ISSN 1063-7788, Physics of Atomic Nuclei, 2017, Vol. 80, No. 2, pp. 321–328. © Pleiades Publishing, Ltd., 2017.

ELEMENTARY PARTICLES AND FIELDS
Theory

Symmetry-Preserving Perturbations of the Bateman Lagrangian and Dissipative Systems*

Rutwig Campoamor-Stursberg**

Instituto de Matemática Interdisciplinar and Departamento Geometría y Topología, Faculted de Ciencias Matematicas Universidad Complutense, Madrid, Spain

Received May 18, 2016

Abstract—Perturbations of the classical Bateman Lagrangian preserving a certain subalgebra of Noether symmetries are studied, and conservative perturbations are characterized by the Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(2)$. Non-conservative albeit integrable perturbations are determined by the simple Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, showing further the relation of the corresponding non-linear systems with the notion of generalized Ermakov systems.

DOI: 10.1134/S1063778817020107

1. INTRODUCTION

Within the frame of classical dynamics, dissipative forces are understood as those types of interactions for which energy is lost during motion. Assuming that the consequence of such interactions is the energy transfer from the dissipative part of a system to the heath bath, successful models to describe various types of phenomena have been developed, albeit the quantization and its interpretation, due to the well-known ambiguities of the Lagrangian (Hamiltonian) formalism, can give rise to inconsistencies between the canonical commutation relations and the equations of motion (see [1–4] and references therein).

Dissipative systems of various types have been considered by different authors by means of the Lagrangian formulation, and first integrals have been obtained by using either the classical Noether theorem or some of its generalizations [5–10]. One of the principal difficulties arises from the correct identification of a Lagrangian or Hamiltonian that displays correctly the physical properties of the system. In this context, although mathematical Lagrangians or Hamiltonians have been shown to be useful, they must be handled with care in order to avoid ambiguities and misleading interpretations within the so-called canonical formalism [11, 12]. A standard procedure to circumvent the difficulties arising from such phenomenological approaches consists in coupling the dissipative system to an environment with additional degrees of freedom, in order that the system-plus-reservoir is a Hamiltonian system [13]. One of the first examples to be analyzed from this perspective was the damped harmonic oscillator, completed with one additional degree of freedom by Bateman in [14]. The added “dual" equation is a time-reversed version of the oscillator, and the corresponding variable fulfills the absorption of the energy dissipated by the damped oscillator [14–16].

In this work we reconsider the classical Bateman system from the perspective of Noether symmetries. Using that the system is linearizable as a system of second-order ordinary differential equations, we analyze the non-linear perturbations of the Bateman Lagrangian that preserve a certain subalgebra of Noether symmetries, and determine under what conditions these perturbations have the additional property that the corresponding Hamiltonian is a constant of the motion. It is shown that such perturbations are non-linear and possess at most four independent Noether symmetries. It is further observed that for the limit in the friction constant $K \to 0$, the symmetry-preserving perturbations of the Bateman system correspond to generalized Ermakov systems associated to the two-dimensional harmonic oscillator $\ddot{\mathbf{q}} + \omega^2 \mathbf{q} = 0$. Generalized Ermakov systems in the sense of Ray and Reid and related to the time-dependent oscillator are obtained as the limit of symmetry-preserving perturbations of a dissipative generalization of the Bateman system.

2. PERTURBATION OF LAGRANGIANS PRESERVING SYMMETRY

Let $L(t, \mathbf{q}, \dot{\mathbf{q}})$ be a regular Lagrangian in n dimensions and let

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \quad 1 \le i \le n \qquad (1)$$

*The text was submitted by the author in English.

**E-mail: rutwig@ucm.es

be the corresponding Euler–Lagrange equations. By the regularity assumption we can always rewrite the equations of motion in normal form, i.e.

$$\ddot{q}^i = \omega_i\left(t, \dot{q}^j, q^j\right) = g^{ij}\left(\frac{\partial L}{\partial q^j} - \frac{\partial^2 L}{\partial t \partial \dot{q}^j} - \frac{\partial^2 L}{\partial \dot{q}^j \partial q^k}\dot{q}^k\right), \quad 1 \leq i \leq n, \tag{2}$$

where g^{ij} denotes the inverse Hessian matrix of L. Starting from the normal form, it constitutes a standard result in the symmetry analysis of differential equations to verify that the system (2) can be reformulated in equivalent form as the first order partial differential equation

$$\mathbf{A}f = \left(\frac{\partial}{\partial t} + \dot{q}^i\frac{\partial}{\partial q^i} + \omega_i\frac{\partial}{\partial \dot{q}^i}\right)f = 0. \tag{3}$$

In this context, a constant of the motion of the system (2) is defined as a function $F\left(t, \mathbf{q}, \dot{\mathbf{q}}\right)$ satisfying the constraint

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \dot{q}^i\frac{\partial F}{\partial q^i} + \ddot{q}^i\frac{\partial F}{\partial \dot{q}^i} = \mathbf{A}\left(F\right) = 0. \tag{4}$$

Constants of the motion can have a different origin depending on the type of symmetries considered [17]. The usual symmetry types considered in the context of Classical Mechanics are point, Noether and pure dynamical symmetries [18]. In the following we will only consider the case of Noether symmetries and the conservation laws associated to them.

