ISSN 1063-7788, Physics of Atomic Nuclei, 2017, Vol. 80, No. 2, pp. 383–388. © Pleiades Publishing, Ltd., 2017.

ELEMENTARY PARTICLES AND FIELDS
Theory

Two-center Coulomb Problem with Calogero Interaction*

T. Hakobyan** and **A. Nersessian*****

Yerevan State University, Armenia
Tomsk Polytechnic University, Russia

Received May 18, 2016

Abstract—We show that the Calogero-type perturbation preserves the integrability and partial separation of variables for the Stark–Coulomb and two-center Coulomb problems, and present the explicit expressions of their constants of motion. We reveal that this perturbation preserves the spectra of initial systems, but leads to the change of degree of degeneracy.

DOI: 10.1134/S106377881702020X

1. INTRODUCTION

The Coulomb problem is maximally superintegrable due to the conservation of the Runge–Lenz vector. The Hamiltonian admits separation of variables in several coordinate systems. Any such coordinate system possesses its own integrable perturbations. Among them there are the Coulomb problem in constant electric field (Coulomb–Stark problem) and the two-center Coulomb problem. They admit a separation of variables, respectively, in parabolic and elliptic coordinates. Note that both systems are not exactly solvable. In the Coulomb–Stark problem, one can get analytically only the perturbative spectrum, while in the two-center Coulomb system, the energy spectrum can be constructed only numerically, except for some special cases [1]. Nevertheless, the separation of variables is crucial in their study.

The well-known rational Calogero model [2], which describes one-dimensional particles, interacting with inverse-square potential,

$$\mathcal{H}_0 = \sum_i \frac{p_i^2}{2} + \sum_{i<j}^{N} \frac{g(g-1)}{(x_i - x_j)^2}, \qquad (1)$$

is another example of maximally superintegrable system [3]. It possesses higher-order (in momenta) integrals of motion, which had been constructed by the Lax pair [4]. The inverse-square potential in one dimension possesses various integrable generalizations [see [5, 6] for the review], which have many applications in physics and mathematics.

The mixture of the Coulomb and Calogero potentials gives rise to a more general integrable N-dimensional system [7]. Recently we have shown together with Olaf Lechtenfeld that the Calogero–Coulomb system is also superintegrable [8]. This property can be understood in the action-angle language. An explicit form of the complete set of constants of motion can be derived by taking proper deformations of the corresponding integrals of the underlying Coulomb system, then forming the symmetric polynomials on them [8, 9]. This method differs from the standard construction [4], so that the deformations of the Liouville integrals do not commute any more. Nevertheless, the functional independence of the constricted integrals of motion is preserved.

In this report we consider in this context the N-dimensional Coulomb–Stark and two-center Coulomb problems with the additional Calogero potential (we will refer them as Calogero–Coulomb–Stark and two-center Calogero–Coulomb problems). Both systems have a highlighted direction, along which the full rotational symmetry of the initial one-center Coulomb problem is broken down to the $SO(N-1)$ symmetry. It is defined, respectively, by the external filed direction and by the line connecting two Coulomb charges. We show that under the proper choice of this highlighted direction, both systems still remain integrable and admit partial separation of variables. In fact, the Schrödinger equation decouples into three parts, only one of which depends on the inverse-square interaction term. The latter can be treated as a deformation of the Schrödinger equation for the $SO(N-1)$ angular momentum, usually referred as an angular Calogero Hamiltonian [10–14].

The report is based on the recent publications in [9, 15].

*The text was submitted by the authors in English.

**E-mail: tigran.hakobyan@ysu.am
***E-mail: arnerses@ysu.am

2. CALOGERO–COULOMB PROBLEM

The Calogero–Coulomb problem is a mixture of the N-particle rational Calogero model (1) and of the N-dimensional Coulomb system [7]:

$$\mathcal{H}_\gamma = \frac{\mathbf{p}^2}{2} + \sum_{i<j}^{N} \frac{g(g-1)}{(x_i - x_j)^2} - \frac{\gamma}{r}. \qquad (2)$$

It inherits most of the properties of the original Coulomb system and possesses hidden symmetries given by an analog of Runge–Lenz vector [8, 9]. It is convenient to describe this system by means of the Dunkl operators which make transparent the analogy with the initial Coulomb problem. Let us consider instead the extended Hamiltonian in this regard,

$$\mathcal{H}^{\rm gen}_\gamma = \frac{\boldsymbol{\pi}^2}{2} - \frac{\gamma}{r} = \frac{\mathbf{p}^2}{2} + \sum_{i<j} \frac{g(g-s_{ij})}{(x_i - x_j)^2} - \frac{\gamma}{r}. \quad (3)$$

