference between the charged and neutral pions. In this case, the scattering-length difference $\Delta a_{\text{CIB}}^{\text{theor}}$ is about 98% of the experimental value $\Delta a_{\text{CIB}}^{\text{expt}}$ in (86). In contrast to this, the theoretical value $\Delta a_{\text{CIB}}^{\text{theor}}$ obtained in earlier studies was about 50% of the experimental value $\Delta a_{\text{CIB}}^{\text{expt}}$, as was indicated above.

Usually, CSB for nuclear forces in nucleon–nucleon systems is quantitatively measured by the difference of the proton–proton and neutron–neutron scattering lengths [13, 28, 36]; that is,

$$\Delta a_{\text{CSB}} \equiv a_{pp} - a_{nn} = |a_{nn}| - |a_{pp}|\,. \tag{88}$$

According to (63) and (83), the experimental value of this difference is

$$\Delta a_{\text{CSB}}^{\text{expt}} = 1.3\,(6)\ \text{fm}, \tag{89}$$

which is about 7% in relative units. This deviation is beyond the experimental errors and is indicative of the breakdown of the hypothesis of charge symmetry of nuclear forces [28, 33–36].

For the difference of the proton–proton and neutron–neutron scattering lengths, the experimental value of the proton–proton scattering length a_{pp} in (63) and the value presented in (81) for the neutron–neutron scattering length a_{nn} and calculated in the present study on the basis of the proposed model lead to the value

$$\Delta a_{\text{CSB}}^{\text{theor}} = 0.9\,(4)\ \text{fm}, \tag{90}$$

which, within the errors, agrees with the experimental value in (89). Thus, CSB of about 0.5% for the pion–nucleon coupling constants (51)–(54), which, in the model being considered, is due to the mass difference between the neutron and proton, leads to quite significant CSB of about 5% for the singlet nucleon–nucleon scattering length. The latter is in fairly good agreement with experimental data.

5. BASIC CONCLUSIONS AND SUMMARY

In the present study, we have proposed a phenomenological model of charge-independence and charge-symmetry breaking for pion–nucleon coupling constants. In this model, the distinctions between the available four types of the elementary pseudovector pion–nucleon coupling constants are described by simple expressions in terms of the splitting proportional to the masses of nucleons and pions involved in the interaction process—namely, by expressions (10)–(13). The proposed model generalizes relation (8) considered earlier in [24] and based on the assumption that, in the approximation of exact charge symmetry, the charge splitting of the pion–nucleon coupling constant is equal to the charge splitting of the pion mass.

As a physics substantiation and a physics interpretation of this model, we can indicate the fact that the pion–nucleon coupling constants $f_{N\pi}$ are a measure of the strength of pion–nucleon interaction; therefore, the assumption that the higher the masses of the particles involved in the interaction process, the greater the strength in question would be reasonable. It follows that charge-independence and charge-symmetry breaking for the pion–nucleon coupling constant in the proposed model is directly related to the mass difference between interacting particles (nucleons and pions). Thus, CIB for the pion–nucleon coupling constant is completely explained by the mass difference between the charged and neutral pions and between the neutron and proton, CSB for the coupling constant being associated with the mass difference between the neutron and proton. We have found that the charged pion–nucleon coupling constants f_c^2 and g_c^2 exceed the neutral pion–nucleon coupling constants f_0^2 and g_0^2 by about 7%, which is indicative of a substantial breakdown of charge independence of nuclear forces in the pion–nucleon coupling constants ($f_c^2 > f_0^2$ and $g_c^2 > g_0^2$). In the case of CSB, the neutron–neutron coupling constant exceeds the proton–proton coupling constant by about 0.5%, which is indicative of CSB in the pion–nucleon coupling constant.

The calculations performed in the present study and the conclusions drawn on the basis of their results have revealed that the proposed model leads to a number of reasonable results and implications that agree well with experimental data. In particular, a relation between the different pion–nucleon coupling constants characterizing nucleon–nucleon interaction in the $^1S_0$ spin-singlet state has been derived within this model. By employing the experimental value of $f^2_{pp\pi^0} = 0.0749\,(7)$ reliably established for the neutral pion–nucleon coupling constant, which characterizes the proton–proton interaction, we have calculated the charged and neutral pion–nucleon coupling constants. The results are $f_c^2 = 0.0802\,(7)$ and $f_0^2 = 0.0750\,(7)$, respectively. We have also calculated the pion–nucleon coupling constants $f^2_{nn\pi^0} = 0.0751\,(7)$ and $f^2_{np\pi} = 0.0785\,(7)$, which characterize, respectively, the neutron–neutron and the neutron–proton interaction.

The value of $g_c^2 = 14.52\,(13)$ found within the proposed model for the charged pseudoscalar pion–nucleon coupling constant is in perfect agreement with the experimental value of $g_c^2 = 14.52\,(26)$ obtained by the Uppsala group for neutron studies [18] and is also in good agreement with a number of other values obtained for the charged pion–nucleon coupling constant [4–6, 8, 19, 20, 23, 24]. The value obtained for the ratio of the neutral pseudoscalar neutron, $g_{nn\pi^0}$, and proton, $g_{pp\pi^0}$ coupling constants, $g_{nn\pi^0}/g_{pp\pi^0} = 1.0028$, also agrees well with other two values found for this quantity on the basis of different models [44, 45].

The results that we obtained within the proposed model for the neutron–proton scattering length and effective range—$a_{np} = -23.4\,(4)$ fm and $r_{np} = 2.70\,(5)$ fm—and for the neutron–neutron scattering length and effective range—$a_{nn} = -18.2\,(4)$ fm and $r_{nn} = 2.84\,(5)$ fm—by employing the experimental low-energy parameters of proton–proton scattering agree within the errors with their experimental counterparts. Thus, we have seen that, within the proposed model, charge-independence breaking for nuclear forces is due almost completely to the mass difference between the charged and neutral pions, the scattering-length difference $\Delta a^{\text{theor}}_{\text{CIB}}$ being, in this case, about 98% of the experimental value $\Delta a^{\text{expt}}_{\text{CIB}}$. By and large, the proposed model leads to a number of results that agree well with experimental data. This concerns both the various forms of the pion–nucleon coupling constants and the calculated low-energy parameters of nucleon–nucleon scattering.

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