Recall that a vector field

$$\mathbf{X} = \xi\left(t, \mathbf{q}\right)\frac{\partial}{\partial t} + \eta^j\left(t, \mathbf{q}\right)\frac{\partial}{\partial q^j} \tag{5}$$

is called a Noether symmetry of (2) if it satisfies the condition

$$\dot{\mathbf{X}}\left(L\right) + \mathbf{A}\left(\xi\right)L - \mathbf{A}\left(V\right) = 0, \tag{6}$$

where $\dot{\mathbf{X}} = \mathbf{X} + \dot{\eta}^j\left(t, \mathbf{q}, \dot{\mathbf{q}}\right)\frac{\partial}{\partial \dot{q}^j}$ with $\dot{\eta}^j = \frac{d\eta^j}{dt} - \dot{q}^j\frac{dx^j}{dt}$ denotes the first prolongation of the vector field $\mathbf{X}$ and $V\left(t, \mathbf{q}\right)$ does not depend on the velocities [19]. Expanding the symmetry condition (6) provides the following partial differential equation

$$\xi\left(t, \mathbf{q}\right)\frac{\partial L}{\partial t} + \eta^j\left(t, \mathbf{q}\right)\frac{\partial L}{\partial q^j} + \dot{\eta}^j\left(t, \mathbf{q}, \dot{\mathbf{q}}\right)\frac{\partial L}{\partial \dot{q}^j} + \frac{d\xi}{dt}L\left(t, \mathbf{q}, \dot{\mathbf{q}}\right) - \frac{\partial V}{\partial t} - \dot{q}^j\frac{\partial V}{\partial q^j} = 0. \tag{7}$$

The classical Noether theorem (see e.g. [18]) states that to any symmetry (5) corresponds a constant of the motion given by

$$J = \xi\left(\dot{q}^k\frac{\partial L}{\partial \dot{q}^k} - L\right) - \eta^k\frac{\partial L}{\partial \dot{q}^k} + V\left(t, \mathbf{q}\right). \tag{8}$$

We observe that a constant of the motion J in particular constitutes an invariant of the prolongation $\dot{\mathbf{X}}$.

2.1. Perturbations Preserving Symmetries

Supposed that a system described by a Lagrangian $L\left(t, \mathbf{q}, \dot{\mathbf{q}}\right)$ possesses an r-dimensional Lie algebra $\mathcal{L}_{\mathrm{NS}}$ of Noether symmetries, for any subalgebra $\mathcal{L}_0 < \mathcal{L}_{\mathrm{NS}}$ of dimension $r_0 < r$ we can ask whether the Lagrangian L can be perturbed to a Lagrangian $\widehat{L} = L + \varepsilon S\left(t, \mathbf{q}, \dot{\mathbf{q}}\right)$ in such manner that the symmetry generators $\mathbf{X}_j$ $(1 \leq j \leq r_0)$ of the subalgebra $\mathcal{L}_0$ are also Noether symmetries of the system associated to $\widehat{L}$. The equations of motion (1) of the perturbed Lagrangian $\widehat{L}$ are thus

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} + \frac{d}{dt}\left(\frac{\partial S}{\partial \dot{q}^i}\right) - \frac{\partial S}{\partial q^i} = 0. \tag{9}$$

If we now require that the extended Lagrangian $\widehat{L}$ admits the subalgebra $\mathcal{L}_0$ of Noether symmetries, the condition (6) leads, for each of the symmetry generators $\mathbf{X}_j$, to the constraint

$$\dot{\mathbf{X}}_k\left(\widehat{L}\right) + \mathbf{A}\left(\xi_k\right)\widehat{L} - \mathbf{A}\left(V_k\right) = \dot{\mathbf{X}}_k\left(L\right) + \mathbf{A}\left(\xi_k\right)L - \mathbf{A}\left(V_k\right) + \dot{\mathbf{X}}_k\left(S\right) + \frac{d\xi_k}{dt}S = 0, \tag{10}$$

where $1 \leq k \leq r_0$. As the $\mathbf{X}_k$ are Noether symmetries of the original Lagrangian L, and we moreover have fixed the functions $V_k(t, \mathbf{q})$,[1)] the Eq. (10) simplifies considerably, and the remaining terms constitute the integrability conditions for the perturbation term $S\left(t, \mathbf{q}, \dot{\mathbf{q}}\right)$:

$$\dot{\mathbf{X}}_k\left(S\right) + \frac{d\xi_k}{dt}S\left(t, \mathbf{q}, \dot{\mathbf{q}}\right) = \xi_k\left(t, \mathbf{q}\right)\frac{\partial S}{\partial t} + \eta_k^j\left(t, \mathbf{q}\right)\frac{\partial S}{\partial q^j} + \dot{\eta}_k^j\left(t, \mathbf{q}, \dot{\mathbf{q}}\right)\frac{\partial S}{\partial \dot{q}^j} + \frac{d\xi_k}{dt}S\left(t, \mathbf{q}, \dot{\mathbf{q}}\right) = 0, \tag{11}$$

where $1 \leq k \leq r_0$. It is immediate to verify that for the case of velocity-independent perturbation terms (i.e. $\frac{\partial S}{\partial \dot{q}^j} = 0$), a Noether symmetry is preserved only if $\frac{\partial \xi_k}{\partial q^j} = 0$ for all $1 \leq k \leq r_0$ and $1 \leq j \leq n$. The perturbation problem possesses non-trivial solutions for subalgebras $\mathcal{L}_0$ of Noether symmetries, the generators of which have components of the type $\xi_k = \varphi_k(t)$. It should be remarked that this constraint is quite usual for the structure of point (in particular Noether)

[1)] By this we mean that the function is determined by the Noether symmetry condition for the non-perturbed Lagrangian L.

symmetries of non-trivial second-order systems of differential equations [20, 21].

Another interesting special case arises for components $\xi_k = \varphi_k(t)$ linear in the variable t. In this case, $\dot{\xi}_k = \frac{d\xi_k}{dt} = \alpha_k$ is a constant and Eq. (11) reduces to

$$\dot{\mathbf{X}}_k(S) + \alpha_k S(t, \mathbf{q}, \dot{\mathbf{q}}) = 0, \qquad (12)$$

meaning that the perturbation terms correspond to the semi-invariants of weight α_k (invariants if $\alpha_k = 0$) of the prolonged vector field $\dot{\mathbf{X}}_k$. This equation can be used, for example, to determine the perturbations preserving the various subalgebras of Noether symmetries of a free Lagrangian.