The modified momentum is expressed in terms of the Dunkl operators by

$$\boldsymbol{\pi} = -i\boldsymbol{\nabla}, \quad \nabla_i = \partial_i - \sum_{j\neq i} \frac{g}{x_i - x_j} s_{ij}. \qquad (4)$$

The operator s_{ij} permutes the ith and jth coordinates. On the symmetric wavefunctions the generalized Hamiltonian $\mathcal{H}^{\rm gen}_\gamma$ reduces to the Calogero–Coulomb Hamiltonian (2).

The Dunkl operators commute mutually like ordinary partial derivatives. However, their commutations with coordinates are nontrivial deformations of the Heisenberg algebra relations [16],

$$[\pi_i, x_j] = -iS_{ij}. \qquad (5)$$

Here the operators S_{ij} for $i \neq j$ are just rescaled permutations: $S_{ij} = -gs_{ij}$, and the S_{ii} are defined by the relation $\sum_j S_{ij} = 1$.

Let us define the deformed angular momentum operator via the Dunkl momentum [10, 17]:

$$L_{ij} = x_i\pi_j - x_j\pi_i. \qquad (6)$$

It satisfies the deformed angular momentum commutation relations [10]. The generalized Calogero–Coulomb Hamiltonian preserves it [8].

The deformed Runge–Lenz vector is given by the expression [9]

$$A_i = \frac{1}{2}\sum_j \{L_{ij}, \pi_j\} + \frac{i}{2}[\pi_i, S] - \frac{\gamma x_i}{r}. \qquad (7)$$

It contains the permutation-group invariant element which vanishes in the absence of the Calogero term

$$S = \sum_{i<j} S_{ij}. \qquad (8)$$

The generalized Hamiltonian preserves the deformed Runge–Lenz vector too [9].

In the absence of the Calogero interaction ($g = 0$), the integrals of the generalized Calogeo–Coulomb Hamiltonian (3) are reduced to the standard integrals of the N-dimensional Coulomb problem.

The Calogero–Coulomb problem can be obtained by the restriction of the extended Hamiltonian (3) to the symmetric wavefunctions. Therefore, its constants of motion can be constructed by taking the symmetric polynomials on the components of the Dunkl angular momentum and Runge–Lenz vector [8, 9]:

$$\mathcal{L}_{2k} = \sum_{i<j} L_{ij}^{2k}, \qquad (9)$$

$$\mathcal{A}_k = \sum_i A_i^k. \qquad (10)$$

The expressions above demonstrate that the Calogero–Coulomb problem is a superintegrable system, like the pure Calogero [3] and Coulomb models.

Note that the square of Dunkl angular momentum is related to the angular part $\mathcal{I}$ of the Calogero Hamiltonian [10]:

$$\mathcal{L}_2 = 2\mathcal{I} + S(S - N + 2). \qquad (11)$$

In two dimension the symmetries of the Calogero–Coulomb system, based on the dihedral group D_2, have been studied also in [18].

3. COULOMB–CALOGERO–STARK PROBLEM

Consider the N-dimensional Coulomb problem in constant electric field F in the presence of the Calogero interaction:

$$\mathcal{H}_{\gamma,F} = \frac{\mathbf{p}^2}{2} + \sum_{i<j}^{N} \frac{g(g-1)}{(x_i - x_j)^2} - \frac{\gamma}{r} + Fx_0, \qquad (12)$$

where x_0 is the normalized center-of-mass coordinate (see Eq. (15) below). The external field is aligned in the direction $(1, 1, \ldots, 1)$, which ensures the permutation invariance of the Hamiltonian. In the absence of the external field, this model is reduced to the Calogero–Coulomb model, considered above.

The generalized Hamiltonian is defined in terms of the Dunkl momentum (4) as follows:

$$\mathcal{H}^{\rm gen}_{\gamma,F} = \frac{\boldsymbol{\pi}^2}{2} - \frac{\gamma}{r} + Fx_0. \qquad (13)$$

First, consider its invariants.