3. THE BATEMAN SYSTEM

The Bateman system, first considered in 1931 [14, 15], constitutes one of the first approaches to develop an effective description of classical dissipative systems coupled to the environment, in order to reconcile the dynamical description of systems subjected to dissipative forces within the interpretation of the classical formalism. In this context, an extensive discussion of the Bateman system, its various different effective canonical descriptions and its implications for the quantization of the system can be found in [16] and references therein.

Bateman observed that a dissipative system as the usual damped harmonic oscillator can be completed to a conservative system in the plane adding a “dual" equation of motion

$$\ddot{q}^1 + K\dot{q}^1 + \omega^2 q^1 = 0, \qquad (13)$$

$$\ddot{q}^2 - K\dot{q}^2 + \omega^2 q^2 = 0, \qquad (14)$$

where K and ω are constants [14]. An admissible Lagrangian for the Bateman system is found to be

$$L_{\mathrm{B}} = \dot{q}^1\dot{q}^2 - \frac{K}{2}\left(\dot{q}^1 q^2 - q^1\dot{q}^2\right) - \omega^2 q^1 q^2, \qquad (15)$$

whereas the Hamiltonian of the system, expressed in terms of the velocities and position variables as

$$H_{\mathrm{B}} = \dot{q}^1\dot{q}^2 + \omega^2 q^1 q^2, \qquad (16)$$

is easily seen to be a constant of the motion of the system, as expected from the completion of the damped oscillator by the second equation of (13). Either starting from the Lagrangian (15) or using the fact that Noether symmetries constitute a particular type of point symmetries [19], it can be easily verified that a Noether symmetry $\mathbf{X}$ of L_{B} has the generic components

$$\xi(t, \mathbf{q}) = G(t), \qquad (17)$$

$$\eta^1(t, \mathbf{q}) = \frac{1}{2}\left(\dot{G}(t) - KG(t)\right) q^1 + \alpha q^1 + F_1(t),$$

$$\eta^2(t, \mathbf{q}) = \frac{1}{2}\left(\dot{G}(t) + KG(t)\right) q^2 - \alpha q^2 + F_2(t),$$

where for $a = 1, 2$ the functions $F_a(t)$ satisfy the second-order ordinary differential equation[2)]

$$\ddot{F}_a(t) + (-1)^{a-1} K\dot{F}_a(t) + \omega^2 F_a(t) = 0, \qquad (18)$$

while $G(t)$is a solution to the third order equation

$$\dddot{G}(t) + \left(4\omega^2 - K^2\right)\dot{G}(t) = 0. \qquad (19)$$

The scalar function $V(t, \mathbf{q})$ of (6) can be generically described as

$$V(t, \mathbf{q}) = \frac{q^1 q^2}{2}\dddot{G}(t) + q^1\left(\dot{F}_2(t) - \frac{K}{2}F_2(t)\right) + q^2\left(\dot{F}_1(t) + \frac{K}{2}F_1(t)\right). \qquad (20)$$

As the Lie algebra $\mathcal{L}_{\mathrm{NS}}$ of Noether symmetries is of dimension 8, we conclude that the Bateman system is linearizable [19, 22]. Following a standard procedure, Eq. (13) can be reformulated in matrix form as

$$\ddot{\mathbf{q}} + \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix}\dot{\mathbf{q}} + \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{pmatrix}\mathbf{q} = \mathbf{0}. \qquad (21)$$

It is well known that any linear system of the form

$$\ddot{\mathbf{q}} = M_1\dot{\mathbf{q}} + M_2\mathbf{q} \qquad (22)$$

with M_1, M_2 two commuting scalar matrices can be reduced to the friction-free form

$$\ddot{\mathbf{x}} = \left(M_2 - \frac{1}{2}M_1^2\right)\mathbf{x} \qquad (23)$$

by means of the change of variables $\mathbf{q} = \exp(M_1 t)\,\mathbf{x}$. In particular, for the Bateman system these reduced equations have the form

$$\ddot{\mathbf{Q}} + \left(\omega^2 - \frac{K^2}{4}\right)\mathbf{Q} = 0, \qquad (24)$$

and thus, in the coordinates $\{Q^1, Q^2\}$, the system is free of friction terms. It is noteworthy to observe that the friction-free representative (24) of the system is deeply connected with the symmetry condition (19). It is straightforward to verify that if $Q(t)$ satisfies the differential Eq. (24), then $G(t) = Q^2(t)$ is a solution of (19).[3)] This explicitly shows that

$$G(t) = C_1 + C_2\sin\left(\sqrt{4\omega^2 - K^2}\,t\right) \qquad (25)$$

[2)]This shows that $\{F_1(t), F_2(t)\}$ is a solution to the equations of motion (13).

[3)]The change of basis from the position variables $\{q^1, q^2\}$ to the $\{Q^1, Q^2\}$ reference actually describes an exponentially expanded coordinate system. This turns out to be an important point in the interpretation of the Bateman system [16].

$$+ C_3 \cos\left(\sqrt{4\omega^2 - K^2 t}\right).$$

The Noether symmetries of the type

$$\mathbf{X} = G(t)\frac{\partial}{\partial t} + \frac{\dot{G}(t) + (2\alpha - K)G(t)}{2} q^1 \frac{\partial}{\partial q^1} + \frac{\dot{G}(t) + (K - 2\alpha)G(t)}{2} q^2 \frac{\partial}{\partial q^2} \quad (26)$$

hence generate a subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(2)$. We observe that, as a consequence of the "duality" of the equations of motion, the Noether symmetry $\mathbf{X}_0 = \frac{\partial}{\partial t}$ associated naturally to the Hamiltonian H_{B} does not arise from the generators of the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra alone, but as a linear combination of the symmetries corresponding to the parameter values $G(t) = 1$, $\alpha = K$ in (17). We will see that this anomaly translates to conservative perturbations of the Bateman system.

Besides the Hamiltonian, a second constant of the motion is easily found from the preceding $\mathfrak{sl}(2,\mathbb{R})$ generators and equals

$$J_{\mathrm{B}} = q^2 \dot{q}^1 - q^1 \dot{q}^2 + K q^1 q^2. \quad (27)$$

For vanishing K, the latter reduces to the well-known component of angular momentum, which moreover allows to obtain the Hamiltonian using exclusively the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra.

4. PERTURBATIONS OF THE BATEMAN SYSTEM

In this paragraph we analyze small perturbations of the system (13) by means of perturbation terms of the type $S(t, \mathbf{q}, \dot{\mathbf{q}})$, and such that the $\mathfrak{sl}(2,\mathbb{R})$-subalgebra of Noether symmetries (26) is preserved. We in particular determine those perturbations for which the corresponding Hamiltonian is still a constant of the motion. In this symmetry analysis, we do not only require that the perturbed system possesses a $\mathfrak{s}(2,\mathbb{R})$ subalgebra of Noether symmetries, but also that the generators are those of (26) identically. For the perturbed Lagrangian

$$\widehat{L}_{\mathrm{B}} = \dot{q}^1 \dot{q}^2 - \frac{K}{2}\left(\dot{q}^1 q^2 - q^1 \dot{q}^2\right) - \omega^2 q^1 q^2 + \varepsilon S(t, \mathbf{q}, \dot{\mathbf{q}}), \quad (28)$$

the imposition of the Noether symmetries (26) with $\alpha = 0$ leads, by (6) and jointly with the auxiliary scalar function $V(t, \mathbf{q}) = \frac{1}{2} q^1 q^2 \ddot{G}(t)$, to the following partial differential equation for the function $S(t, \mathbf{q}, \dot{\mathbf{q}})$ (see Eq. (7)):

$$\left[\frac{\partial S}{\partial t} + \frac{K}{2}\left(\dot{q}^2 \frac{\partial S}{\partial \dot{q}^2} - \dot{q}^1 \frac{\partial S}{\partial \dot{q}^1} - q^1 \frac{\partial S}{\partial q^1} \right.\right. \quad (29)$$

$$\left.\left. + q^2 \frac{\partial S}{\partial q^2}\right)\right] G(t) + \left[q^1 \frac{\partial S}{\partial \dot{q}^1} + q^2 \frac{\partial S}{\partial \dot{q}^2}\right] \ddot{G}(t) + \frac{1}{2}\left[\left(K q^2 - \dot{q}^2\right)\frac{\partial S}{\partial \dot{q}^2} - \left(K q^1 + \dot{q}^1\right)\frac{\partial S}{\partial \dot{q}^1} + q^1 \frac{\partial S}{\partial q^1} + q^2 \frac{\partial S}{\partial q^2} + 2S(t, \mathbf{q}, \dot{\mathbf{q}})\right] \dot{G}(t) = 0.$$

In order to ensure that this equation is satisfied for the three symmetries in (26), we separate the equation into a system consisting of the coefficients of $G(t)$, $\ddot{G}(t)$ and $\dot{G}(t)$ respectively. We hence obtain

$$\frac{\partial S}{\partial t} + \frac{K}{2}\left(\dot{q}^2 \frac{\partial S}{\partial \dot{q}^2} - \dot{q}^1 \frac{\partial S}{\partial \dot{q}^1} - q^1 \frac{\partial S}{\partial q^1} + q^2 \frac{\partial S}{\partial q^2}\right) = 0,$$

$$q^1 \frac{\partial S}{\partial \dot{q}^1} + q^2 \frac{\partial S}{\partial \dot{q}^2} = 0,$$

$$\left(K q^2 - \dot{q}^2\right)\frac{\partial S}{\partial \dot{q}^2} - \left(K q^1 + \dot{q}^1\right)\frac{\partial S}{\partial \dot{q}^1} + q^1 \frac{\partial S}{\partial q^1} + q^2 \frac{\partial S}{\partial q^2} + 2S(t, \mathbf{q}, \dot{\mathbf{q}}) = 0. \quad (30)$$

The general solution to the two first equations has the form

$$S(t, \mathbf{q}, \dot{\mathbf{q}}) = \widehat{S}\left(q^1 \exp\left(\frac{Kt}{2}\right), q^2 \exp\left(\frac{-Kt}{2}\right), \dot{q}^1 q^2 - q^1 \dot{q}^2\right). \quad (31)$$

Introducing now the auxiliary variables $u = q^1 \times \exp\left(\frac{Kt}{2}\right)$, $v = q^2 \exp\left(\frac{-Kt}{2}\right)$, $w = \dot{q}^1 q^2 - q^1 \dot{q}^2$ and evaluating the previous solution in the third equation of (30) further leads to the linear partial differential equation

$$u \frac{\partial \widehat{S}}{\partial u} + v \frac{\partial \widehat{S}}{\partial v} + 2Kuv \frac{\partial \widehat{S}}{\partial w} + 2\widehat{S}(u, v, w) = 0, \quad (32)$$

By elementary methods, the general solution is found to be

$$\widehat{S}(u, v, w) = \frac{S_0\left(\frac{v}{u}, w + Kuv\right)}{u^2} = \frac{1}{\left(q^1\right)^2 \exp(Kt)} S_0\left(\frac{q^2}{q^1 \exp(Kt)}, \dot{q}^1 q^2 - q^1 \dot{q}^2 + K q^1 q^2\right). \quad (33)$$