The entire Dunk angular momentum tensor (6) is not an integral of motion any more. Instead, its

components, which are orthogonal to the external field, are preserved,

$$L_{ij}^{\perp} = L_{ij} + \frac{1}{N}\sum_k (L_{jk} - L_{ik}). \qquad (14)$$

Alternatively, one can express them in terms of the Jacobi coordinates, which separate the center-of-mass from the relative motion. They are defined by the orthogonal map [12, 19]

$$x_0 = \frac{1}{\sqrt{N}}(x_1 + \ldots + x_N),$$
$$\tilde{x}_k = \frac{1}{\sqrt{k(k+1)}}(x_1 + \ldots + x_k - kx_{k+1}), \qquad (15)$$

where $1 \le k \le N-1$. The first coordinate describes the center of mass, while the others, marked by tilde, characterize the relative motion.

Denote now by $\tilde{L}_{ij}$ the components of the deformed *relative* angular momentum, rotated by the Jacobi transformation. The algebra generated by $L_{ij}^{\perp}$, in fact, coincides with the $\tilde{L}_{ij}$, which are responsible for the relative motion ($1 \le i, j \le N-1$). In the absence of Calogero interaction, they form the $SO(N-1)$ subalgebra, which describes the rotations in the hyperspace, orthogonal to the center-of-mass direction.

Apart from the deformed relative angular momenta, the modified component of the Runge–Lenz vector (7) along the field direction, defined by

$$A = x_0\Big(2\mathcal{H}_{\gamma,F}^{\text{gen}} + \frac{\gamma}{r}\Big) - \Big(rp_r + \frac{N-1}{2i}\Big)p_0 - \frac{F}{2}\left(r^2 + 3x_0^2\right) \qquad (16)$$

is preserved too.

This invariant commutes with the deformed relative angular momentum. In the $g = 0$ limit, one can extract from these symmetry generators the standard Liouville integrals of the Coulomb–Stark system. The $N-2$ integrals can be chosen to be the quadratic Casimir elements of the naturally embedded algebras $SO(2) \subset \ldots \subset SO(N-1)$. They are described in the relative angular coordinates and momenta. The last two integrals are given by the Hamiltonian and the modified component of the Runge–Lenz vector, which had been constructed for $N = 3$ in [20].

Out of the $g = 0$ point, we deal with the deformed quantities, and the Liouville property cannot be extended straightforwardly. Nevertheless, in the presence of a constant uniform electric field, the generalized Calogero–Coulomb model (13) still remains an integrable system.

The integrals of the pure Calogero–Coulomb system (12) obtained by the restriction to the symmetric wavefunctions, must be symmetric too. Since the longitudenal component of the Runge–Lenz vector (16) obeys this condition, it remains as a correct integral for this system,

$$[A, \mathcal{H}_{\gamma,F}] = 0. \qquad (17)$$

We should take symmetric expressions of the kinematical constants of motion too, as in the absence of the electric field [9]. For this purpose it is more suitable to use the angular momentum in Jacobi coordinates:

$$[\mathcal{H}_{\gamma,F}, \tilde{\mathcal{L}}_{2k}] = 0, \quad \tilde{\mathcal{L}}_{2k} = \sum_{1\le i<j\le N-1} \tilde{L}_{ij}^{2k}. \qquad (18)$$

The first member of this family is the square of the relative Dunkl angular momentum. It is related to the angular part of the Calogero model with reduced center of mass $\tilde{\mathcal{I}}$, which we call the relative angular Calogero Hamiltonian, by the same formula as Eq. (11) above. So, we have proved the integrability of the Calogero–Coulomb–Stark system.

Separation of Variables in Parabolic Coordinates

It is well known that the Coulomb–Stark system admits separation of variables in parabolic coordinates. It appears that the Calogero–Coulomb–Stark system admits complete separation of variables in parabolic coordinates for $N = 2, 3$ and partial separation for $N > 3$ [15].

In the Jacobi coordinates (15), the last system acquires the following form:

$$\mathcal{H}_{\gamma,F} = \frac{p_0^2}{2} - \frac{\gamma}{\sqrt{x_0^2 + \tilde{x}^2}} + Fx_0 + \tilde{\mathcal{H}}_0, \qquad (19)$$

where the last term is the Calogero Hamiltonian (1) with reduced center of mass. We pass to the parabolic coordinates $(\xi, \eta, \varphi_\iota)$, where φ_ι are the relative angular variables, and

$$\xi = r + x_0, \quad \eta = r - x_0. \qquad (20)$$

In new coordinates the Hamiltonian (19) is expressed as follows:

$$\mathcal{H}_{\gamma,F} = -\frac{2}{\xi+\eta}(\gamma + B_\xi + B_\eta) + \frac{\tilde{\mathcal{I}}}{\xi\eta} + \frac{F}{2}(\xi - \eta), \qquad (21)$$

where we have shorten the kinetic term using the notation

$$B_\xi = \frac{1}{\xi^{\frac{N-3}{2}}}\frac{\partial}{\partial\xi}\xi^{\frac{N-1}{2}}\frac{\partial}{\partial\xi}. \qquad (22)$$