Let us define $\widehat{u} = v u^{-1} = q^2 \left(q^1\right)^{-1} \exp(-Kt)$ and $W = \dot{q}^1 q^2 - q^1 \dot{q}^2 + K q^1 q^2$. Written in terms of these new auxiliary variables, the perturbed Lagrangian is given by

$$\widehat{L}_{\mathrm{B}} = \dot{q}^1 \dot{q}^2 - \frac{K}{2}\left(\dot{q}^1 q^2 - q^1 \dot{q}^2\right) \quad (34)$$

$$- \omega^2 q^1 q^2 + \frac{\varepsilon \widehat{u}}{q^1 q^2} S_0 (\widehat{u}, W)$$

and the corresponding equations of the motion are[4)]

$$\ddot{q}^1 + K\dot{q}^1 + \omega^2 q^1 - \frac{\varepsilon \widehat{u}^2}{q^1 (q^2)^2} \left(\frac{\partial S_0}{\partial \widehat{u}} - W \frac{\partial^2 S_0}{\partial \widehat{u} \partial W} \right) - \frac{\varepsilon \widehat{u} \dot{W}}{q^2} \frac{\partial^2 S_0}{\partial W^2} = 0, \qquad (35)$$

$$\ddot{q}^2 - K\dot{q}^2 + \omega^2 q^2 + \frac{\varepsilon \widehat{u}}{(q^1)^2 q^2} \left(2S_0 (\widehat{u}, W) + \widehat{u} \left(\frac{\partial S_0}{\partial \widehat{u}} - W \frac{\partial^2 S_0}{\partial \widehat{u} \partial W} \right) - 2W \frac{\partial S_0}{\partial W} \right) + \frac{\varepsilon \widehat{u} \dot{W}}{q^1} \frac{\partial^2 S_0}{\partial W^2} = 0. \qquad (36)$$

For any generic choice of $S_0 (\widehat{u}, W)$, the system is clearly non-linear and certainly $\mathfrak{sl}(2, \mathbb{R})$ invariant. A long but routine computation shows that the number of independent Noether symmetries for the Lagrangian (34) is at most four, proving the non-linearity of the system [19]. As a matter of fact, the exact number of Noether symmetries is determined by the properties of the Hamiltonian $\widehat{H}_\mathrm{B}$ associated to $\widehat{L}_\mathrm{B}$, and given explicitly by

$$\widehat{H}_\mathrm{B} = \dot{q}^1 \dot{q}^2 + \omega^2 q^1 q^2 + \frac{\widehat{u}}{q^1 q^2} \left(\left(\dot{q}^1 q^2 - q^1 \dot{q}^2 \right) \frac{\partial S_0}{\partial W} - S_0 (\widehat{u}, W) \right). \qquad (37)$$

In these conditions, the two following possibilities can occur:

If $\frac{d\widehat{H}_\mathrm{B}}{dt} \neq 0$, the Noether symmetry algebra of $\widehat{L}_\mathrm{B}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

If $\frac{d\widehat{H}_\mathrm{B}}{dt} = 0$, the Noether symmetry algebra of $\widehat{L}_\mathrm{B}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(2)$.

Computing the time derivative of $\widehat{H}_\mathrm{B}$ and simplifying the resulting expression, we obtain

$$\frac{d\widehat{H}_\mathrm{B}}{dt} = \frac{K\widehat{u}}{q^1 q^2} \left(F(\widehat{u}, W) + \widehat{u} \frac{\partial F}{\partial \widehat{u}} \right). \qquad (38)$$

Therefore, it follows that the condition $\frac{d\widehat{H}_\mathrm{B}}{dt} = 0$ is satisfied only if

$$F(\widehat{u}, W) = \frac{F_1(W)}{\widehat{u}}, \qquad (39)$$

holds, from which we conclude that the Hamiltonian of (28) is a constant of the motion if and only if the perturbation factor has the form

$$S(t, \mathbf{q}, \dot{\mathbf{q}}) = \frac{S_0 \left(\dot{q}^1 q^2 - q^1 \dot{q}^2 + K q^1 q^2 \right)}{q^1 q^2}. \qquad (40)$$

Now we observe that for perturbation terms of the latter form, the vector field $\mathbf{Y} = q^1 \frac{\partial}{\partial q^1} - q^2 \frac{\partial}{\partial q^2}$ is automatically a Noether symmetry of $\widehat{L}_\mathrm{B}$, showing that the algebra of Noether symmetries is four-dimensional. This, as observed before for the classical Bateman system, follows from the fact that the infinitesimal generator $\frac{\partial}{\partial s}$ cannot be expressed only in terms of the generators of the $\mathfrak{sl}(2, \mathbb{R})$ subalgebra, and is, in last instance, a consequence of the different sign of the friction term in the equations of motion.

In any case, independently on the conservative nature of the perturbation, the non-linear system (35), (36) possesses two independent constants of the motion, and can thus be integrated, at least formally, as the precise form of the invariants may be quite involved for velocity-dependent perturbation factors. For the special case with $\frac{\partial S_0}{\partial \dot{\mathbf{q}}} = 0$, which can be seen as a perturbation of the potential, however, the equations of motion (35), (36) are given in normal form:

$$\ddot{q}^1 + K\dot{q}^1 + \omega^2 q^1 - \frac{\varepsilon \widehat{u}^2}{q^1 (q^2)^2} \frac{\partial S_0}{\partial \widehat{u}} = 0, \qquad (41)$$

$$\ddot{q}^2 - K\dot{q}^2 + \omega^2 q^2 + \frac{\varepsilon \widehat{u}}{(q^1)^2 q^2} \left(2S_0 (\widehat{u}) + \widehat{u} \frac{\partial S_0}{\partial \widehat{u}} \right) = 0. \qquad (42)$$

and the two independent invariants can be expressed as

$$J_1 = \frac{1}{2} W^2 + 2\widehat{u} S_0 (\widehat{u}), \qquad (43)$$

$$J_2 = \widehat{H}_\mathrm{B} - K \int \frac{\widehat{u} \left(F(\widehat{u}) + \widehat{u} F'(\widehat{u}) \right)}{q^1 q^2} dt. \qquad (44)$$

4.1. Perturbations as Generalized Ermakov Systems with Friction

Despite the fact that the Hamiltonian (37) is not a conserved quantity for generic functions (33), the systems (35), (36) are of interest because of their two independent invariants derived from the $\mathfrak{sl}(2, \mathbb{R})$ Noether symmetry algebra. There is however another property worthy to be observed: for the limit $K \to 0$, the Bateman system goes over to a two-dimensional uncoupled harmonic oscillator. In this sense, for $K \to 0$ the equations of motion (35), (36) correspond to those of a perturbation of the harmonic oscillator:

$$\ddot{q}^1 + \omega^2 q^1 - \frac{\varepsilon}{(q^1)^3} \left(\frac{\partial S_0}{\partial \widetilde{u}} - \widetilde{W} \frac{\partial^2 S_0}{\partial \widetilde{u} \partial \widetilde{W}} \right) \qquad (45)$$

[4)] Note that because of the terms in $\dot{W}$, the equations of motion are generally not given in normal form.

$$-\frac{\varepsilon\widetilde{u}\dot{\widetilde{W}}}{q^1}\frac{\partial^2 S_0}{\partial\widetilde{W}^2}=0,$$

$$\ddot{q}^2+\omega^2q^2+\frac{\varepsilon}{(q^1)^3}\Big(2S_0\left(\widetilde{u},\widetilde{W}\right)+\widetilde{u}\left(\frac{\partial S_0}{\partial\widetilde{u}}-\widetilde{W}\frac{\partial^2 S_0}{\partial\widetilde{u}\partial\widetilde{W}}\right)-2\widetilde{W}\frac{\partial S_0}{\partial\widetilde{W}}\Big)+\frac{\varepsilon q^2\dot{W}}{(q^1)^2}\frac{\partial^2 S_0}{\partial W^2}=0, \quad (46)$$

where the variables $\widetilde{u}$ and $\widetilde{W}$ are defined as

$$\widetilde{u}=\lim_{K\to 0}\widehat{u}=\frac{q^2}{q^1},$$

$$\widetilde{W}=\lim_{K\to 0}\widehat{W}=\dot{q}^1q^2-q^1\dot{q}^2. \quad (47)$$

It further follows from (38) that

$$\lim_{K\to 0}\frac{d\widehat{H}_{\mathrm{B}}}{dt}=0, \quad (48)$$

showing that (45), (46) correspond to the equations of motion of a Hamiltonian system. It turns out that these equations are those of a generalized Ermakov system with a velocity-dependent potential and constant frequency ω as considered by many authors, with the particularity of being Hamiltonian [23, 24]. In this sense, the perturbation of the Bateman system given by the equations of the motion (35), (36) can be interpreted as a generalization of Ermakov systems possessing friction terms proportional to the velocities. It is important to observe that these non-linear systems with Lagrangian (34) cannot be obtained from perturbing either the two dimensional harmonic oscillator or a system formed by two uncoupled damped oscillators, because of the time-reversal nature of the second equation of motion.

4.2. Conservative Perturbations of the Bateman System

Considering now the conservative perturbations of the Bateman system, i.e., restricting to functions of the form (40), the equations of the motion, brought into normal form, are given by

$$\ddot{q}^1+K\dot{q}^1+\omega^2q^1+\frac{1}{q^1\left(q^2\right)^2}\left(S_0\left(W\right)-W\frac{dS_0}{dt}\right)=0, \quad (49)$$

$$\ddot{q}^2-K\dot{q}^2+\omega^2q^2+\frac{1}{\left(q^1\right)^2q^2}\left(S_0\left(W\right)-W\frac{dS_0}{dt}\right)=0. \quad (50)$$

In this form, the resemblance with generalized Ermakov systems for $K=0$ is immediately seen. The invariants of the conservative perturbation are easily deduced from the either the symmetry algebra $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{so}(2)$ or the equations of motion:

$$J_1=W-2\frac{dS_0}{dW}=\left(\dot{q}^1q^2-q^1\dot{q}^2+Kq^1q^2\right)-2\frac{dS_0}{dW}, \quad (51)$$

$$J_2=\widehat{H}_{\mathrm{B}}=\dot{q}^1\dot{q}^2+\omega^2q^1q^2-\frac{S_0\left(W\right)}{q^1q^2}+W\frac{dS_0}{dW}. \quad (52)$$

Finally, if $S_0(W)$ is a constant function, the perturbation term is merely

$$S\left(t,\mathbf{q}\right)=\frac{\alpha}{q^1q^2}, \quad (53)$$

where α is a nonzero constant. In particular, the invariant J_1 coincides exacly with that of the unperturbed Bateman system (27), suggesting that the perturbation term (53) provides a natural non-linear version of the Bateman system.

5. DISSIPATIVE GENERALIZATIONS OF THE BATEMAN SYSTEM

There are various possibilities of generalizing the Bateman system, to the price of losing the corresponding Hamiltonian as a constant of the motion. The most obvious and simple generalization of the system (13) is given by replacing the constants K and ω by time-dependent functions, resulting in the time-dependent Lagrangian

$$L\left(t,\mathbf{q},\dot{\mathbf{q}}\right)=\dot{q}^1\dot{q}^2-\varphi\left(t\right)\left(\dot{q}^1q^2-q^1\dot{q}^2\right)-\omega^2\left(t\right)q^1q^2, \quad (54)$$

where $\varphi(t)$, $\omega(t)$ are undetermined functions. In this case, the equations of motion are

$$\ddot{q}^1+2\varphi\left(t\right)\dot{q}^1+\left(\omega^2\left(t\right)+\dot{\varphi}\left(t\right)\right)q^1=0, \quad (55)$$

$$\ddot{q}^2-2\varphi\left(t\right)\dot{q}^2+\left(\omega^2\left(t\right)-\dot{\varphi}\left(t\right)\right)q^2=0. \quad (56)$$