Further we proceed by extending straightforwardly the steps, applied for the usual Coulomb system in external field in [21]. Employing the following ansatz to the total wavefunction

$$\Psi(\xi,\eta,\varphi_\iota)=\Phi_1(\xi)\Phi_2(\eta)\psi(\varphi_\iota), \tag{23}$$

we decouple Schrödinger equation $\mathcal{H}_{\gamma,F}\Psi=E\Psi$ into three parts. The two of them depend, respectively, on ξ and η,

$$\left(B_\xi+\frac{E}{2}\xi-\frac{F}{4}\xi^2-\frac{\tilde q(\tilde q+N-3)}{4\xi}+\lambda_1\right)\times\Phi_1(\xi)=0, \tag{24a}$$

$$\left(B_\eta+\frac{E}{2}\eta+\frac{F}{4}\eta^2-\frac{\tilde q(\tilde q+N-3)}{4\eta}+\lambda_2\right)\times\Phi_2(\eta)=0. \tag{24b}$$

Here the Coulomb charge is fractioned into two parts,

$$\lambda_1+\lambda_2=\gamma. \tag{24c}$$

The last equation describes the spectrum and eigenstates of the relative angular Calogero model [11]:

$$\tilde{\mathcal{I}}(\varphi_\iota,\partial_{\varphi_\iota})\psi_{\tilde q}(\varphi_\iota)=\frac{\tilde q(\tilde q+N-3)}{2}\psi_{\tilde q}(\varphi_\iota). \tag{24d}$$

In particular, the spectrum is determined by the numbers

$$\tilde q=\frac{gN(N-1)}{2}+3l_3+\ldots+Nl_N \quad \text{with} \quad l_i=0,1,2,\ldots. \tag{25}$$

For integer values of the coupling g, the angular energy spectrum is that of a free particle with angular momentum $\tilde q$ on the $(N-2)$-dimensional sphere, but has a significantly lower degeneracy due to the restriction to the symmetric wavefunctions [11, 13].

The longitudinal component of the Runge–Lenz vector (16) separates the two equations (24a) and (24b):

$$A\Psi=(\lambda_2-\lambda_1)\Psi. \tag{26}$$

The second invariant, given by the relative angular Hamiltonian $\tilde{\mathcal{I}}$, is common in both cases and separates the relative angular degrees of freedom.

In the absence of the electric field, $F=0$, the fractional Coulomb charges λ_i with $i=1,2$ take discrete values depending on the parabolic quantum numbers $n_i=0,1,2,\ldots$

$$\lambda_i=\gamma\frac{n_i+\frac12(\tilde q+1)}{n+\frac12(N-3)}. \tag{27}$$

They specify also the partial wavefunctions $\Phi_1(\xi)=\Phi_{n_1\tilde q}(\xi)$ and $\Phi_2(\eta)=\Phi_{n_2\tilde q}(\eta)$. The principal quantum number, which characterizes the spectrum

$$E_n=-\frac{\gamma^2}{2\left(n+\frac{N-3}{2}\right)^2},\qquad n=n_r+l+1,\quad n_r=0,1,2,\ldots, \tag{28}$$

is expressed now via parabolic quantum numbers,

$$n=n_1+n_2+\tilde q+1. \tag{29}$$

As in the usual Coulomb problem [22], the electric field completely removes the degeneracy in the orbital momentum, but preserves the degeneracy with respect to q.

4. TWO-CENTER CALOGERO–COULOMB SYSTEM

Consider now the integrable two-center Coulomb system in the presence of the inverse-square Calogero potential. In order to construct the Hamiltonian of this system, we should replace, as in the previous sections, the momentum operators by the Dunkl momenta, and then restrict the Hamiltonian to the symmetric wavefunction. In order to assure the permutation symmetry, we align the axis, connecting two Coulomb charges, along the center-of-mass coordinate. In the Jacobi coordinates (15), the distances to the charges are given by

$$r_1=\sqrt{\tilde x^2+(x_0-a)^2},\qquad r_2=\sqrt{\tilde x^2+(x_0+a)^2}. \tag{30}$$

The generalized two-center Calogero–Coulomb Hamiltonian is

$$\mathcal{H}^{\rm gen}_{\gamma_1,\gamma_2}=\frac{\boldsymbol{\pi}^2}{2}-\frac{\gamma_1}{r_1}-\frac{\gamma_2}{r_2}. \tag{31}$$

On the symmetric wavefunctions, it produces the two-center Calogero–Coulomb system,

$$\mathcal{H}_{\gamma_1,\gamma_2}=\frac{\mathbf{p}^2}{2}+\sum_{i<j}^{N}\frac{g(g-1)}{(x_i-x_j)^2}-\frac{\gamma_1}{r_1}-\frac{\gamma_2}{r_2}. \tag{32}$$

Like the Calogero–Coulomb–Stark Hamiltonian, it possesses the symmetry given by the deformed angular momentum generators perpendicular to the predefined direction (14).

The modified Runge–Lenz integral of the $g=0$ Hamiltonian has been constructed in [23, 24]. The construction can be extended to the case of nonzero coupling values by [15]

$$A=\mathcal{L}_2+a^2p_0^2-2ax_0\left(\frac{\gamma_1}{r_1}-\frac{\gamma_2}{r_2}\right), \tag{33}$$

where $\mathcal{L}_2$ is the Dunkl angular momentum square, defined in Eqs. (9).

Separation of Variables in Elliptic Coordinates

Now, let us show that in complete analogy with the previous case, the two center Calogero–Coulomb system (32) admits complete separation of variables in the elliptic coordinates for $N = 2, 3$ and the partial separation for $N > 3$.

The map from the Jacobi variables (x_0, y) to the elliptic coordinates (ξ, η) looks as follows (similar to the usual hydrogen atom case [25]):

$$\xi = \frac{r_1 + r_2}{2a}, \quad \eta = \frac{r_1 - r_2}{2a}, \tag{34}$$

where r_i is the distance from the i-th Coulomb charge (30). The relative angles φ_i remain unchanged. The new coordinates belong to the regions $\xi \geq 1$ and $-1 \leq \eta \leq 1$. The two-center Calogero–Coulomb Hamiltonian (32) in elliptic coordinates reads

$$\begin{aligned}\mathcal{H}_{\gamma_1,\gamma_2} = &\frac{1}{2a^2(\xi^2-\eta^2)}(B_\eta - B_\xi) \\ &+ \frac{\tilde{\mathcal{I}}(\varphi_\iota, \partial_{\varphi_\iota})}{a^2(\xi^2-1)(1-\eta^2)} \\ &- \frac{\gamma_1}{a(\xi+\eta)} - \frac{\gamma_2}{a(\xi-\eta)},\end{aligned} \tag{35}$$

where the operator B_ξ from the kinetic energy part acquires the following form:

$$B_\xi = \frac{1}{(\xi^2-1)^{\frac{N-3}{2}}}\frac{\partial}{\partial\xi}(\xi^2-1)^{\frac{N-1}{2}}\frac{\partial}{\partial\xi}. \tag{36}$$

Then, choosing the wavefunction

$$\Psi(\xi, \eta, \varphi_\iota) = \Phi_1(\xi)\Phi_2(\eta)\psi(\varphi_\iota), \tag{37}$$

we can separate the variables in the Schrödinger equation into the three parts. The first two equations are

$$\begin{aligned}\Big(B_\xi - \frac{\tilde{q}(\tilde{q}+N-3)}{\xi^2-1} + 2a(\gamma_1+\gamma_2)\xi \\ + 2a^2E\xi^2 - \lambda\Big)\Phi_1(\xi) = 0,\end{aligned} \tag{38a}$$

$$\begin{aligned}\Big(B_\eta - \frac{\tilde{q}(\tilde{q}+N-3)}{\eta^2-1} + 2a(\gamma_1-\gamma_2)\eta \\ + 2a^2E\eta^2 - \lambda\Big)\Phi_2(\eta) = 0.\end{aligned} \tag{38b}$$

The third equation is inherited from the Stark case (24d). It describes the energy eigenstates of the relative angular Calogero Hamiltonian and its spectrum, depending on the composite quantum number $\tilde{q}$ (25). In the absence of the Calogero term, it determines the spectrum and energy states of a free particle system on $(N-2)$-dimensional sphere.