The Hamiltonian of the system, expressed again in velocity-position variables, is given by

$$H=\dot{q}^1\dot{q}^2+\omega^2\left(t\right)q^1q^2, \quad (57)$$

and, as expected, the time derivative vanishes only for constant functions $\varphi(t)$ and $\omega(t)$

$$\frac{dH}{dt}=2\omega\left(t\right)\dot{\omega}\left(t\right)q^1q^2+\dot{\varphi}\left(t\right)\times\left(\dot{q}^1q^2-q^1\dot{q}^2\right)\neq 0. \quad (58)$$

From the point of view of symmetries and conservation laws, there are, however, close similarities to the

Bateman system. Also in this case, a generic Noether symmetry of (54) has the components

$$\xi(t,\mathbf{q}) = \theta(t),$$
$$\eta^1(t,\mathbf{q}) = \frac{1}{2}\left(\dot\theta(t) - \varphi(t)\theta(t)\right)q^1 + \alpha q^1 + F_1(t),$$
$$\eta^2(t,\mathbf{q}) = \frac{1}{2}\left(\dot\theta(t) + \varphi(t)\theta(t)\right)q^2 - \alpha q^2 + F_2(t), \qquad (59)$$

where $F_1(t)$ and $F_2(t)$ are solutions to the equations of motion (55) and (56) respectively, while θ satisfies the third-order equation

$$\dddot\theta + 4\left(\omega^2(t) - \varphi^2(t)\right)\dot\theta + 4\Big(\omega(t)\dot\omega(t) - \varphi(t)\dot\varphi(t)\Big)\theta(t) = 0. \qquad (60)$$

Solutions to the latter equation are of the type $\theta(t) = U^2(t)$, where $U(t)$ satisfies

$$\ddot U(t) + \left(\omega^2(t) - \varphi^2(t)\right)U(t) = 0. \qquad (61)$$

This shows that the system (55), (56) is always linearizable, independently of the values of $\varphi(t)$ and $\omega(t)$. The invariants of the system can be shown to equal

$$J_1 = \dot q^1 q^2 - q^1\dot q^2 + 2\varphi(t)q^1q^2, \qquad (62)$$
$$J_2 = \dot q^1\dot q^2 - \varphi(t)\left(\dot q^1q^2 - q^1\dot q^2 - \varphi(t)q^1q^2\right) + \int^{q^1q^2}\left(\omega^2(s) - \varphi^2(s)\right)ds. \qquad (63)$$

As before, we can consider a perturbed Lagrangian

$$\widehat L(t,\mathbf{q},\dot{\mathbf{q}}) = \dot q^1\dot q^2 - \varphi(t)\left(\dot q^1q^2 - q^1\dot q^2\right) - \omega^2(t)q^1q^2 + \varepsilon S(t,\mathbf{q},\dot{\mathbf{q}}) \qquad (64)$$

and impose that the three symmetries $\mathbf{X} = \xi(t,\mathbf{q})\frac{\partial}{\partial t} + \eta^\alpha(t,\mathbf{q})\frac{\partial}{\partial q^\alpha}$ associated with the function $\theta(t)$ of (60) are Noether symmetries. Using the same ansatz of separating the resulting symmetry condition (6) for the perturbation factor $S(t,\mathbf{q},\dot{\mathbf{q}})$, we are led to the following system of partial differential equations:

$$q^1\frac{\partial S}{\partial\dot q^1} + q^2\frac{\partial S}{\partial\dot q^2} = 0, \qquad (65)$$
$$2S(t,\mathbf{q},\dot{\mathbf{q}}) + 2\varphi(t)\left(q^2\frac{\partial S}{\partial\dot q^2} - q^1\frac{\partial S}{\partial\dot q^1}\right) - \dot q^1\frac{\partial S}{\partial\dot q^1} - \dot q^2\frac{\partial S}{\partial\dot q^2} - q^1\frac{\partial S}{\partial q^1} - q^2\frac{\partial S}{\partial q^2} = 0,$$
$$\frac{\partial S}{\partial t} + 2\dot\varphi(t)\left(q^2\frac{\partial S}{\partial\dot q^2} - q^1\frac{\partial S}{\partial\dot q^1}\right) + \varphi(t)\left(\dot q^2\frac{\partial S}{\partial\dot q^2} - \dot q^1\frac{\partial S}{\partial\dot q^1} - q^1\frac{\partial S}{\partial q^1} + q^2\frac{\partial S}{\partial q^2}\right) = 0.$$

The general solution is again found by successive solving of these equations, and can be written in the form

$$S(t,\mathbf{q},\dot{\mathbf{q}}) = \frac{\exp\left(-2\int\varphi(t)\,dt\right)}{(q^1)^2} \times S_0\bigg(\frac{q^2}{q^1}\exp\left(-2\int\varphi(t)\,dt\right), \dot q^1q^2 - q^1\dot q^2 + 2\varphi(t)q^1q^2\bigg). \qquad (66)$$

It can be verified that if we require $S(t,\mathbf{q},\dot{\mathbf{q}})$ to be independent on the time t, the only possibility is given by

$$S(t,\mathbf{q},\dot{\mathbf{q}}) = \frac{S_0\left(\dot q^1q^2 - q^1\dot q^2 + 2\varphi(t)q^1q^2\right)}{q^1q^2}. \qquad (67)$$

Although perturbation terms of the latter form imply the existence of an additional symmetry, so that the Lagrangian (64) possesses exactly four independent Noether symmetries, this does not mean that $\frac{\partial}{\partial t}$ is also a Noether symmetry, as can be seen from the symmetry components in (59). In this case, the invariant of the Lagrangian (64) linear in the velocities is the natural generalization of (62):

$$J_1 = \dot q^1q^2 - q^1\dot q^2 + 2\varphi(t)q^1q^2 - 2\frac{dS_0}{dt}. \qquad (68)$$

There exists in addition a second invariant quadratic in the velocities, the explicit integral form of which is omitted, and being dependent on both $\omega^2(t)$ and $\dot\varphi(t)$.