Obviously, the partial states $\Phi_{1,2}$ in the first two equations depend on the energy level E and the $\tilde{q}$. The parameter λ in the first two equations separates the variables ξ and η. It coincides with the eigenvalue of the slightly redefined Runge–Lenz invariant for the two center Calogero–Coulomb system (33) with the Dunkl angular momentum square replaced by the doubled angular Calogero Hamiltonian,

$$A = 2\mathcal{I} + a^2p_0^2 - 2ax_0\left(\frac{\gamma_1}{r_1} - \frac{\gamma_2}{r_2}\right). \tag{39}$$

We follow the steps done above for the parabolic case. First, use the total wavefunction Ψ instead of the partial ones, $\Phi_{1,2}$, in Eqs. (38a) and (38b). Next, cancel out the energy E by taking appropriate combinations of both equations. This yields the eigenstate equation for the modified Runge–Lenz invariant: $A\Psi = \lambda\Psi$.

ACKNOWLEDGMENTS

This work was partially supported by the Armenian State Committee of Science Grants 15RF-039 and 15T-1C367. It was done within programs of ICTP Network NET68 and of the Regional Training Network on Theoretical Physics sponsored by Volkswagenstiftung under Contract no. 86 260.

REFERENCES

1. I. V. Komarov and S. Yu. Slavyanov, J. Phys. B **1**, 1066 (1968); Yu. Demkov and I. Komarov, Theor. Math. Phys. **38**, 174 (1979).
2. F. Calogero, J. Math. Phys. **10**, 2191 (1969); J. Math. Phys. **12**, 419 (1971).
3. S. Wojciechowski, Phys. Lett. A **95**, 279 (1983).
4. J. Moser, Adv. Math. **16**, 197 (1975).
5. M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. **71**, 313 (1981); Phys. Rep. **94**, 313 (1983).
6. A. P. Polychronakos, J. Phys. A **39**, 12793 (2006); hep-th/0607033.
7. A. Khare, J. Phys. A **29**, L45 (1996).
8. T. Hakobyan, O. Lechtenfeld, and A. Nersessian, Phys. Rev. D **90**, 101701(R) (2014); arXiv:1409.8288.
9. T. Hakobyan and A. Nersessian, Phys. Rev. A **92**, 022111 (2015); arXiv:1504.00760.
10. M. Feigin and T. Hakobyan, J. High Energy Phys. **1511**, 107 (2015); arXiv:1409.2480.
11. M. Feigin, O. Lechtenfeld, and A. Polychronakos, J. High Energy Phys. **1307**, 162 (2013), arXiv:1305.5841.
12. T. Hakobyan, A. Nersessian, and V. Yeghikyan, J. Phys. A **42**, 205206 (2009), arXiv:0808.0430.
13. F. Correa and O. Lechtenfeld, J. High Energy Phys. **10**, 191 (2015) arXiv:1508.04925.

14. T. Hakobyan, D. Karakhanyan, and O. Lechtenfeld, Nucl. Phys. B **886**, 399 (2014); arXiv:1402.2288; T. Hakobyan, S. Krivonos, O. Lechtenfeld, and A. Nersessian, Phys. Lett. A **374**, 801 (2010); arXiv:908.3290; T. Hakobyan, O. Lechtenfeld, A. Nersessian, and A. Saghatelian, J. Phys. A **44**, 055205 (2011); arXiv:1008.2912; O. Lechtenfeld, A. Nersessian, and V. Yeghikyan, Phys. Lett. A **374**, 4647 (2010); arXiv:1005.0464.
15. T. Hakobyan and A. Nersessian, Phys. Rev. D **93**, 045025 (2016); arXiv:1509.01077.
16. C. F. Dunkl, Trans. Am. Math. Soc. **311**, 167 (1989).
17. M. V. Feigin, Theor. Math. Phys. **135**, 497 (2003).
18. V. X. Genest, A. Lapointe, and L. Vinet, Phys. Lett. A **379**, 923 (2015); arXiv:1405.5742.
19. T. Hakobyan, O. Lechtenfeld, and A. Nersessian, Nucl. Phys. B **858**, 250 (2012); arXiv:1110.5352.
20. P. J. Redmond, Phys. Rev. B **133**, 1352 (1964).
21. K. Helfrich, Theor. Chim. Acta **24**, 271 (1972).
22. L. Landau and E. Lifshitz, *Course of Theoretical Physics,* Vol. 3: *Quantum Mechanics* (Pergamon, Oxford, 1977; Nauka, Moscow, 1989).
23. C. Coulson and A. Joseph, Int. J. Quantum Chem. **1**, 337 (1967).
24. H. A. Erikson and E. L. Hill, Phys. Rev. **76**, 29 (1949).
25. S. P. Alliluev and A. V. Matveenko, Sov. Phys. JETP **24**, 1260 (1967).