Taking the limit $\lim_{K\to 0}\widehat L(t,\mathbf{q},\dot{\mathbf{q}})$, i.e., neglecting the damping terms, the equations of motion of the Lagrangian (64) can be seen to be a generalized (Hamiltonian) Ermakov system, having a similar structure to those obtained in the literature from the time-dependent harmonic oscillator (see [24, 25] and references therein). Within this interpretation, the symmetry-preserving non-linear perturbations of (54) can be considered as a dissipative but integrable generalization of Ermakov–Ray–Reid systems.

6. CONCLUSIONS

The perturbation problem for Lagrangians preserving a fixed subalgebra of Noether symmetries has been studied in the context of the damped harmonic oscillator extended to a conservative Hamiltonian system [14]. For the classical Bateman system, it has been shown that perturbations preserving the Hamiltonian as a constant of the motion are characterized by a four dimensional algebra of Noether symmetries isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(2)$, the fourth Noether symmetry being a direct consequence of the time-reserved version of the damped oscillator in the Bateman system. It follows that an exact Noether symmetry algebra isomorphic $\mathfrak{sl}(2,\mathbb{R})$ determines perturbations such that the Hamiltonian is lost as an invariant. However, being integrable perturbations, these non-linear systems can be interpreted as a further generalization of Ermakov systems, with the particularity of possessing a friction term proportional to the velocities. Purely dissipative generalizations of the Bateman system and their symmetry preserving perturbations are also analyzed.

A problem that arises naturally and deserves to be inspected more closely is the interrelation between the various canonical descriptions of the conservative perturbations (40) of the Bateman system, along the same lines developed in [16]. As shown there, the additional variable introduced to complete the damped harmonic oscillator does not correspond to a position variable. In this context, for the non-linear perturbation (49), (50), this anomaly is preserved, in addition to the non-trivial coupling of the variables $\mathbf{q}$ in the perturbation factor. Whether these symmetry-preserving expansions of the Bateman Hamiltonian have some potential physical meaning, as those studied recently in [26] in the context of optical resonators, constitutes currently an unsolved question.

As the perturbation problem is not restricted to linearizable systems, it can further be of potential interest for the analysis of other (dissipative) Lagrangian systems involving irreversible nonlinear heat transport [27, 28] with few Noether symmetries. In this context, comparison of the conservation laws resulting from the perturbed system could serve as an additional tool in the analysis of equilibrium problems.

ACKNOWLEDGMENTS

The author was partially supported by the research project MTM2013-43820-P of the Ministerio de Economía y Competitividad (MINECO).

REFERENCES

1. E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies,* 4th ed. (Cambridge Univ. Press, Cambridge, 1947).
2. M. Päsler, *Prinzipe der Mechanik* (Walter de Gruyter, Berlin, 1968).
3. R. W. Leven, B.-P. Koch, and B. Pompe, *Chaos in Dissipativen Systemen* (Akademie, Berlin, 1989).
4. P. Bracken, Acta Appl. Math. **57**, 83 (1999).
5. B. D. Vujanovic, Int. J. Non-Linear Mech. **5**, 269 (1970).
6. L. Y. Bahar and H. G. Kwanty, Hadron. J. **3**, 1264 (1980).
7. W. Sarlet and F. Cantrijn, SIAM Rev. **23**, 467 (1981).
8. F. Badrakhan, J. Sound Vibrat. **82**, 227 (1982).
9. M. Tsamparlis and A. Paliathanasis, J. Phys. A **44**, 175202 (2011).
10. V. A. Dorodnitsyn and N. H. Ibragimov, Commun. Nonlin. Sci. Numer. Simul. **19**, 328 (2014).
11. M. Henneaux, Ann. Phys. (N. Y.) **140**, 45 (1982).
12. B. D. Vujanovic and S. E Jones, *Variational Methods in Nonconservative Phenomena* (Academic, Boston, 1989).
13. M. Razavy, *Classical and Quantum Dissipative Systems* (Imperial College Press, London, 2005).
14. H. Bateman, Phys. Rev. **38**, 815 (1931).
15. P. M. Morse and H. Feshbach, *Methods in Theoretical Physics* (McGraw Hill, New York, 1953), Vol. 1.
16. D. Schuch, J. Guerrero, F. F. López Ruiz, and V. Aldaya, Phys. Scr. **90**, 045209 (2015).
17. W. H. Steeb, *Continuous Symmetries, Lie Algebras, Differential Equations and Computer Algebra,* 2nd ed. (World Scientific, Singapore, 2007).
18. E. J. Saletan and A. H. Cromer, *Theoretical Mechanics* (Wiley, New York, 1971).
19. L. V. Ovsyannikov, *Group Analysis of Differential Equations* (Academic, New York, 1982).
20. R. Campoamor-Stursberg, Commun. Nonlin. Sci. Numer. Simul. **19**, 2602 (2014).
21. T. G. Mkhize, S. Moyo, and S. V. Meleshko, Math. Methods Appl. Sci. **38**, 1824 (2015).
22. R. Campoamor-Stursberg, Commun. Nonlin. Sci. Numer. Simul. **17**, 1178 (2012).
23. C. Athorne, Phys. Lett. A **159**, 375 (1991).
24. F. Haas and J. Goedert, Phys. Lett. A **332**, 25 (2004).
25. J. R. Ray and J. L. Reid, J. Math. Phys. **22**, 91 (1981).
26. C. M. Bender, M. Gianfreda, S. K. Ozdemir, et al., Phys. Rev. A **88**, 062111 (2013).
27. J. Verhas, Adv. Thermodyn. **3**, 250 (1990).
28. S. Sieniutycz, *Conservation Laws in Variational Thermo-Hydrodynamics* (Kluwer, Amsterdam, 1994).